Anomalous transport from lattice QCD

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Introduction

- Quantum anomalies + EM fields / Vorticity → **Anomalous transport phenomena**

- Examples:
  - Chiral Magnetic Effect (CME)
  - Chiral Separation Effect (CSE)
  - Chiral Vortical Effect (CVE)
  - Chiral Electrical Separation Effect (CESE)
  - ...

- Need non-perturbative methods to study effect of strong interactions:
  Lattice QCD

Lattice QCD in a nutshell

- QCD partition function

$$\mathcal{Z} = \int \mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-S_E} = \int \mathcal{D}A \,\det M\, e^{-S_G}$$

with S_E the Wick-rotated, finite temperature QCD action

$$S_E = S_G + S_F = \int_0^{1/T} \mathrm{d}\tau \int \mathrm{d}^3x \left[\frac{\mathrm{Tr}\, G^2}{2g^2} + \sum_f \bar{\psi}_f \underbrace{(\not{D} + m_f)}_{\equiv M} \psi_f \right]$$

- Observables

$$\langle \mathcal{O} \rangle = \int \mathcal{D}\mathcal{U} \,\det M\, e^{-S_G}\, \mathcal{O}$$

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- **Lattice QCD**:
  - Discretize space-time → lattice spacing a
  - Numerically evaluate path integral → Monte Carlo methods
  - Inherently in **equilibrium**

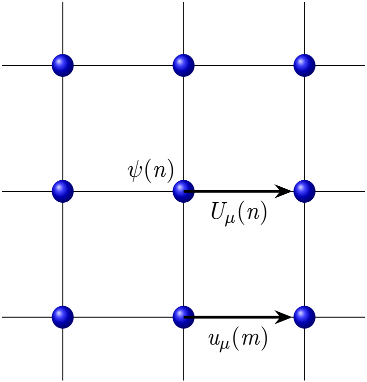
Lattice QCD in a nutshell

- Lattice $N_s^3 \times N_t,\ T = (aN_t)^{-1},\ V = a^3 N_s^3$
- **Continuum limit**: $a \to 0$ (equiv. $N_t \to \infty$) while V, T fix

- Fermions $\psi(n) \leftrightarrow$ points
- Gluons $\leftrightarrow$ links
  $U_\mu(n) = e^{iA_\mu(n)} \in \mathrm{SU}(3)$
- Background B

  $u_\mu \in \mathrm{U}(1)$ links

  No sign problem!



Lattice QCD in a nutshell

- Fermionic expectation values:
  - Sea quarks → determinant
  - Valence quarks → operator

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- **Quenched** approximation: $\det M = \text{const.}$ (exact when $m_q \to \infty$)
  Perturbatively: neglect virtual sea quark loops

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- Different fermion discretizations (doublers appear!):
  - Wilson: doublers are given a **cutoff dependent mass** to decouple in the continuum limit
  - Staggered: Dirac and flavor structure is **mixed with coordinate dependence** to reduce doubling problem
  - Overlap, Domain Wall, ...

Chiral Separation Effect (CSE)

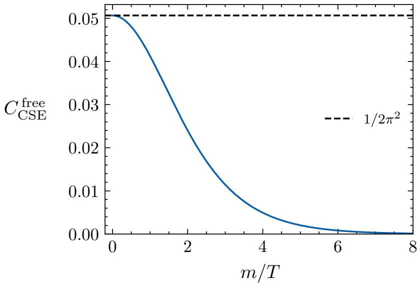
Conductivity

- Magnetic field + finite density → Axial current

$$J^A_{\mathsf{CSE}} = \sigma_{\mathrm{CSE}}\, eB = \boxed{C_{\mathsf{CSE}}}\, eB\mu + \mathcal{O}(\mu^3)$$

- Free fermions Son, Zhitnitsky '04 Metlitski, Zhitnitsky '05:

$$C_{\mathsf{CSE}} = C_{\mathsf{CSE}}(m/T) \xrightarrow{m\to 0} \frac{1}{2\pi^2}$$



- Effect of interactions?

Lattice studies

- Finite μ on the lattice $\rightarrow$ Sign problem $\rightarrow$ how to avoid it?

