Finanziato dall'Unione europea NextGenerationEU | Ministero dell'Università e della Ricerca | Italiadomani PIANO NAZIONALE DI RIPRESA E RESILIENZA | UNIVERSITÀ DEGLI STUDI FIRENZE DIPARTIMENTO DI FISICA E STRONOMIA

Transport properties from quantum kinetic theory

David Wagner

Conference on Chirality, Vorticity and Magnetic Field
26.07.2024

INFN FIRENZE

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Quantum kinetic theory: What is it, why does it always look so complicated, and why should I care? A short introduction to the method and a selection of applications

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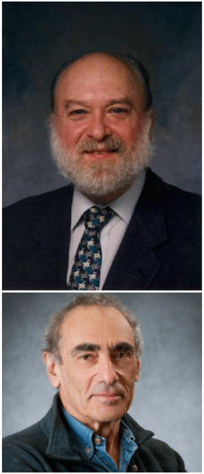
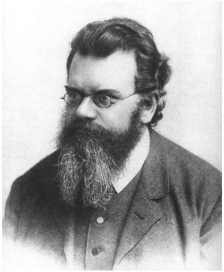
Introduction to the framework

What is quantum kinetic theory?

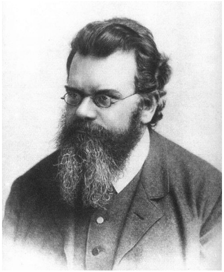


What is ~~quantum~~ kinetic theory?



What is ~~quantum~~ kinetic theory?

- Nonequilibrium statistical description of a dilute gas
  - Nonequilibrium: Describes process of thermalization selfconsistently, concept of temperature emerges from dynamics
  - Statistical: Quantity of interest: single-particle distribution function
  - Dilute: Main assumption: System consists of essentially pointlike free particles scattering via short-range interactions



What is ~~quantum~~ kinetic theory?

Ihr solltet mein Papier lesen!

L. Boltzmann, Sitz.-Ber. Akad. Wiss. Wien (II) 66, 275–370 (1872)
english translation: The Kinetic Theory of Gases, 262-349 (2003)

- Nonequilibrium statistical description of a dilute gas
  - Nonequilibrium: Describes process of thermalization selfconsistently, concept of temperature emerges from dynamics
  - Statistical: Quantity of interest: single-particle distribution function
  - Dilute: Main assumption: System consists of essentially pointlike free particles scattering via short-range interactions

Connection to macroscopic currents

- Kinetic theory is an effective microscopic description
  - Provides (infinitely) more information than macroscopic approaches, such as thermo- and hydrodynamics
  - Can be used to extract information about any macroscopic current of interest

Kinetic representation of currents

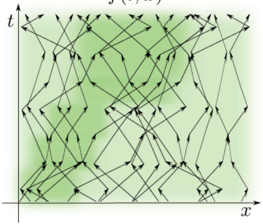
$$N^\mu(t,\mathbf{x}) = \int \mathrm{d}K\, k^\mu f(t,\mathbf{x},\mathbf{k}) \;,$$

$$T^{\mu\nu}(t,\mathbf{x}) = \int \mathrm{d}K\, k^\mu k^\nu f(t,\mathbf{x},\mathbf{k}) \;,$$

$$S^\mu(t,\mathbf{x}) = -\int \mathrm{d}K\, k^\mu f(t,\mathbf{x},\mathbf{k}) \left[\ln f(t,\mathbf{x},\mathbf{k}) - 1\right]$$

$\mathrm{d}K := \mathrm{d}^3k/[(2\pi\hbar)^3 k^0]$

Evolution of the distribution function



L. Rezzolla, O. Zanotti, 978-0-19-174650-5 (2013)

Boltzmann equation (classical version)

$$\left(\frac{\partial}{\partial t} + \frac{\mathbf{k}}{m} \cdot \boldsymbol{\nabla} + \mathbf{F} \cdot \boldsymbol{\nabla}_{\mathbf{k}} \right) f(\mathbf{x}, \mathbf{k}) = C[f]$$

- Left-hand side: Advection through $(\mathbf{x}, \mathbf{k})$-phase space
- Right-hand side: Collision term
  - Depends on higher-order distribution functions, e.g. $f_2(\{\mathbf{x}_1, \mathbf{k}_1\}; \{\mathbf{x}_2, \mathbf{k}_2\})$
  - Has to be truncated ($\rightarrow$ BBGKY hierarchy)
  - *Stoßzahlansatz*: Replace $f_2(\{\mathbf{x}_1, \mathbf{k}_1\}; \{\mathbf{x}_2, \mathbf{k}_2\}) \rightarrow f(\mathbf{x}_1, \mathbf{k}_1) f(\mathbf{x}_2, \mathbf{k}_2)$

Collision term

$$C[f] = \frac{1}{2} \int \mathrm{d}K_1 \, \mathrm{d}K_2 \, \mathrm{d}K' \mathcal{W} \left(f_1 f_2 - f f' \right)$$

Connecting to quantum theory

- How to translate these ideas to quantum mechanics?
  - Try to build on the conserved currents and find $W(x,k,t)$ such that

$$\mathrm{Tr}\left[\widehat{\rho}(\hat{\mathbf{x}},\hat{\mathbf{p}},t)\,\widehat{A}(\hat{\mathbf{x}},\hat{\mathbf{p}})\right] = \int \mathrm{d}^3x \int \frac{\mathrm{d}^3k}{(2\pi\hbar)^3} A(\mathbf{x},\mathbf{k}) W(\mathbf{x},\mathbf{k},t)$$

- Choice “closest” to classical kinetic theory: **Wigner function**

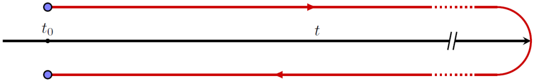
$$W(\mathbf{x},\mathbf{k},t) = \int \mathrm{d}^3v\, e^{-\frac{i}{\hbar}\mathbf{k}\cdot\mathbf{v}} \left\langle \mathbf{x}+\frac{\mathbf{v}}{2}\middle|\,\widehat{\rho}\,\middle|\mathbf{x}-\frac{\mathbf{v}}{2}\right\rangle$$

E. P. Wigner, Phys. Rev. 40, 749-760 (1932)
H.-W. Lee, Physics Reports 259, 147-211 (1995)

- Price to pay: Wigner function is not positive semidefinite
- Relativistic field-theoretical version (scalar field):

$$W(x,k) = \int \mathrm{d}^4v\, e^{-\frac{i}{\hbar}k\cdot v} \left\langle \widehat{\phi}^\dagger\left(x+\frac{v}{2}\right)\widehat{\phi}\left(x-\frac{v}{2}\right)\right\rangle \equiv G^<(x,k)$$

