Lecture 2: Theory of the discrete series.

H is reductive $\Rightarrow$ H unimodular $\Rightarrow$

$X = G/H$ has left invariant measure dx

$L^2(G/H)$ carries the left regular repⁿ L;

$$L_g\varphi(x) = \varphi(g^{-1}x) \quad (\varphi \in L^2(G/H),\ x \in X,\ g \in G).$$

Harmonic Analysis: Plancherel decomposition of $L^2(G/H)$ in terms of irreducible unitary reps.

Basic Representation theory.

Setting: V Fréchet (or complete loc cvx).

Def: A continuous repⁿ of G in V is a group homom $\pi: G \to GL(V)$ s.t. $(g,v) \mapsto \pi(g)v$, $G \times V \to V$ is cts.

1) invariant subspace of V: a linear subspace $W \subseteq V$ s.t.

$$\pi(g)W \subset W \quad (\forall g \in G)$$

2) π is irreducible $:\Longleftrightarrow$ $0, V$ are the only closed invariant subspaces of V

3) $V^\infty := \{ v \in V \mid g \mapsto \pi(g)v,\ G \to V, \text{ is } C^\infty \}$

smooth vectors, V^∞ is G-invariant.

4) Derived rep^n: $\pi_* : \mathfrak{g} \longrightarrow \mathrm{End}(V^\infty)$, $\pi_*(X)v = \frac{d}{dt}\Big|_{t=0} \pi(\exp tX)v$

5) $U(\mathfrak{g})$-rep^n : $\pi_* : U(\mathfrak{g}) \longrightarrow \mathrm{End}(V^\infty)$

6) $V_K = \{ v \in V \mid \dim \mathrm{span}(\pi(K)v) < \infty \}$ (K-finite vectors)

7) <u>Fact</u>: $V^\infty \cap V_K$ is dense in V.

8) <u>Def</u>: for $\delta \in \hat{K}$, put $V[\delta] = \mathrm{image}(V_\delta \otimes \mathrm{Hom}(V_\delta, V))$

Then $V_K = \bigoplus_{\delta \in \hat{K}} V[\delta]$ (isotypical components)

9) <u>Def</u>: V admissible $:\Longleftrightarrow$ $\forall_{\delta \in \hat{K}} \dim V[\delta] < \infty$

10) <u>Lemma</u> V admissible $\Longrightarrow$ $V_K \subset V^\infty$ (is a $(\mathfrak{g}, K)$-module)

11) **Def** A $(\mathfrak{g},K)$-module is a linear space $V/\mathbb{C}$ equipped with a $\mathfrak{g}$-rep & a K-rep s.t.

- $\forall_{v\in V_K}$ $K\times \operatorname{span}\{K.v\}\to \operatorname{span}\{K.v\}$ is C^∞
- $\forall_{X\in\mathfrak{k},\, v\in V}$: $Xv = \frac{d}{dt}\Big|_{t=0} \exp tX.v$
- $\forall_{k\in K,\, X\in U(\mathfrak{g})}$ $k\circ X = (\operatorname{Ad}(k)X)\circ k$

12) **Lemma** If (π,V) is admissible then V_K is admissible $(\mathfrak{g},K)$-module. Furthermore,

a) the map $W\longmapsto W\cap V_K$ defines a bijection between the closed G-invariant subspaces of V and the $(\mathfrak{g},K)$-invariant subspaces of V_K. The inverse is given by $U\longmapsto \overline{U}$.

b) (π,V) is irreducible iff V_K is irred $(\mathfrak{g},K)$-module

13) An intertwining operator from (π_1, V_1) to (π_2, V_2) is a continuous linear map $T: V_1 \to V_2$ s.t $\forall (g \in G)$:

$$\begin{tikzcd} V_1 \arrow[r, "T"] \arrow[d, "\pi_1(g)"'] \arrow[dr, phantom, "\circlearrowright"] & V_2 \arrow[d, "\pi_2(g)"] \\ V_1 \arrow[r, "T"'] & V_2 \end{tikzcd}$$

14) Exercise: Versions of Schur's lemma:

- for V admiss^{le} $(\mathfrak{g}, K)$-module: V irred^{le} $\Rightarrow \mathrm{End}_{\mathfrak{g},K}(V) = \mathbb{C}I$
- for (π, V) admissible: π irred^{le} $\Rightarrow \mathrm{End}_G(V) = \mathbb{C}I$.