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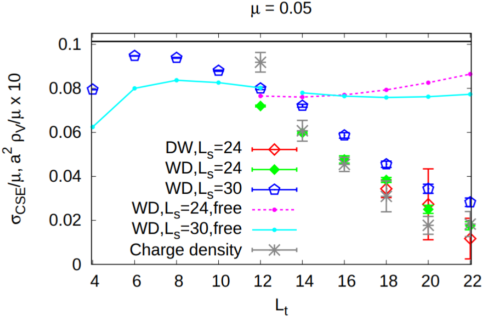

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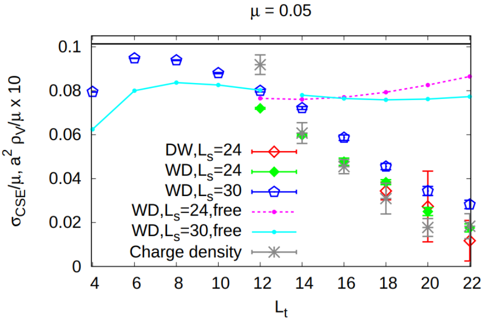


- Taylor expansion: Staggered physical point QCD
  🔗 Brandt, Endrődi, EGV, Markó '23

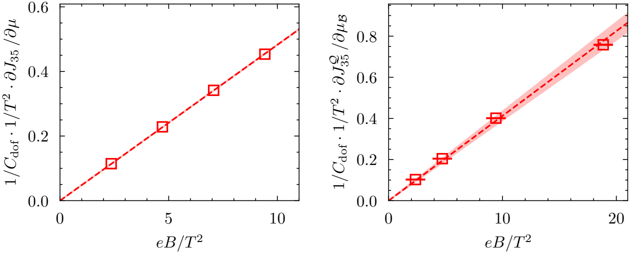
Taylor expansion

- Remember $J^A_{\mathsf{CSE}} = C_{\mathsf{CSE}}\, eB\mu + \mathcal{O}(\mu^3)$
- Measure derivatives of the currents:

$$C_{\mathsf{CSE}}\, eB_3 = \frac{\mathrm{d}\langle J_3^A \rangle}{\mathrm{d}\mu}\Big|_{\mu=0} \sim \left\langle J_4^V J_3^A \right\rangle_{\mu=0}$$

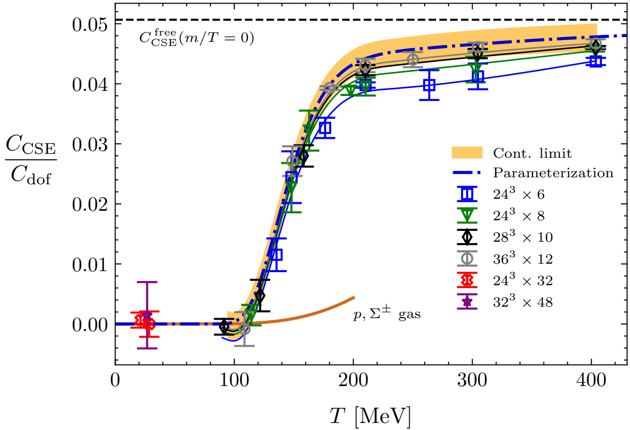
- Simulations at $\mu = 0 \rightarrow$ no sign problem
- Numerical derivative (linear fit) w.r.t. B to obtain C_{CSE}:

free fermions　　　　full QCD



CSE in QCD

- First full QCD result Brandt, Endrődi, EGV, Markó '23



Chiral Magnetic Effect (CME)

CME in equilibrium

- Magnetic field + chiral density → Vector current

  Fukushima, Kharzeev, Warringa '08

$$J^V_{\mathsf{CME}} = C_{\mathsf{CME}}\, eB\mu_5 + \mathcal{O}(\mu_5^3)$$

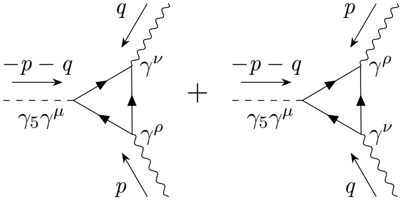
- Now understood as out-of-equilibrium effect, for free fermions

$$J^V_{\mathsf{CME}}(t) = \frac{1}{2\pi^2}\, eB\mu_5(t)$$

- In-equilibrium: careful regularization needed!