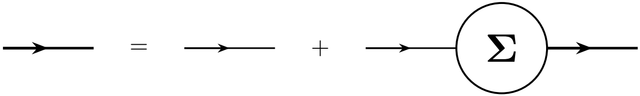
Equations of motion



$$G = \begin{pmatrix} G^{++} & G^{+-} \\ G^{-+} & G^{--} \end{pmatrix} = \begin{pmatrix} G^{F} & G^{<} \\ G^{>} & G^{\bar{F}} \end{pmatrix}$$

- Real-time QFT: Expectation values of operators can be represented as time evolution along a closed-time path
- EoM: (contour-ordered) Dyson-Schwinger equation

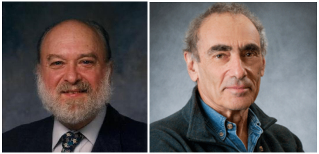
$$G_0^{-1} G^{AB}(x_1, x_2) = -i c^{AB} \delta^{(4)}(x_1 - x_2) + i \int \mathrm{d}^4 x' \Sigma^{AC}(x_1, x') c_{CD} G^{DB}(x', x_2)$$



- Wigner transform: $-i\hbar \partial_1^\mu G^{AB}(x_1, x_2) \to \left(k^\mu - \frac{i\hbar}{2}\partial^\mu\right) G^{AB}(x, k)$

Approximations

- $\hbar$-gradient expansion: Assume that the two-point functions are sufficiently localized in central coordinate $x = (x_1 + x_2)/2$
  - *Notion of a particle should make sense!*
- Allows to approximate memory integrals



$$\int \mathrm{d}^4 x' G(x_1, x')\Sigma(x', x_2) \longrightarrow G(x,k)\Sigma(x,k) - \frac{i\hbar}{2}\left\{G(x,k), \Sigma(x,k)\right\}_{\mathrm{PB}}$$

(Gradient-expanded) Kadanoff-Baym equations

$$G_0^{-1} G^<(x,k) = \frac{i}{2}\left[\Sigma^>(x,k)G^<(x,k) - \Sigma^<(x,k)G^>(x,k)\right] + \frac{\hbar}{4}\left[\left\{\Sigma^>(x,k), G^<(x,k)\right\}_{\mathrm{PB}} - \left\{\Sigma^<(x,k), G^>(x,k)\right\}_{\mathrm{PB}}\right]$$

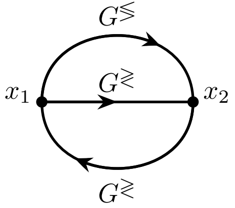
L. P. Kadanoff, G. Baym, ISBN 9780429493218 (1989)

$$\{f,g\}_{\mathrm{PB}} := (\partial_\mu f)(\partial_k^\mu g) - (\partial_k^\mu f)(\partial_\mu g)$$

Example: Scalar field

- Approximate self-energy to lowest nontrivial order
- Separate real and imaginary parts of KB equations

$\Sigma^{\gtrless} =$



Quantum kinetic equations (lowest order)

$$\left(k^2 - m^2\right) G^<(x,k) = 0 \quad \Longrightarrow \quad G^<(x,k) = 2\pi\hbar^2\delta(k^2 - m^2) f(x,k) \; ,$$

$$k \cdot \partial f(x,k) = \frac{1}{2} \int \mathrm{d}K_1 \, \mathrm{d}K_2 \, \mathrm{d}K' \, (2\pi\hbar)^4 \delta^{(4)}(k_1 + k_2 - k - k') \times \frac{|M|^2}{16} \left(f_1 f_2 \widetilde{f} \widetilde{f}' - \widetilde{f}_1 \widetilde{f}_2 f f' \right)$$

“Ok, but why the complicated setup?”

Including electromagnetic fields

- Standard definition of Wigner function is not gauge invariant due to fields at different positions
  - Remedy: Include gauge link $U(x_1, x_2) = \exp\left[-\frac{i}{\hbar}(x_1 - x_2) \cdot \int_{-1/2}^{1/2} \mathrm{d}t A(x_1 + x_2 + t(x_1 - x_2))\right]$

Wigner function with EM fields

$$G^<(x,k) = \int \mathrm{d}^4 v\, e^{-\frac{i}{\hbar}k\cdot v} \left\langle \widehat{\phi}^\dagger\left(x + \frac{v}{2}\right) U\left(x + \frac{v}{2}, x - \frac{v}{2}\right) \widehat{\phi}\left(x - \frac{v}{2}\right)\right\rangle$$

- Main effect: Wigner representation of momentum changes

$$-i\hbar\partial_1^\mu G^< \to \left\{k^\mu - \frac{\hbar}{2} j_1(\Delta) F^{\mu\nu}\partial_{k,\nu} - \frac{i\hbar}{2}\left[\partial^\mu - j_0(\Delta) F^{\mu\nu}\partial_{k,\nu}\right]\right\} G^<(x,k)$$

D. Vasak, M. Gyulassy, H. T. Elze, Annals Phys. 173, 462-492 (1987)

$\Delta := (\hbar/2)\partial \cdot \partial_k$

Including electromagnetic fields

- Standard definition of Wigner function is not gauge invariant due to fields at different positions
  - Remedy: Include gauge link
    $U(x_1,x_2) = \exp\left[-\frac{i}{\hbar}(x_1-x_2)\cdot\int_{-1/2}^{1/2}\mathrm{d}t A(x_1+x_2+t(x_1-x_2))\right]$

Wigner function with EM fields

$$G^<(x,k) = \int \mathrm{d}^4 v \ldots \frac{v}{2}\big)\big)\Big\rangle$$

Similar effects in curved spacetime

Y.-C. Liu, L.-L. Gao, K. Mameda, X.-G. Huang,Phys. Rev. D 99, 085014 (2019)

see also **S. Lin's talk, 22.07. 16:30**

- Main

$$-i\hbar\partial_1^\mu G^< \ldots G^<(x,k)$$

(Classical) kinetic theory in Newton-Cartan geometry

P. Matus, R. Biswas, P. Surówka, F. Peña-Benítez, 2407.08805 (2024)

see also **P. Surówka's talk, today 14:30**

D. Vasak, M. Gy

$\Delta := (\hbar/2)\partial\cdot\partial_k$

Including spin

- Case of nonzero spin: Wigner function becomes **matrix-valued**
  - → Additional components encode spin degrees of freedom

Wigner function (spin $1/2$)