15) Def: a Harish-Chandra" module is a finitely generated admissible $(\mathfrak{g}, K)$-module.

16) A motivating result, due to HC. Suppose that $(\pi, \mathcal{H})$ is irreducible unitary. Then π is admissible.

Let $(\pi_1, \mathcal{H}_1)$, $(\pi_2, \mathcal{H}_2)$ be irreducible unitary.
If $\mathcal{H}_{1K}$, $\mathcal{H}_{2K}$ are equivalent as $(\mathfrak{g}, K)$-modules, then π_1 and π_2 are unitarily equivalent.

Defn $\mathfrak{Z} = \mathfrak{Z}(\mathfrak{g}) := $ center $U\mathfrak{g}$.

Defn (π, V) is called quasisimple if $\mathfrak{Z}$ acts by scalars on V^∞ ...
i.e., $\exists\ \chi \in \hat{\mathfrak{Z}} = \mathrm{Hom}(\mathfrak{Z}, \mathbb{C})$: $Z - \chi(Z) I = 0$ on V for all $Z \in \mathfrak{Z}$. (χ is called infinitesimal character of π).

Thm (HC) Let $(\pi, \mathcal{H})$ be irreducible unitary. Then π is quasi-simple.

Back to $\mathbb{D}(G/H)$

$D \in \mathbb{D}(G/H)$, formal adjoint $D' \in \mathbb{D}(G/H)$ is defined by

$$\langle Df, g \rangle = \langle f, D'g \rangle \quad (f, g \in C_c^\infty(\mathbb{R}^n)).$$

(Here $\langle f, g \rangle = \int_X f(x)\overline{g(x)}\,dx$).

Thm $D = D' \Rightarrow D$ is essentially self-adjoint with operator core $L^2(X)^\infty$.

Cor The $\mathbb{R}$-algebra $\mathbb{D}_s(G/H) = \{ D \in \mathbb{D}(G/H) \mid D' = D \}$ is finitely generated and its elements strongly commute.

Proof $\mathbb{D}(G/H) \simeq S(\mathfrak{a}^{d*})^{W(\mathfrak{a}^d)}$ is finitely generated as a $\mathbb{C}$-algebra. From this, the first assertion follows. By a thm of Nelson, the final conclusion follows. $\square$.

Def A unitary repⁿ $(\pi, \mathcal{H})$ of G belongs to the discrete series of G/H if it is irreducible & there exists a unitary G-intertwining map $i : \mathcal{H} \longrightarrow L^2(G/H)$.

Not. $(G/H)^\wedge_{ds}$ = set of unit^y equivalence classes of reps from the discrete series of G/H.

Def. for $\xi \in (G/H)^\wedge_{ds}$, $L^2(G/H)_\xi :=$ image of $(\mathcal{H}_\xi \otimes \mathrm{Hom}_G(\mathcal{H}_\xi, L^2(G/H)))$.

Rem
$$L^2_d(G/H) = \bigoplus^{\perp}_{\xi \in (G/H)^\wedge_{ds}} L^2(G/H)_\xi . \quad (\text{type } \xi)$$

Lemma R induces injective homⁿ of algebras $\mathfrak{Z} \hookrightarrow \mathbb{D}(G/H)$.
Accordingly, $\mathbb{D}(G/H)$ is a finite $\mathfrak{Z}$-module.

Cor $\forall_\xi$ $L^2(G/H)^\infty_{\xi K}$ is a finite $\mathbb{D}(G/H)$-module which allows a direct sum decomposition into $(\mathfrak{g}, K)$-submodules on which $\mathbb{D}(G/H)$ acts by scalars.