- Well known example: triangle anomaly (with massive fermions)

$$\Gamma^{\mu\nu\rho}_{AVV}(p+q,p,q) =$$



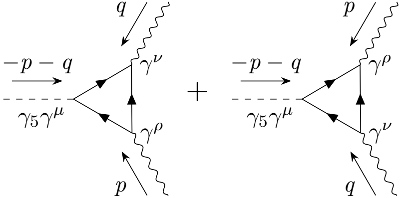
- Uncareful regularization:

$$(p+q)_\mu \Gamma^{\mu\nu\rho}_{AVV}(p+q,p,q) = mP_5^{\nu\rho}(p,q)$$

Regulator sensitivity: anomaly

- Well known example: triangle anomaly (with massive fermions)

$$\Gamma^{\mu\nu\rho}_{AVV}(p+q,p,q) =$$



- Uncareful regularization:

$$(p+q)_\mu \Gamma^{\mu\nu\rho}_{AVV}(p+q,p,q) = m P_5^{\nu\rho}(p,q)$$

- Pauli-Villars regularization[1]:

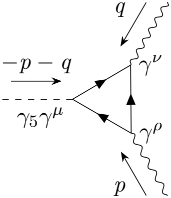
$$\begin{aligned}(p+q)_\mu \Gamma^{\mu\nu\rho}_{AVV}(p+q,p,q) =& m P_5^{\nu\rho}(p,q) + \sum_{s=1}^{3} c_s m_s P_{5,s}^{\nu\rho}(p,q) \\ \rightarrow& m P_5^{\nu\rho}(p,q) + \frac{\epsilon^{\alpha\beta\nu\rho} q_\alpha p_\beta}{4\pi^2}\end{aligned}$$

[1] new particles c_s ($s = 0,1,2,3,\ s=0$ physical fermion $m_0 \equiv m$) and $m_{s>0} \rightarrow \infty$

CME regulator sensitivity

- C_{CME} (and C_{CSE}[2]) can also be written with the triangle diagram

$$\Gamma^{\mu\nu\rho}_{AVV}(p+q,p,q) =$$



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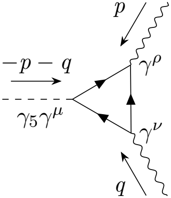

with $J_3 \sim A_3, \quad B_3 \sim q_1 A_2, \quad \mu_5 = A_0^5$

- Vanishing external momentum:

$$C_{\mathsf{CME}} = \lim_{p,q,p+q\to 0} \frac{1}{q_1}\Gamma^{023}_{AVV}(p+q,p,q) = \frac{1}{2\pi^2} + \sum_{s=1}^{3}\frac{c_s}{2\pi^2} = 0$$

- C_{CME} is zero due to anomalous contribution!

[2]For CSE this issue is not present

Absence of CME in equilibrium

- Bloch's theorem N. Yamamoto '15
  Conserved currents cannot flow in equilibrium ground state

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Some other approaches:

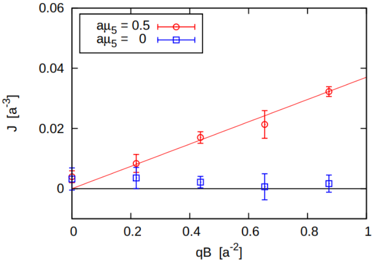
- Triangle diagram Hou, Liu, Ren '11
- Dirac eigenvalues + Lattice (free overlap) Buividovich '14
- Weyl-Wigner formalism Zubkov '16 Banerjee, Lewkowicz, Zubkov '21
- Vacuum polarization in background B Brandt, Endrődi, EGV, Markó '24
  + Lattice (full QCD staggered)
- ...

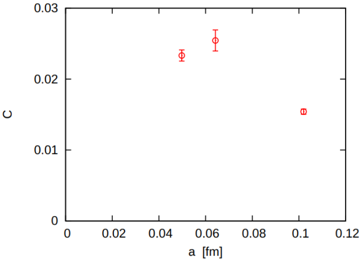
Dynamical lattice simulations

- Simulations at finite μ_5 are possible!
- Quenched and full QCD: Wilson

$C_{\text{CME}} \approx 0.025 \neq 0?$ $1/(2\pi^2) \approx 0.05$

A. Yamamoto '11 A. Yamamoto '11





- Non-conserved vector current!
  Also used in hadron spectroscopy, transport ...
- Does it matter for CME?