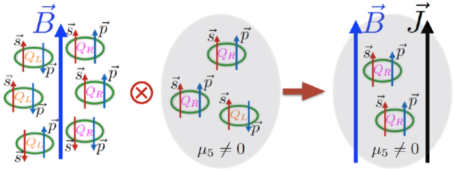
$$G^< = \frac{1}{4}\left(\mathcal{F} + i\gamma_5\mathcal{P} + \not{\mathcal{V}} + \gamma_5\not{\mathcal{A}} + \frac{i}{4}[\gamma_\mu,\gamma_\nu]\mathcal{S}^{\mu\nu}\right)$$

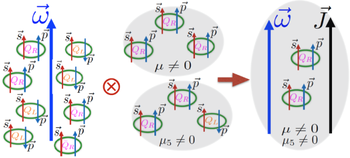
- Underlying equations (Dirac, Proca, $\cdots$) can be solved perturbatively in $\hbar$ expansion
  - → Gradient contributions appear!
    - $\mathcal{V}^\mu \sim -\frac{\hbar}{2m}\partial_\nu\mathcal{S}^{\nu\mu}$, $k_\mu\mathcal{S}^{\mu\nu} \sim -\frac{\hbar}{2}\partial^\nu\mathcal{F}$, $\mathcal{P} \sim -\frac{\hbar}{4m^2}\epsilon^{\mu\nu\alpha\beta}k_\mu\partial_\nu\mathcal{S}_{\alpha\beta}$, $\cdots$
  - ▶ Responsible for a lot of the interesting transport phenomena

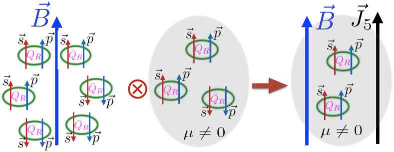
H. T. Elze, M. Gyulassy, D. Vasak, Nucl. Phys. B 276, 706-728 (1986)
D. Vasak, M. Gyulassy, H. T. Elze, Annals Phys. 173, 462-492 (1987)
S. Mrowczynski, U. W. Heinz, Annals Phys. 229, 1-54 (1994)
Y. A. Markov, M. A. Markova, Theor. Math. Phys. 108, 977-991 (1996)
Y. A. Markov, M. A. Markova, Theor. Math. Phys. 111, 601-612 (1997)

Equilibrium: Chiral effects

Sessions dedicated to CME: 23.07.24, 11:00-13:00 & 14:30-16:00







D. Kharzeev, J. Liao, S. Voloshin, G. Wang, Prog. Part. Nucl. Phys. 88, 1-28 (2016)

- Intuition:
  - CME: Imbalance in chirality induces current along $\mathbf{B}$
  - CVE: Imbalance in chirality& charge induces current along $\boldsymbol{\omega}$
  - CSE: Imbalance in charge induces axial current along $\mathbf{B}$

CME, CVE, & CSE from QKT

- Quantum kinetic theory: these effects arise naturally as gradient contributions at global equilibrium

1) Assume global equilibrium, $f_s^{(0)} \coloneqq [\exp(k \cdot u/T - \mu_s/T) + 1]^{-1}$

$$\mathcal{V}^\mu(x,k) \approx k^\mu \delta(k^2)(f_R^{(0)} + f_L^{(0)}) \, , \; \mathcal{A}^\mu(x,k) \approx k^\mu \delta(k^2)(f_R^{(0)} - f_L^{(0)})$$

2) Obtain gradient contributions

$$\frac{\hbar}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\alpha \mathcal{A}_\beta = -k_{[\mu}\mathcal{V}_{\nu]} \, , \; \frac{\hbar}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\alpha \mathcal{V}_\beta = -k_{[\mu}\mathcal{A}_{\nu]}$$

3) Compute currents

$$J^\mu(x) = \int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\, \mathcal{V}^\mu(x,k) = n u^\mu + \xi\omega^\mu + \xi_B B^\mu \, ,$$

$$J_5^\mu(x) = \int \frac{\mathrm{d}^4k}{(2\pi\hbar)^4}\, \mathcal{A}^\mu(x,k) = n_5 u^\mu + \xi_5\omega^\mu + \xi_{B5} B^\mu$$

J.-H. Gao, Z.-T. Liang, S. Pu, Q. Wang, X.-N. Wang, Xin-Nian, Phys. Rev. Lett. 109, 232301 (2012)

N. Weickgenannt, X.-L. Sheng, E. Speranza, Q. Wang, D. H. Rischke, Phys. Rev. D 100, 056018 (2019)

Y. Hidaka, S. Pu, Q. Wang, D.-L. Yang, Prog. Part. Nucl. Phys., 127, 103989 (2022)

Z. Chen, S. Lin, Phys. Rev. D 105, 014015 (2022)

CME, CVE, & CSE from QKT

- Quantum kinetic theory: these effects arise naturally as gradient contributions at global equilibrium

1) Assume global equilibrium, $f_s^{(0)} \coloneqq [\exp(k \cdot u/T - \mu_s/T) + 1]^{-1}$

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3) Compute currents

CVE CME

$$J^\mu(x) = \int \frac{\mathrm{d}^4 k}{(2\pi\hbar)^4}\, \mathcal{V}^\mu(x,k) = n u^\mu + \xi \omega^\mu + \xi_B B^\mu \, ,$$

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ACVE CSE

J.-H. Gao, Z.-T. Liang, S. Pu, Q. Wang, X.-N. Wang, Xin-Nian, Phys. Rev. Lett. 109, 232301 (2012)
N. Weickgenannt, X.-L. Sheng, E. Speranza, Q. Wang, D. H. Rischke, Phys. Rev. D 100, 056018 (2019)
Y. Hidaka, S. Pu, Q. Wang, D.-L. Yang, Prog. Part. Nucl. Phys., 127, 103989 (2022)
Z. Chen, S. Lin, Phys. Rev. D 105, 014015 (2022)

CME, CVE, & CSE from QKT

- Quantu … ent contrib …

1) Assum … $^{-1}$

$$\mathcal{V}^\mu(x,k) \approx \ldots (f_R \ldots + f_L \ldots)\,, \quad \mathcal{A}^\mu(x,k) \approx \ldots (f_R \ldots - f_L^{(0)})$$

CME and negative magnetoresistance:
Analysis in Dirac semimetals with Keldysh technique
see **R. Abramchuk's talk, 23.07. 12:00**

2) Obtain gradient cor…

$$\frac{\hbar}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\alpha \ldots$$