Not^n For $\chi \in \mathbb{D}(G/H)^\wedge$, put

$$\mathcal{E}_\chi(G/H) := \{ f \in C^\infty(G/H) \mid Df = \chi(D) f \ (D \in \mathbb{D}(G/H)) \}$$

Goal: For each $\chi \in \mathbb{D}(G/H)^\wedge$ describe the irreducible $(\mathfrak{g}, K)$ – submodules of $\mathcal{E}_\chi(G/H)_K \cap L^2(G/H)^\infty$

Flensted – Jensen's idea: Use duality $G/H \longleftrightarrow G^d/K^d$

For simplicity: assume $G < G_{\mathbb{C}}$ and define G^d, K^d, H^d as Lie subgps of $G_{\mathbb{C}}$ with Lie algebra's $\mathfrak{g}^d, \mathfrak{k}^d, \mathfrak{h}^d$.

Note: $G_+ = \exp(\mathfrak{p} \cap \mathfrak{q})(K \cap H)$, so $G_+ \subset G \cap G^d$.

for $f \in C^\infty(G/H)_K$ and $x \in G_+$ the function $k \longmapsto f(kx)$ has a unique analytic extension to $f_x : K_{\mathbb{C}} \longrightarrow \mathbb{C}$.

Thm (F-J) $\exists!$ map $f \mapsto {}^d f$, $C^\infty(G/H)_K \longrightarrow C^\infty(G^d/K^d)_{H^d}$

s.t.

1) ${}^d f = f$ on G_+

2) for all $x \in G_+$, $h \in H^d$, ${}^d f(hx) = f_x(h)$.

For all $D \in \mathbb{D}(G/H)$,

$$^d(Df) = {}^d D \, {}^d f.$$

Cor $f \mapsto {}^d f$ gives $\mathcal{E}_\chi(G/H)_K \hookrightarrow \mathcal{E}_{{}^d\chi}(G^d/K^d)$

where ${}^d\chi \in \mathbb{D}(G^d/K^d)^\wedge$ is defined by

$$^d\chi({}^d D) = \chi(D).$$

Intermezzo Poisson transform on G/K.

Setting: $\mathfrak{a} \subset \mathfrak{p}$ max. abelian, $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$, $W \simeq N_K(\mathfrak{a})/Z_K(\mathfrak{a})$

$$\Sigma^+ \rightsquigarrow \mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha, \quad N = \exp \mathfrak{n}.$$

- **Iwasawa decomp:** $G = KAN \quad (\simeq K \times A \times N)$
- $M = Z_K(\mathfrak{a})$ normalizes A and N
- $P := MAN < G$ closed subgp (minimal parabolic)
- For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ the char$^{\text{r}}$ $\chi_\lambda \in \mathbb{D}(G/K)^\wedge$ is def$^{\text{d}}$ by

$$\chi_\lambda(D) = \gamma(D, \lambda), \quad (D \in \mathbb{D}(G/K)).$$

- **Notation:** $\mathcal{E}_\lambda(G/K) = \mathcal{E}_{\chi_\lambda}(G/K)$.
- **Exponential:** for $\xi \in \mathfrak{a}^*_{\mathbb{C}}$, put $a^\xi := e^{\xi(\log a)} \quad (a \in A)$

Induced Representation

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ define $(1 \otimes \lambda \otimes 1) : P \longrightarrow \mathbb{C}$ by

$$(1 \otimes \lambda \otimes 1)(man) = a^{\lambda} \quad (= e^{\lambda(\log a)}).$$

We define $\pi_\lambda \doteq \mathrm{Ind}_P^G (1 \otimes (-\lambda) \otimes 1)$ to be the representation on

$$C^0(G/P : -\lambda) = \{ f \in C^0(G) \mid f(xman) = a^{\lambda - \rho_{\mathfrak{n}}} f(x)\}$$
$$(x \in G,\ m \in M,\ a \in A,\ n \in N)$$

given by $(\pi_\lambda(g) f)(x) = L_g f(x) = f(g^{-1}x)$.

Def The Poisson transform $\mathcal{P}_\lambda : C(G/P : -\lambda) \longrightarrow C^\infty(G/K)$ is defined by

$$\mathcal{P}_\lambda \varphi(x) = \int_K \varphi(xk)\, dk \quad (x \in G).$$

Lemma $\mathcal{P}_\lambda$ maps $C(G/P:-\lambda)$ into $\mathcal{E}_\lambda(G/K)$, $\underline{2-R}$

& intertwines π_λ with L.