Taylor expansion

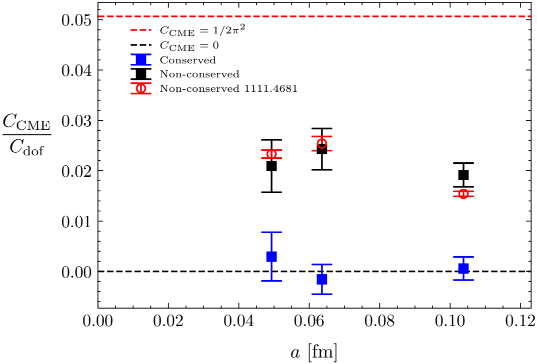
- Now Taylor expansion in μ_5: $J^V_{\mathsf{CME}} = C_{\mathsf{CME}}\, eB\mu_5 + \mathcal{O}(\mu_5^3)$
- Measure derivatives of the currents:

$$C_{\mathsf{CME}}\, eB_3 = \frac{\mathrm{d}\langle J_3^V \rangle}{\mathrm{d}\mu_5}\Big|_{\mu_5=0} \sim \left\langle J_4^A \, J_3^V \right\rangle_{\mu_5=0}$$

- We can use conserved or non-conserved J_3^V
- Direct crosscheck with $C_{\mathrm{CME}} \neq 0$ setup!

Conserved currents & CME

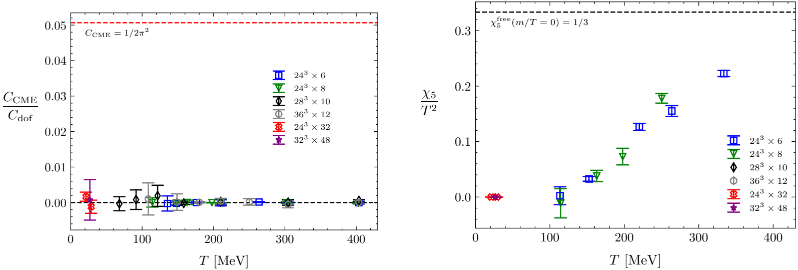
- Quenched QCD (Wilson) Brandt, Endrődi, EGV, Markó '24



- Crucial to use a conserved vector current

- **CME vanishes** in equilibrium, also QCD
  (full QCD staggered)

Brandt, Endrődi, EGV, Markó ’24



- Chiral density is **finite** at $\mu_5 \neq 0$

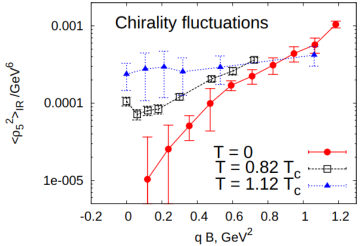
$$\rho_5(\mu_5) = \frac{T}{V} \left. \frac{\mathrm{d}^2 \log \mathcal{Z}}{\mathrm{d}\mu_5^2} \right|_{\mu_5=0} \mu_5 + \mathcal{O}(\mu_5^3) = \chi_5 \mu_5 + \mathcal{O}(\mu_5^3)$$

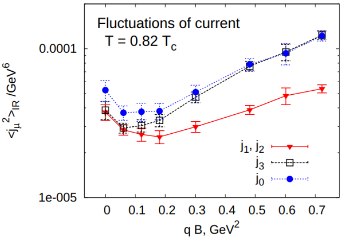
Indirect CME signals

- Fluctuations of chiral density and current enchanced by B

(Two-color QCD quenched overlap fermions)