Anomalous Hall instability in chiral MHD:
see **S. Wang's talk, today 15:00**

3) Compute currents

$$J^\mu(x) = \int \frac{d^4k}{\ldots}$$

$$J_5^\mu(x) = \int \ldots$$

Chiral effects in astrophysics

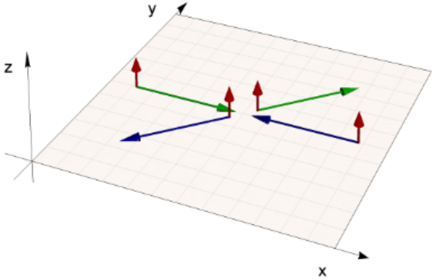
K. Kamada, N. Yamamoto, D.-L. Yang, Prog. Part. Nucl. Phys. 129, 104016 (2023)

see also **D.-L. Yang's talk, 23.07. 14:30**

J.-H. Gao, Z.-T. Liang, S. Pu, Q. Wang, X.-N. Wang, Xin-Nian, Phys. Rev. Lett. 109, 232301 (2012)
N. Weickgenannt, X.-L. Sheng, E. Speranza, Q. Wang, D. H. Rischke, Phys. Rev. D 100, 056018 (2019)
Y. Hidaka, S. Pu, Q. Wang, D.-L. Yang, Prog. Part. Nucl. Phys., 127, 103989 (2022)
Z. Chen, S. Lin, Phys. Rev. D 105, 014015 (2022)

Beyond equilibrium: Collisions

Angular momentum and collisions

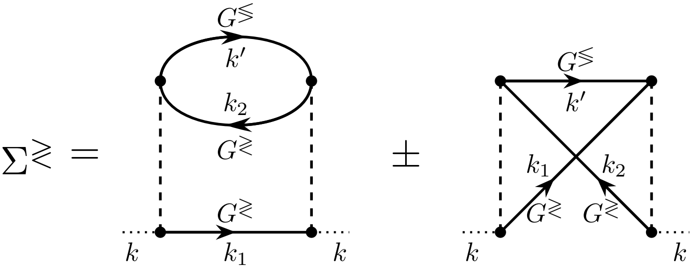


W. Florkowski, A. Kumar, R. Ryblewski, Prog. Part. Nucl. Phys. 108, 103709 (2019)

- Assume that collisions take place in a point
  - → Total orbital angular momentum vanishes
  - → Spin is conserved on its own
  - → No exchange of spin and orbital angular momenta
- Collisions must be **nonlocal** for spin equilibration!

Collisions from QKT

$$k \cdot \partial G^{<}(x,k) = \frac{1}{2}\left[\Sigma^{<}(x,k)G^{>}(x,k) - \Sigma^{<}(x,k)G^{<}(x,k)\right]$$



- Collisions determined by self-energies
- Crucial quantum enhancement: *All internal lines have to be evaluated to order $\mathcal{O}(\hbar)$!*
  - $\longrightarrow$ Introduces gradient corrections inside the collision integral!

Collisions from QKT

DW, NW, DHR, Phys.Rev.D 106 (2022) 11, 116021

Boltzmann equation with collisions

$$\mathfrak{f}(x,k,\mathfrak{s}) = \delta(k^2-m^2)f(x,k,\mathfrak{s}) := \frac{1}{2}\left[\mathcal{F}(x,k) - \mathfrak{s}\cdot\mathcal{A}(x,k)\right]$$

$$\begin{aligned} k\cdot\partial f(x,k,\mathfrak{s}) = \frac{1}{2}\int & \mathrm{d}\Gamma_1\mathrm{d}\Gamma_2\mathrm{d}\Gamma'\mathrm{d}\bar{S}(k)\delta^{(4)}(k_1+k_2-k-k')\mathcal{W} \\ & \times\left[f(x+\Delta_1-\Delta,k_1,\mathfrak{s}_1)f(x+\Delta_2-\Delta,k_2,\mathfrak{s}_2)\right. \\ & \left.\quad -f(x,k,\bar{\mathfrak{s}})f(x+\Delta'-\Delta,k',\mathfrak{s}')\right] \end{aligned}$$

$\mathrm{d}\Gamma := 2\mathrm{d}^4k\delta(k^2-m^2)\mathrm{d}S(k)$

Collisions from QKT

DW, NW, DHR, Phys.Rev.D 106 (2022) 11, 116021

Boltzmann equation with collisions

$$\mathfrak{f}(x,k,\mathfrak{s}) = \delta(k^2-m^2) f(x,k,\mathfrak{s}) := \frac{1}{2}\left[\mathcal{F}(x,k) - \mathfrak{s}\cdot\mathcal{A}(x,k)\right]$$

$$k\cdot\partial f(x,k,\mathfrak{s}) = \frac{1}{2}\int \mathrm{d}\Gamma_1 \mathrm{d}\Gamma_2 \mathrm{d}\Gamma' \mathrm{d}\bar{S}(k)\delta^{(4)}(k_1+k_2-k-k')\mathcal{W}$$
$$\times\left[f(x+\Delta_1-\Delta,k_1,\mathfrak{s}_1) f(x+\Delta_2-\Delta,k_2,\mathfrak{s}_2) - f(x,k,\bar{\mathfrak{s}}) f(x+\Delta'-\Delta,k',\mathfrak{s}')\right]$$

- Contributions inside the collision term have gradient corrections

$$f(x,k,\mathfrak{s}) + \Delta^\mu\partial_\mu f(x,k,\mathfrak{s}) \approx f(x+\Delta,k,\mathfrak{s})$$

- A (momentum- and spin-dependent) **spacetime shift Δ^μ** enters
  $\rightarrow$ Particles do not scatter at the same spacetime point!
- This enables a conversion of orbital and spin angular momenta

---

$\mathrm{d}\Gamma := 2\mathrm{d}^4k\,\delta(k^2-m^2)\mathrm{d}S(k)$

More on collisional QKT:

S. Mrowczynski, U. W. Heinz, Annals Phys. 229, 1-54 (1994)

K. Morawetz, P. Lipavsky, V. Spicka, N.-H. Kwong, Phys. Rev. C 59, 3052–3059 (1999)

D.-L. Yang, K. Hattori, Y. Hidaka, Yoshimasa JHEP 07, 070 (2020)

K. Hattori, Y. Hidaka, N. Yamamoto, D.-L. Yang, JHEP 02, 001 (2021)

N. Weickgenannt, E. Speranza, X.-L. Sheng, Q. Wang, D. H. Rischke, Phys. Rev. D 104, 016022 (2021)

X.-L. Sheng, N. Weickgenannt, E. Speranza, D. H. Rischke, Q. Wang, Phys. Rev. D 104, 016029 (2021)

J. Hu, 2110.12339 (2021)

S. Lin, Phys. Rev. D 105, 076017 (2022)

S. Fang, S. Pu, D.-L. Yang, Phys. Rev. D 106, 016002 (2022)

Y. Hidaka, S. Pu, Q. Wang, D.-L. Yang, Prog. Part. Nucl. Phys. 127, 103989 (2022)

DW, N. Weickgenannt, D. H. Rischke, Phys. Rev. D 106, 116021 (2022)