Fact: $f \mapsto f|_K$, $C(G/P:-\lambda) \longrightarrow C(K/M)$ defines a topological linear isomorphism. L on $C(G/P:-\lambda)$ may be transfered to a rep'n π_λ of G on $C(K/M)$. ('compact picture').

$$\begin{tikzcd} \mathcal{P}_\lambda : C(G/P:-\lambda) \arrow[r] \arrow[d, "\simeq", "\text{restr}"'] & \mathcal{E}_\lambda(G/K) \\ C(K/M) \arrow[ur, "\mathcal{P}_\lambda"'] & \end{tikzcd}$$

Def: $\mathcal{B}'(K/M) = [C^\omega(K/M, dk]'$ (hyper functions on K/M)

Thm Helgason's conjecture. $\mathcal{P}_\lambda$ admits a unique extension to a continuous linear map $\mathcal{B}'(K/M) \longrightarrow \mathcal{E}_\lambda(G/K)$, which intertwines π_λ and L. For $e(\lambda) \neq 0$ $\left(\Leftarrow \langle \mathrm{Re}\,\lambda, \alpha \rangle \geq 0 \ \forall_{\alpha \in \Sigma^+}\right)$ this extension is a topological linear isomorphism $\mathcal{B}'(K/M) \longrightarrow \mathcal{E}_\lambda(G/K)$.

Proven by KKMOOT*, late 1970's. The inverse is given by $c(\lambda)^{-1}\beta_\lambda$, with c the HC c-function, and β_λ a boundary value map. (generalizes the classical Poisson kernel on S^1).

*) Kashiwara, Kowata, Minemura, Okamoto, Oshima, Tanaka

Results of Flensted-Jensen.

Thm $\operatorname{rk} G/H = \operatorname{rk} K/K\cap H \Longrightarrow (G/H)^\wedge_{ds} \neq \phi.$

Method of proof: • by the rk condition $\mathfrak{q}$ has a Cartan subspace $\mathfrak{t} \subset \mathfrak{q}$ with $\mathfrak{t} \subset \mathfrak{k} \cap \mathfrak{q}$.

- $\mathfrak{a}^d = i\mathfrak{t}$ is maximal abelian in $\mathfrak{p}^d$
  and $\mathfrak{a}^d \subset \mathfrak{p}^d \cap \mathfrak{h}^d \rightsquigarrow \Sigma(\mathfrak{a}^d),\ W^d$

- $$(H^d \backslash G^d / P^d)_{\text{closed}} \xleftrightarrow{1-1} W^d_{K\cap H} \backslash W^d$$
  $$(K\cap H)\, v\, P^d \longleftarrow\!\!\shortmid\ v$$

- Flensted-Jensen's functions

$$\psi_{v,\lambda} = \mathcal{P}_\lambda\left(\delta_{(K\cap H)v/M^d} \right), \quad \lambda \in (\mathfrak{a}^d)^*$$

if λ satisfies an integrality condition, then

$\psi_{v,\lambda} = {}^d f_{v,\lambda}$ for $f \in \mathcal{E}_\chi(G/H)_K \cap L^2(G/H)^\infty$,

$${}^d\chi = \gamma^d(\cdot, \lambda).$$

Classification by Oshima & Matsuki, 1982.

<u>Thm</u> : $(G/H)^\wedge_{ds} \neq \emptyset \iff \mathrm{rk}\, G/H = \mathrm{rk}\, K/K\cap H.$

O & M showed: functions $f \in \mathcal{E}_\chi(G/H)_K \cap L^2(G/H)^\infty$ satisfy: ($\chi \longleftrightarrow \lambda$)

1) $\beta^d_\lambda(f^d)$ supported in closed $K\cap H$ orbits on G^d/P^d.

2) $\langle \lambda, \alpha \rangle \in \mathbb{R}$ $\forall \alpha \in \Sigma(\mathfrak{a}^d)$. (real)

3) $\langle \lambda, \alpha \rangle \neq 0$ $\forall \alpha \in \Sigma(\mathfrak{a}^d)$. (regular)

This is sufficient for our purposes.