Buividovich, Chernodub, Luschevskaya, Polikarpov '09





- Similar in quenched QCD

Braguta, Buividovich, Kalaydzhyan, Kuznetsov, Polikarpov '10

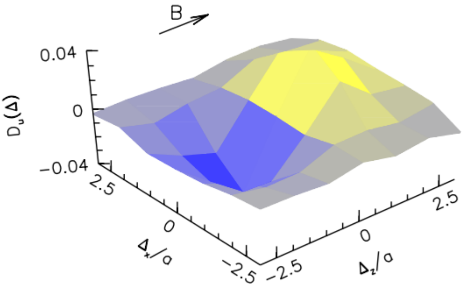
Charge separation, chirality and B

- Correlator of chirality and electric dipole moment proportional to B for weak fields (Two-color QCD quenched overlap fermions)

  🔗 Buividovich, Chernodub, Luschevskaya, Polikarpov '09

- Close to topological fluctuations, local charge separation by B in the transverse plane (Full QCD staggered)

  🔗 Bali et al '14

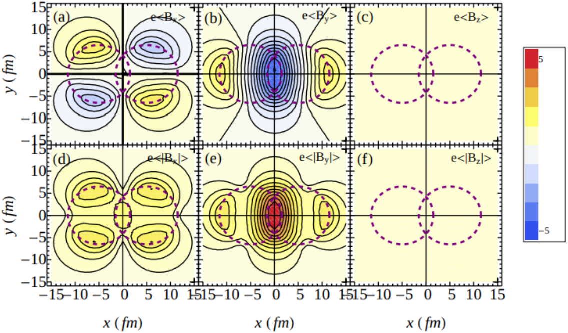


CME: what is next?

CME & inhomogeneous B field

- So far we considered uniform B fields
- Magnetic fields in heavy-ion collisions are **inhomogeneous**

Deng, Huang '12

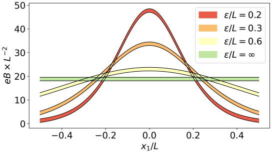


- How is CME affected?

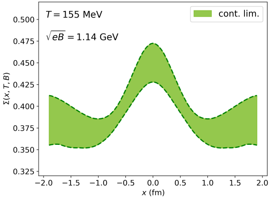
Lattice simulation

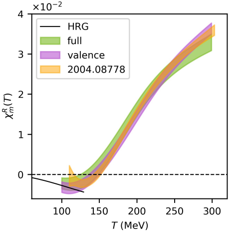
- 1-d magnetic field profile:

$$\vec{B}(x) = \frac{B}{\cosh^2\left(\frac{x_1}{\varepsilon}\right)}\hat{x}_3$$



- Used to check impact in phase diagram and calculate magnetic susceptibility





Brandt et al '23

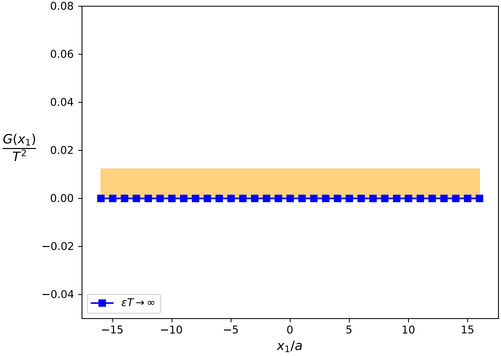
Brandt, Endrődi, Markó, Valois '24

Free fermions

- Local linear response of current profile $J(x_1)$ to *homogeneous* μ_5

$$G(x_1) \equiv \left. \frac{\mathrm{d}\langle J_3^V(x_1)\rangle}{\mathrm{d}\mu_5} \right|_{\mu_5=0}$$

- Free staggered fermions



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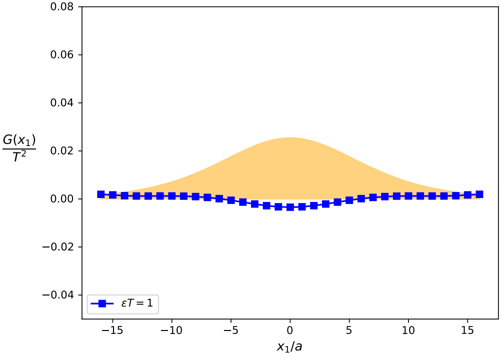