X.-L. Sheng, Q. Wang, D. H. Rischke, Phys. Rev. D 106, L111901 (2022)

DW, N. Weickgenannt, E. Speranza, Phys. Rev. D 108, 116017 (2023)

N. Yamamoto, D.-L. Yang, Phys. Rev. D 109, 056010 (2024)

D

E

$x, k)]$

$)\mathcal{W}$

$\mathfrak{s}_2)$

ions

$$f(x,k,\mathfrak{s}) + \Delta^\mu \partial_\mu f(x,k,\mathfrak{s}) \approx f(x+\Delta,k,\mathfrak{s})$$

- A (momentum- and spin-dependent) **spacetime shift Δ^μ** enters
  $\rightarrow$ Particles do not scatter at the same spacetime point!
- This enables a conversion of orbital and spin angular momenta

$\mathrm{d}\Gamma := 2\mathrm{d}^4 k \delta(k^2 - m^2)\mathrm{d}S(k)$

Back to equilibrium

- Local-equilibrium distribution function makes local collision term vanish (without spacetime shifts Δ^μ)
- Has to depend on the **collisional invariants**
  - $\rightarrow$ Charge, four-momentum and total angular momentum

Back to equilibrium

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- Has to depend on the **collisional invariants**
  $\rightarrow$ Charge, four-momentum and total angular momentum

Local-equilibrium distribution function

$$f_{\text{eq}}(x,k,\mathfrak{s}) = \exp\left(\alpha_0 - \beta_0 E_{\mathbf{k}} + \frac{\hbar}{2}\Omega_{0,\mu\nu}\Sigma_{\mathfrak{s}}^{\mu\nu}\right)$$

- Necessary conditions on Lagrange multipliers for a vanishing **nonlocal** collision term: $\partial^\mu \alpha_0 = 0$, $\partial^{(\mu}(\beta_0 u^{\nu)}) = 0$, $\Omega_0^{\mu\nu} = -\frac{1}{2}\partial^{[\mu}(\beta_0 u^{\nu]})$
- Same conditions as for **global** equilibrium, where $k\cdot\partial f_{\text{eq}} = 0$

$\Sigma_{\mathfrak{s}}^{\mu\nu} := -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}k_\alpha \mathfrak{s}_\beta$, $E_{\mathbf{k}} := k\cdot u$

Back to equilibrium

Alternative way of introducing $\Omega_0^{\mu\nu}$: Statistical operator
Then explicitly pseudogauge dependent!
see also **M. Buzzegoli's talk, 23.07. 17:30**

- on term
-
  $\rightarrow$ Charge, four-momentum and total angular momentum

Local-equilibrium distribution function

$$f_{\text{eq}}(x,k,\mathfrak{s}) = \exp\left(\alpha_0 - \beta_0 E_{\mathbf{k}} + \frac{\hbar}{2}\Omega_{0,\mu\nu}\Sigma_{\mathfrak{s}}^{\mu\nu}\right)$$

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- Same conditions as for **global** equilibrium, where $k \cdot \partial f_{\text{eq}} = 0$

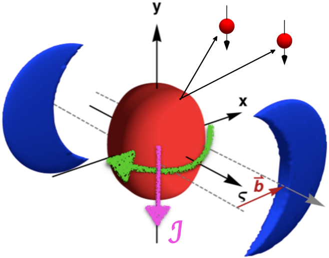
$\Sigma_{\mathfrak{s}}^{\mu\nu} := -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}k_\alpha \mathfrak{s}_\beta$, $E_{\mathbf{k}} := k \cdot u$

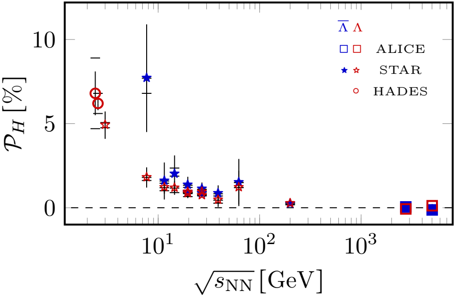
Polarization & spin hydrodynamics

Sessions dedicated to polarization: 22.07.24, 11:00-13:00
24.07.24, 14:30-16:00
26.07.24, 11:00-13:00

Sessions dedicated to hydrodynamics: 23.07.24, 16:30-18:00
25.07.24, 14:30-16:00
26.07.24, 14:30-16:00

Global Λ-Polarization

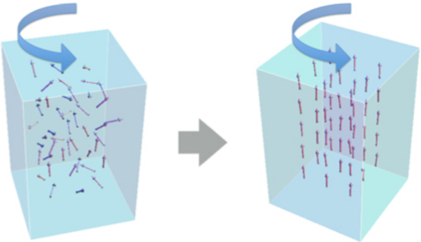




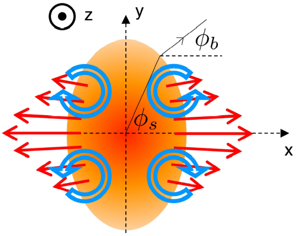
- "Global": Integrated polarization along the direction of orbital angular momentum
- Can be explained by assuming spins in equilibrium

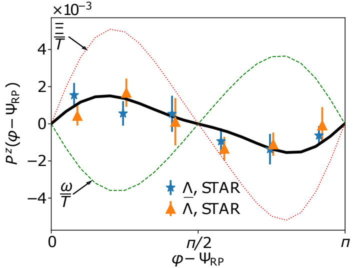
→ "Polarization through rotation"

- Analogous to Barnett effect



Local Λ-Polarization

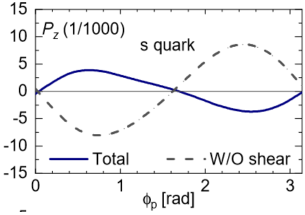




F. Becattini, M. Buzzegoli, G. Inghirami, I. Karpenko, A. Palermo, PRL 127, 272302 (2021)

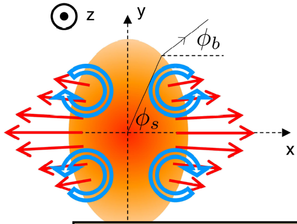
- "Local": Angle-dependent polarization along beam-direction
- Could only be explained recently by incorporating shear effects

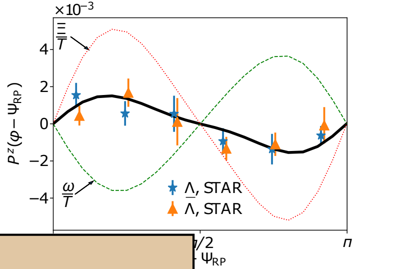
→ Simple picture of equilibrated spins not complete



