The advection-diffusion equation in the density frame

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We investigate an alternative approach, to the MIS relativistic approach, developed to describe fluids without an underlying boost symmetry. This *density frame* approach has no non-hydrodynamic modes and no additional parameters compared to the Landau theory of first order hydrodynamics, at the price of not being fully boost invariant. We show that the density frame equations of motion follow Landau ones if the ideal equations are used to rewrite lab-frame time derivatives appearing in the dissipative strains as spatial derivatives. With this rewrite the equations are first order in time and are stable. In addition, we also show that the density frame equations can be derived from the relativistic kinetic theory.

1 Density frame equations from Landau frame using lowest order equation of motion

In 1st order Landau frame hydrodynamics, the conserved current is $\partial_\mu J^\mu = 0$ with $J^\mu \equiv n_{\rm LF} u^\mu + j^\mu_{\rm D,LF}$ where the first term in J^μ is ideal advection and the second term is the diffusive correction $j^\mu_{\rm D,LF} \equiv -T\sigma\Delta^{\mu\nu}\partial_\nu\hat{\mu}_{\rm LF}$ with $\hat{\mu}_{\rm LF} \equiv \mu/T$ being the scaled chemical potential, thermodynamically conjugate to charge density $n_{\rm LF}$. We derive the density frame equations of motion by first considering fluids without a boost symmetry, and then specializing the equations to Lorentz covariant fluids. The advection-diffusion equation in the density frame consists of the conservation law

$$\partial_t N + \partial_i J^i = 0\,, \qquad \text{where} \qquad J^i \equiv N v^i + J^i_D \qquad \text{with diffusive current } J^i_D\,. \tag{1}$$

J^i_D is expanded in spatial gradients of the conserved charge, or its thermodynamic conjugate $\hat{\mu}$. The most general form of J^i_D at first order in gradients of $\hat{\mu}$ is

$$J^i_D = -\frac{\sigma_\parallel(\beta^0, v)}{\beta^0}\hat{v}^i\hat{v}^j\partial_j\hat{\mu} - \frac{\sigma_\perp(\beta^0, v)}{\beta^0}\left(\delta^{ij} - \hat{v}^i\hat{v}^j\right)\partial_j\hat{\mu}\,. \tag{2}$$

Comparing Landau frame and density frame forms give

$$N = n_{\rm LF}u^0 + j^0_{\rm D,LF}\,, \qquad \text{and} \qquad J^i_D = J^i - Nv^i = (\Delta^i{}_\alpha - v^i\Delta^0{}_\alpha) j^\alpha_{\rm D,LF}\,. \tag{3}$$

We use lowest order equation of motion: $\partial_t\hat{\mu} \simeq -v^j\partial_j\hat{\mu}$ to approximate the Landau frame expression for the diffusive current

$$j^\alpha_{\rm D,LF} \simeq -T\sigma\left(\Delta^{\alpha j} - \Delta^{\alpha 0}v^j\right)\partial_j\hat{\mu}\,. \tag{4}$$

Substituting eq. (4) into eq. (3) gives: $J^i_D = -T\,\sigma^{ij}\,\partial_j\hat{\mu}$ where: $T\sigma^{ij} = T\sigma\left(\Delta^i{}_\alpha - v^i\Delta^0{}_\alpha\right)\left(\Delta^j{}_\beta - v^j\Delta^0{}_\beta\right)\Delta^{\alpha\beta} = T\sigma\left(\delta^{ij} - v^iv^j\right)$. Comparison with the general form in eq. (2) shows that

$$\frac{\sigma_\parallel(\beta^0, v)}{\beta^0} = \frac{T\sigma(\beta)}{\gamma^2}\,, \qquad \frac{\sigma_\perp(\beta^0, v)}{\beta^0} = T\sigma(\beta)\,. \tag{5}$$

Hence, the density frame equation of motion is $\partial_t N + \partial_i(Nv^i) = \partial_i\left(T\sigma^{ij}\partial_j\hat{\mu}\right)$ and when $\hat{\mu}$ is written in terms of the charge $N = \chi\mu u^0$, we arrive at advection-diffusion equation:

$$\boxed{\partial_t N + \partial_i(Nv^i) = \partial_i\left(D^{ij}\partial_j N\right)}\,, \tag{6}$$

with $D^{ij} = \frac{D}{\gamma}\left(\delta^{ij} - v^iv^j\right)$. $D = T\sigma/\chi$ is the scalar diffusion coefficient of the Landau frame. The γ factors in the diffusion matrix can be easily understood physically. The diffusion coefficient has units of distance squared per time. The rate of transverse diffusion is suppressed relative to a fluid at rest by one factor of γ due to time dilation. The rate of longitudinal diffusion is suppressed by three factors of γ due to time dilation and length contraction, i.e. each spatial step in the random walk is length contracted by γ and the steps add in square.