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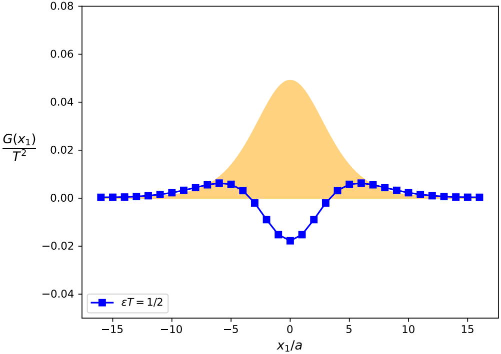

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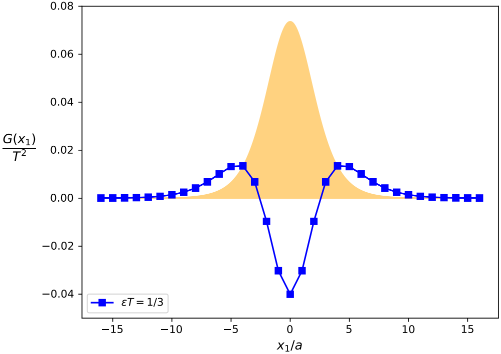

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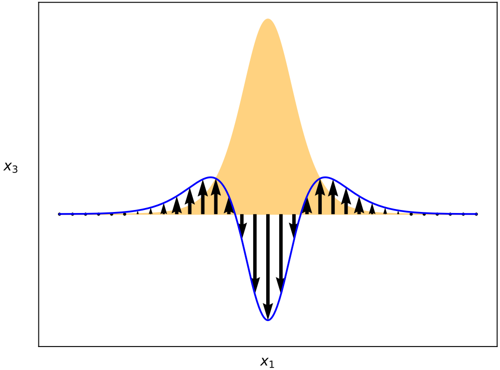
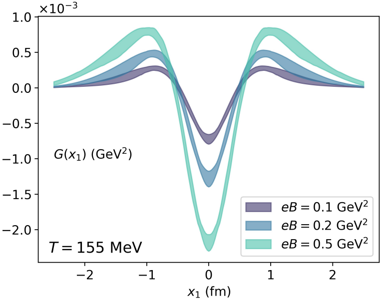
$$G(x_1) \equiv \left. \frac{\mathrm{d}\langle J_3^V(x_1)\rangle}{\mathrm{d}\mu_5} \right|_{\mu_5=0}$$

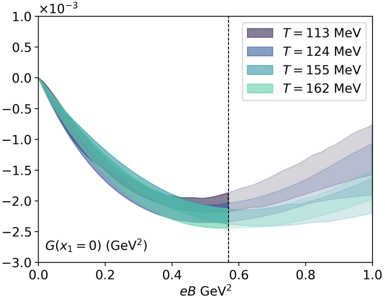
- x_1-dependent current flowing in x_3 $\rightarrow$ integrates to zero



- Non-trivial localized CME in QCD! (full QCD staggered)

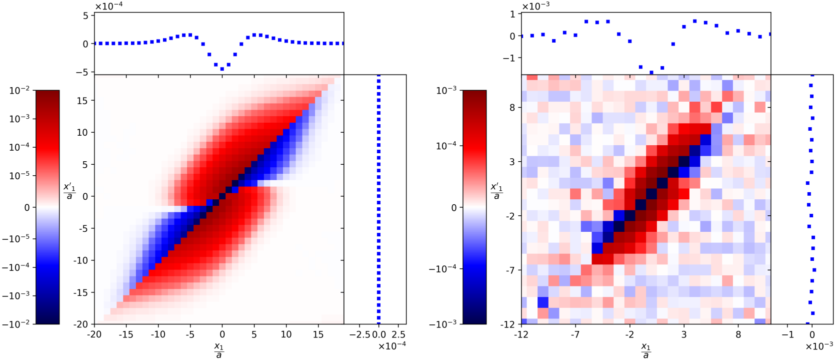
Brandt, Endrődi, EGV, Markó, Valois (in prep.)