B. Fu, S. Y. F. Liu, L. Pang, H. Song, Y. Yin, Phys. Rev. Lett. 127, 142301 (2021)

Local Λ-Polarization



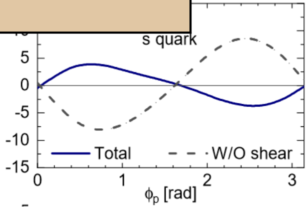


mi, I. Karpenko, A. Palermo,

Many great talks at this conference
see also **A. Palermo's talk, 22.07. 09:45**

- “Lo
  pola
  beam-direction
- Could only be explained recently by incorporating shear effects

→ Simple picture of equilibrated spins not complete



B. Fu, S. Y. F. Liu, L. Pang, H. Song, Y. Yin, Phys. Rev. Lett. 127, 142301 (2021)

Open questions

Pauli-Lubanski vector in (global) equilibrium

$$S^\mu_\varpi = -\epsilon^{\mu\nu\alpha\beta} k_\nu \frac{\int \mathrm{d}\Sigma_\lambda k^\lambda f_0(1-f_0)\varpi_{\alpha\beta}}{8m \int \mathrm{d}\Sigma_\lambda k^\lambda f_0}$$

- Traditional approaches to computing the polarization
  - assume equilibrated spin degrees of freedom
  - neglect dissipative quantities

$\varpi_{\mu\nu} := \frac{1}{2}[\partial_\mu (u_\nu/T) - \partial_\nu (u_\mu/T)]$

Open questions

Pauli-Lubanski vector in (global) equilibrium

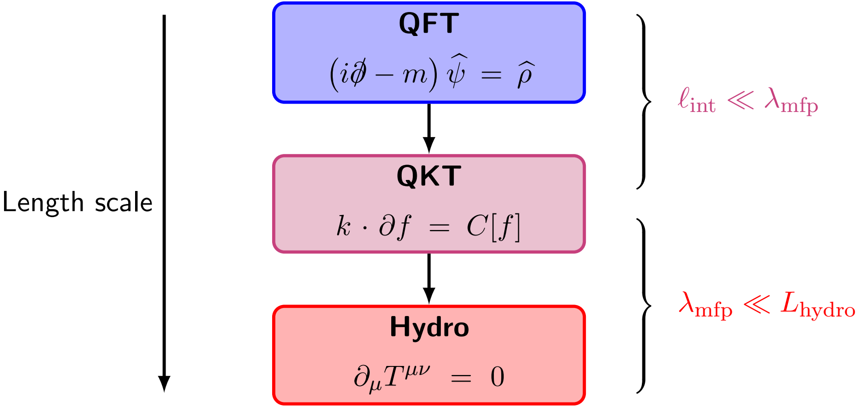
$$S^{\mu}_{\varpi} = -\epsilon^{\mu\nu\alpha\beta} k_{\nu} \frac{\int \mathrm{d}\Sigma_{\lambda} k^{\lambda} f_0 (1 - f_0) \varpi_{\alpha\beta}}{8m \int \mathrm{d}\Sigma_{\lambda} k^{\lambda} f_0}$$

- Traditional approaches to computing the polarization
  - assume equilibrated spin degrees of freedom
  - neglect dissipative quantities
- Not clear so far:
  - (I) How fast do spin degrees of freedom equilibrate?
  - (II) How do dissipative effects influence polarization?

→ Can be answered through spin hydrodynamics from quantum kinetic theory

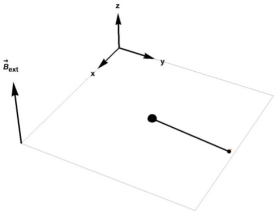
$\varpi_{\mu\nu} := \frac{1}{2}[\partial_{\mu} (u_{\nu}/T) - \partial_{\nu} (u_{\mu}/T)]$

Spin hydrodynamics



Comparison: Nuclear magnetic resonance (NMR)

- NMR: Large constant B-field in z-direction and short-lived alternating field in x, y-plane
- Identify materials by relaxation times T_1, T_2



https://en.wikipedia.org/wiki/Bloch_equations

Bloch equations

$$T_2 \dot{M}_{x,y} + M_{x,y} = \mu_2 \left(\mathbf{M} \times \mathbf{B}\right)_{x,y} \; ,$$
$$T_1 \dot{M}_z + M_z = \mu_1 \left(\mathbf{M} \times \mathbf{B}\right)_z + M_0 \; .$$

$\mu_1 := T_1 \frac{gq}{2m}$, $\mu_2 := T_2 \frac{gq}{2m}$

Spin waves in a fluid at rest

Conservation equations

$$\partial_\mu T^{(\mu\nu)} = 0 + \mathcal{O}(\hbar^2) \,,$$
$$\partial_\lambda S^{\lambda\mu\nu} = \frac{1}{\hbar} T^{[\nu\mu]} + \mathcal{O}(\hbar^2) \,.$$

- No backreaction of spin on fluid evolution, fluid profile serves as input for spin potential

Equations of motion for the spin potential

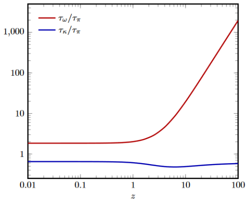
$$\tau_\kappa \dot{\kappa}^{\langle\mu\rangle} + \kappa^\mu = \mu_\kappa \epsilon^{\mu\nu\alpha\beta} u_\nu \nabla_\alpha \omega_\beta \,,$$
$$\tau_\omega \dot{\omega}^{\langle\mu\rangle} + \omega^\mu = -\mu_\omega \epsilon^{\mu\nu\alpha\beta} u_\nu \nabla_\alpha \kappa_\beta \,.$$

- Relaxation times τ_κ, τ_ω determined by **nonlocal** collisions

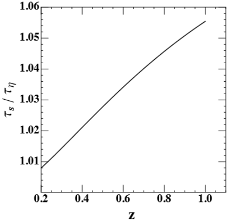
V. E. Ambrus, R. Ryblewski, R. Singh, Phys. Rev. D 106, 014018(2022)
J. Hu, Z. Xu, Phys. Rev. D 107, 016010 (2023)
DW, M. Shokri, D. H. Rischke, 2405.00533 (2024)

Spin relaxation timescales

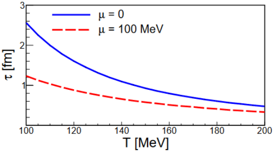
- Relaxation time depends on interaction model, can potentially get large
  - $\rightarrow$ Spin relaxation can be a slow process, dynamics cannot be neglected!