2 Density frame equations from relativistic kinetic theory

Let's assume relaxation time approximation and consider single species of classical relativistic particles, which carry the charge of the system $p^\mu\partial_\mu f = -\mathcal{C}_p\,\delta f$ where $\mathcal{C}_p$ is a momentum dependent parameter controlling the collision rate in the rest frame of the medium. In global equilibrium the phase space distribution function is characterized by constant μ, T, u^μ. If the density of the charged particles depends slowly on space and time then $\hat{\mu}(t,\mathbf{x})$ is no longer a constant but reflects this dependence: $f_0(t,\mathbf{x},\mathbf{p}) = e^{\hat{\mu}(t,\mathbf{x})}e^{\beta^\mu p_\mu}$. In the density frame $\hat{\mu}(t,\mathbf{x})$ is adjusted to reproduce the charge density in the lab frame J^0, while in the Landau frame $\hat{\mu}(t,\mathbf{x})$ is adjusted to reproduce the charge in the rest frame, $n(t,\mathbf{x}) = -u_\mu J^\mu$. These two definitions agree when gradients are neglected, and in this case $f_{\rm eq}(t,\mathbf{x},\mathbf{p})$ is a solution to the Boltzmann equation. $\hat{\mu}(t,\mathbf{x})$ obeys the equations of ideal advection equation at lowest order

$$u^\mu\partial_\mu\hat{\mu} \simeq 0\,. \tag{7}$$

We parameterize $f = f_0 + \delta f(t,\mathbf{x},\mathbf{p})$ and solve for δf order by order in the gradients

$$f_0\,p^\mu\partial_\mu\hat{\mu} = -\mathcal{C}_p\,\delta f\,. \tag{8}$$

In the Landau frame one decomposes the gradient into its temporal and spatial components as

$$\partial_\mu\hat{\mu} = -u_\mu u^\alpha\partial_\alpha\hat{\mu} + \Delta_\mu{}^\alpha\,\partial_\alpha\hat{\mu}\,. \tag{9}$$

Neglecting the temporal term in Eq. (9) by exploiting the lowest order eom, Eq. (7), we substitute into Eq. (8), which leads to first viscous correction in the Landau frame $\delta f_{\rm LF} = -\mathcal{C}_p^{-1}\,f_0\,p^\alpha\nabla_\alpha\hat{\mu}_{\rm LF}$ where $\nabla_\alpha = \Delta_\alpha{}^\mu\partial_\mu$. Evaluating the diffusive current

$$j^\mu_{\rm D,LF} = \int_p \frac{d^3p}{(2\pi)^3}\frac{p^\mu}{p^0}\delta f_{\rm LF}\,, \qquad \text{yields expected Landau frame current} \qquad j^\mu_{\rm D,LF} = -T\sigma\Delta^{\mu\nu}\partial_\nu\hat{\mu}_{\rm LF}\,. \tag{10}$$

The conductivity is defined as $T\sigma\Delta^{\mu\nu} \equiv \Delta^\mu{}_\alpha\Delta^\nu{}_\beta I^{\alpha\beta}$ where $I^{\alpha\beta} \equiv \int\frac{d^3p}{(2\pi)^3p^0}\mathcal{C}_p^{-1}f_0\,p^\alpha p^\beta$.

In the density frame one uses the lowest order equations in the lab frame $\partial_t\hat{\mu} = -v^i\partial_i\hat{\mu}$ which yields $\delta f = -\mathcal{C}_p^{-1}f_0(p^i - p^0v^i)\,\partial_i\hat{\mu}$. J^i_D in the density frame is the difference between the current and the ideal advection J^0v^i, $J^i_D = \int\frac{d^3p}{(2\pi)^3p^0}\left(p^i - p^0v^i\right)f$. Substituting the approximate distribution function $f_0 + \delta f$ leads

$$J^i_D = \mathcal{K}^{ij}(-\partial_j\hat{\mu})\,, \qquad \text{where} \qquad \mathcal{K}^{ij} \equiv \int\frac{d^3p}{(2\pi)^3p^0}\mathcal{C}_p^{-1}f_0\,(p^i - p^0v^i)(p^j - p^0v^j)\,. \tag{11}$$

Noting that $p^i - p^0v^i = \left(\Delta^i{}_\alpha - v^i\Delta^0{}_\alpha\right)\Delta^\alpha{}_\beta\,p^\beta$, we find that $\mathcal{K}^{ij}$ has the expected density frame form

$$\boxed{\mathcal{K}^{ij} = \left(\Delta^i{}_\alpha - v^i\Delta^0{}_\alpha\right)\left(\Delta^j{}_\beta - v^j\Delta^0{}_\beta\right)T\sigma\Delta^{\alpha\beta} = T\sigma\left(\delta^{ij} - v^iv^j\right)}\,. \tag{12}$$

References

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