- Local linear response of current profile $J(x_1)$ to *inhomogeneous* $\mu_5(x_1')$

$$H(x_1, x_1') \equiv \left. \frac{\delta \langle J_3^V(x_1) \rangle}{\delta \mu_5(x_1')} \right|_{\mu_5=0}$$



Free fermions QCD

Out-of-equilibrium from the lattice

- Spectral func. ρ $\updownarrow$ Retarded prop. G_R $\xrightarrow[\text{theory}]{\text{linear response}}$ Kubo formulas $\rightarrow \xi$ transport coeffs.

$$\xi \sim \lim_{\omega \to 0} \frac{\rho(\omega)}{\omega}$$

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- Spectral representation of Euclidean correlators

$$G_E(\tau) = \int \mathrm{d}\omega \, \rho(\omega) \, K(\omega, \tau)$$

Lattice ($G_E(\tau)$) — Want ($\rho(\omega)$) — Known ($K(\omega, \tau)$)

- On the lattice: $N_t \sim \mathcal{O}(10)$ ill-posed inverse problem
- Many methods on the market $\rightarrow$ transport coefficients on the lattice

CME & Ohmic conductivity

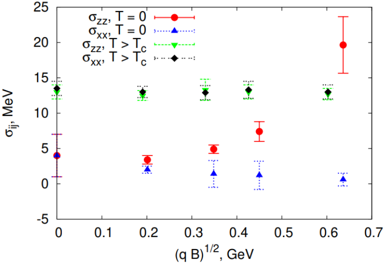
- $E \cdot B$ generates chiral density

$$j^i_{\rm CME} = \sigma^{ij}_{\rm CME} E^j, \quad \sigma^{ij}_{\rm CME} = C(T,B)\, B^i B^j$$

- CME can enhance **parallel** electric conductivity $\sigma_\parallel \parallel \vec{B}$

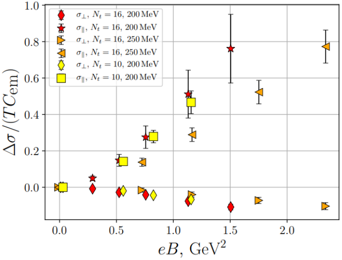
Buividovich et al '10

(two-color QCD quenched overlap)



Astrakhantsev et al '19

(full QCD staggered)



Out-of-equilibrium CME

- Linear response to $\delta\mu_5(t)$ at finite B

- Kubo formula for $G_R(t) = i\theta(-t)\left\langle [j_{45}(t), j_3(0)] \right\rangle$

$$C_{\rm CME}^{\rm neq} \sim \frac{1}{eB} \lim_{\omega\to 0} \frac{\rho(\omega)}{\omega}$$

- Access out-of-equilibrium $C_{\rm CME}$ directly!
  Banerjee, Lewkowicz, Zubkov '22 Buividovich '24

Summary & Outlook

Take-home messages

- Lattice QCD is a powerful tool to study the impact of strong interaction in anomalous transport
- First determination of anomalous conductivity in QCD for CSE
- CME in equilibrium is full of subtleties, both in the continuum and the lattice
- Conserved currents on the lattice are crucial to study CME
- CME vanishes in global equilibrium, also in QCD

Future plans

- Effect of inhomogeneous magnetic fields in anomalous transport
- Access out-of-equilibrium conductivities from the lattice via spectral reconstruction

Backup slides

Anomalous transport

- Transport effects:

$$\begin{pmatrix} \vec{J} \\ \vec{J}_5 \end{pmatrix} = \begin{pmatrix} \sigma_{\mathsf{Ohm}} & \sigma_{\mathsf{CME}} \\ \sigma_{\mathsf{CESE}} & \sigma_{\mathsf{CSE}} \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix}$$

- Chiral Vortical Effect: vector/axial current generated by rotation + $\mu + \mu_5$:

$$\vec{J} = \frac{1}{\pi^2} \mu_5 \mu \vec{\omega}$$

$$\vec{J}_5 = \left[\frac{1}{6} T^2 + \frac{1}{2\pi^2} (\mu_5^2 + \mu^2) \right] \vec{\omega}$$

Currents in staggered

- Staggered "gammas" (free fermions and quark chemical potential):

$$\Gamma_\nu(n,m) = \frac{1}{2}\eta_\nu(n)[e^{a\mu\delta_{\nu,4}}\delta_{n+\hat{\nu},m} + e^{-a\mu\delta_{\nu,4}}\delta_{n-\hat{\nu},m}]$$

$$\Gamma_5(n,m) = \frac{1}{4!}\sum_{i,j,k,l}\epsilon_{ijkl}\Gamma_i\Gamma_j\Gamma_k\Gamma_l$$

$$\Gamma_{\nu 5}(n,m) = \frac{1}{3!}\sum_{i,j,k}\epsilon_{ijk}\Gamma_i\Gamma_j\Gamma_k \quad i,j,k \neq \nu$$