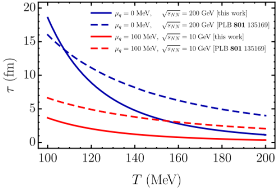
DW, M. Shokri, D. H. Rischke, 2405.00533 (2024)



J. Hu, Phys. Rev. D 105, 096021 (2022)



A. Ayala et al, Phys. Rev. D 109, 074018 (2024)



A. Ayala, D. de la Cruz, L. A. Hernández, J. Salinas, Phys. Rev. D 102, 056019 (2020)

Spin relaxation timescales

- Relaxation time depends on interaction model, can potentially get large
  - → Spin relaxation can be a slow process, dynamics cannot be neglected!

More on spin relaxation:

Y. Hidaka, M. Hongo, M. A. Stephanov, H.-U. Yee, Phys. Rev. C 109, 054909 (2024)

M. Hongo, X.-G. Huang, M. Kaminski, M. Stephanov, H.-U. Yee, JHEP 08, 263 (2022)

A. Ayala et al, Phys. Rev. D 109, 074018 (2024)

A. Ayala, D. de la Cruz, L. A. Hernández, J. Salinas, Phys. Rev. D 102, 056019 (2020)

A. Ayala, D. de la Cruz, S. Hernández-Ortíz, L. A. Hernández, J. Salinas, Phys. Lett. B 801, 135169 (2020)

J. I. Kapusta, E. Rrapaj, S. Rudaz, Phys. Rev. C 101, 024907 (2020)

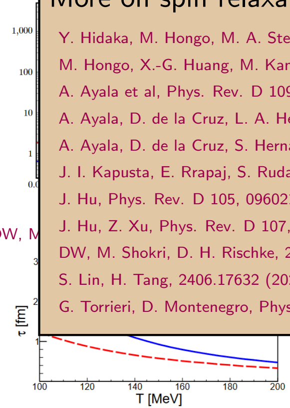
J. Hu, Phys. Rev. D 105, 096021 (2022)

J. Hu, Z. Xu, Phys. Rev. D 107, 016010 (2023)

DW, M. Shokri, D. H. Rischke, 2405.00533 (2024)

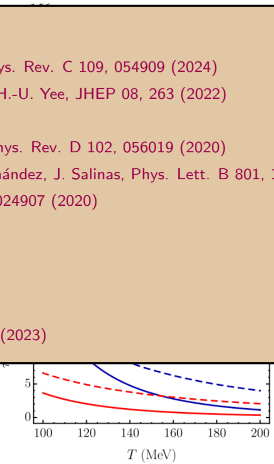
S. Lin, H. Tang, 2406.17632 (2024)

G. Torrieri, D. Montenegro, Phys. Rev. D 107, 076010 (2023)



DW, M

A. Ayala et al, Phys. Rev. D 109, 074018 (2024)

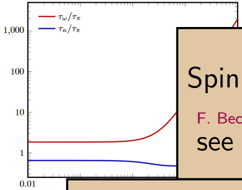


A. Ayala, D. de la Cruz, L. A. Hernández, J. Salinas, Phys. Rev. D 102, 056019 (2020)

Spin relaxation timescales

- Relaxation time de ially get large
  - → Spin relaxation neglected!

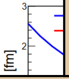
Updates on spin hydrodynamics
see **M. Shokri's talk, 23.07. 16:30**



Spin hydro from quantum statistics

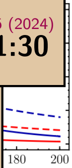
F. Becattini, A. Daher, X.-L. Sheng, Phys. Lett. B 850, 138533 (2024)
see also **A. Daher's talk, 23.07. 17:00**

DW, M. Sh)



Vortical waves in fluids with AVH charges

S. Morales-Tejera, V. E. Ambruș, M. N. Chernodub, 2403.19755 & 2403.19756 (2024)
see also **S. Morales-Tejera's talk, 22.07. 11:30**



Anomalous hydro transport in Weyl semimetals

A. Amoretti, D. K. Brattan, L. Martinoia, I. Matthaiakakis, J. Rongen, JHEP 02, 071 (2024)
see also **L. Martinoia's talk, 25.07. 15:30**

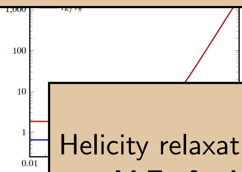
Hernández, J. Salinas,
20)

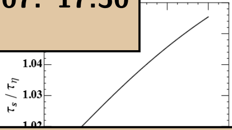
Spin relaxation timescales

l, can potentially get

nics cannot be neglected!

Ideal spin hydro on rotating background
see **A. Chiarini's poster & talk, 22.07. 17:30**

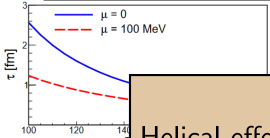


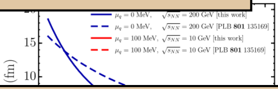


Helicity relaxation time in fermionic plasmas
see **V.E. Ambruș's poster & talk, 22.07. 18:05**

DW, M.

022)





Helical effects in fermionic plasmas
see **V.E. Ambruș's poster & talk, 22.07. 18:10**

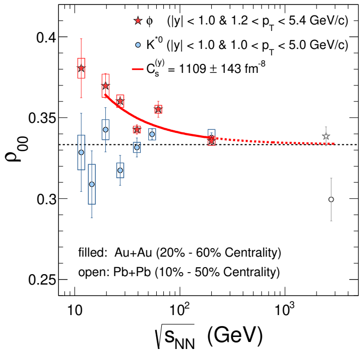
A. Ayala et al, Phys.

linas,

That's not all Folks: Alignment

Alignment of ϕ-mesons

- Spin-1 particles feature tensor polarization ($\hat{=}$ alignment)
- Intuition: "Polarization counts the difference between spin-projections $+1$ and -1, alignment counts spin-projection 0"
  - Larger than expected
  - Many theoretical developments
    Z.-T. Liang, X.-N. Wang, Phys. Lett. B 629, 20-26 (2005)
    X.-L. Sheng, L. Oliva, Q. Wang, Phys. Rev. D 101, 096005 (2020)
    X.-L. Xia, H. Li, X-G. Huang, H.-Z. Huang, PLB 817 (2021) 136325
    X.-L. Sheng, L. Oliva, Z.-T. Liang, Q. Wang, X.-N. Wang, Phys. Rev. D 109, 036004 (2024)
    F. Li, S. Y. F. Liu, 2206.11890 (2022)
    DW, N. Weickgenannt, E. Speranza, Phys. Rev. Res. 5, 013187 (2023)
    J.-H. Chen, Z.-T. Liang, Y.-G. Ma, X.-L. Sheng, Q. Wang, 2407.06480



STAR collaboration, Nature 614, 244-248 (2023)