- Conserved vector current and anomalous axial current:

$$j_\nu^V = \bar{\chi}\Gamma_\nu\chi$$

$$j_\nu^A = \bar{\chi}\Gamma_{\nu 5}\chi$$

- Staggered observable has a tadpole term, for example CSE

$$\left.\frac{\mathrm{d}\left\langle J_3^A\right\rangle}{\mathrm{d}\mu}\right|_{\mu=0} \sim \left\langle J_4^V J_3^A\right\rangle_{\mu=0} + \left\langle\frac{\partial J_3^A}{\partial\mu}\right\rangle_{\mu=0}$$

Currents in Wilson

- Local currents (don't fulfill a WI/AWI)

$$j_\mu^{VL} = \bar{\psi}\gamma_\mu\psi$$
$$j_\mu^{AL} = \bar{\psi}\gamma_\mu\gamma_5\psi$$

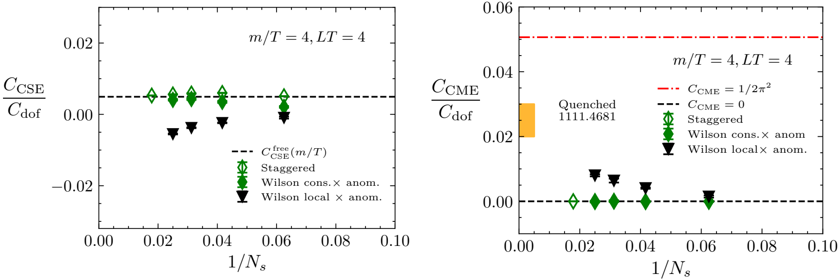
- Conserved vector current and anomalous axial current:

$$j_\mu^{VC}(n) = \frac{1}{2}\left[\bar{\psi}(n)(\gamma_\mu - r)\psi(n+\hat{\mu})) + \bar{\psi}(n)(\gamma_\mu + r)\psi(n-\hat{\mu}))\right]$$
$$j_\mu^{AA}(n) = \frac{1}{2}\left[\bar{\psi}(n)\gamma_\mu\gamma_5\psi(n+\hat{\mu})) + \bar{\psi}(n)\gamma_\mu\gamma_5\psi(n-\hat{\mu})\right]$$

- For correlators like $\left\langle J_4^V J_3^A \right\rangle$ we can use different combinations, for example $\left\langle J_4^{VC} J_3^{AA} \right\rangle$, $\left\langle J_4^{VL} J_3^{AA} \right\rangle$, ...

Results for free fermions

- Consistency check in the free case
- For $m/T = 4$ (similar behavior for other m/T's)



- Using the correct currents is **crucial**

Chirality free fermions

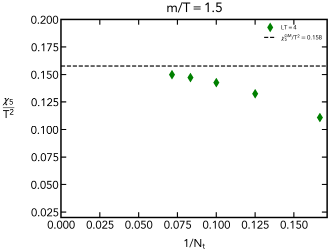
- Naturally

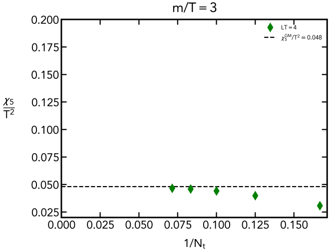
$$n5\Big|_{\mu_5=0} = \frac{T}{V}\frac{\mathrm{d}\log\mathcal{Z}}{\mathrm{d}\mu_5} = 0$$

- But

$$n_5(\mu_5) = \frac{T}{V}\left.\frac{\mathrm{d}^2\log\mathcal{Z}}{\mathrm{d}\mu_5^2}\right|_{\mu_5=0}\mu_5 + \mathcal{O}(\mu_5^2) = \chi_5\mu_5 + \mathcal{O}(\mu_5^2)$$

- χ_5 can be calculated with PV and compared to lattice





- μ_5 does induce chirality in our system

Inhomogenous B free fermions