Alignment in QKT

- Spin-1 Wigner function has 16 components, 9 of which are independent

$$W^{\mu\nu} = E^{\mu\nu} f_E + \frac{k^{(\mu}}{2k} F_S^{\nu)} + F_K^{\mu\nu} + K^{\mu\nu} f_K + i \frac{k^{[\mu}}{2k} F_A^{\nu]} + i\epsilon^{\mu\nu\alpha\beta} \frac{k_\alpha}{m} G_\beta$$

  - $f_K \longrightarrow$ particle-number density
  - $G^\mu \longrightarrow$ polarization
  - $F_K^{\mu\nu} \longrightarrow$ tensor polarization/ alignment
- Several approaches on the market (incomplete list)
  - Coalescence
    Y.-G. Yang, R.-H. Fang, Q. Wang, X.-N. Wang, Phys. Rev. C, 97, 034917 (2018)
  - Strong force fields
    X.-L. Sheng, L. Oliva, Z.-T. Liang, Q. Wang, X.-N. Wang, Phys. Rev. Lett. 131, 042304 (2023)
    X.-L. Sheng, S. Pu, Q. Wang, Phys. Rev. C 108, 054902 (2023)
    S. Fang, S. Pu, D.-L. Yang, Phys. Rev. D 109, 034034 (2024)
    X.-L. Sheng, L. Oliva, Z.-T. Liang, Q. Wang, X.-N. Wang, Phys. Rev. D 109,036004 (2024)
  - Holography
    Y.-Q. Zhao, X.-L. Sheng, S.-W. Li, D. Hou,2403.07468
    X.-L. Sheng, Y.-Q. Zhao, S.-W. Li, F. Becattini, D. Hou, Defu, 2403.07522
  - Hydrodynamic gradients
    DW, N. Weickgenannt, E. Speranza, Phys. Rev. Res. 5, 013187 (2023)
    W.-B. Dong, Y.-L. Yin, x.-L. Sheng, S.-Z. Yang, Q. Wang, Phys. Rev. D 109, 056025 (2024)

Summary and (personal) outlook

- Quantum kinetic theory is a versatile effective microscopic theory
  - At (global) equilibrium: allows to recover, e.g., CME, CVE, CSE, $\cdots$
  - At (local) equilibrium: provides a basis for ideal (spin) hydrodynamics
  - Near local equilibrium: provides a basis for dissipative (spin) hydrodynamics
  - Away from equilibrium: can be studied as a full-fledged nonequilibrium transport theory
- Future applications & developments (I would like to see):
  - Study of spin hydro from QKT in various setups, in particular polarization dynamics
  - Numerical implementation of kinetic equations to first order in $\hbar$
  - Application to astrophysical scenarios, e.g., spin transport in neutron stars

Appendix

Conserved currents in QKT

Conserved currents

$$\frac{1}{2}T^{(\mu\nu)} = \int \mathrm{d}\Gamma k^\mu k^\nu f \ ,$$

$$S^{\lambda\mu\nu} = \frac{1}{2m}\int \mathrm{d}\Gamma k^\lambda \epsilon^{\mu\nu\alpha\beta} k_\alpha \mathfrak{s}_\beta f \ .$$

$$T^{[\mu\nu]} = \frac{1}{2}\int [\mathrm{d}\Gamma]\widetilde{\mathcal{W}}\Delta^{[\mu} k^{\nu]} \left(f_1 f_2 - f f'\right)$$

Conservation laws

$$\int \mathrm{d}\Gamma k^\mu C[f] = 0$$

$$\frac{\hbar}{2m}\int \mathrm{d}\Gamma \epsilon^{\mu\nu\alpha\beta} k_\alpha \mathfrak{s}_\beta C[f] = \frac{\hbar}{m}\int \frac{\mathrm{d}^4 k}{(2\pi\hbar)^4} k^{[\mu}\mathcal{D}^{\nu]}_{\mathcal{V}}$$

$[\mathrm{d}\Gamma] \coloneqq \mathrm{d}\Gamma_1\, \mathrm{d}\Gamma_2\, \mathrm{d}\Gamma\, \mathrm{d}\Gamma'$

Polarization observables in kinetic theory

Vector Polarization (Pauli-Lubanski Pseudovector)

$$S^\mu(k) := \mathrm{Tr}\left[\hat{S}^\mu\,\hat{\rho}(k)\right] = \frac{1}{N(k)}\int \mathrm{d}\Sigma_\lambda k^\lambda \int \mathrm{d}S(k)\mathfrak{s}^\mu f(x,k,\mathfrak{s})$$

Tensor Polarization

$$\begin{aligned}
\rho_{00}(k) &= \frac{1}{3} - \sqrt{\frac{2}{3}}\epsilon^{(0)}_\mu(k)\epsilon^{(0)}_\nu(k)\Theta^{\mu\nu}(k) \\
\Theta^{\mu\nu}(k) &:= \frac{1}{2}\sqrt{\frac{3}{2}}\mathrm{Tr}\left[\left(\hat{S}^{(\mu}\hat{S}^{\nu)} + \frac{4}{3}K^{\mu\nu}\right)\hat{\rho}(k)\right] \\
&= \frac{1}{2}\sqrt{\frac{3}{2}}\frac{1}{N(k)}\int \mathrm{d}\Sigma_\lambda k^\lambda \int \mathrm{d}S(k) K^{\mu\nu}_{\alpha\beta}\mathfrak{s}^\alpha\mathfrak{s}^\beta f(x,k,\mathfrak{s})
\end{aligned}$$

$N(k) := \int \mathrm{d}\Sigma_\gamma k^\gamma \int \mathrm{d}S(k) f(x,k,\mathfrak{s})$, $\quad \hat{S}^\mu := -(1/2m)\epsilon^{\mu\nu\alpha\beta}\hat{J}_{\nu\alpha}\hat{P}_\beta$

Moment expansion

- Split distribution function $f = f_{\text{eq}} + \delta f$
- Perform moment expansion including spin degrees of freedom

Irreducible moments

$$\rho_r^{\mu_1\cdots\mu_\ell}(x) := \int \mathrm{d}\Gamma E_{\mathbf{k}}^r k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\delta f(x,k,\mathfrak{s})$$

$$\tau_r^{\mu,\mu_1\cdots\mu_\ell}(x) := \int \mathrm{d}\Gamma \mathfrak{s}^\mu E_{\mathbf{k}}^r k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\delta f(x,k,\mathfrak{s})$$

$$\psi_r^{\mu\nu,\mu_1\cdots\mu_\ell}(x) := \int \mathrm{d}\Gamma K^{\mu\nu}_{\alpha\beta}\mathfrak{s}^\alpha\mathfrak{s}^\beta E_{\mathbf{k}}^r k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\delta f(x,k,\mathfrak{s})$$

- Equations of motion can be derived from Boltzmann equation
- Knowing the evolution of all moments is equivalent to solving the Boltzmann equation

$k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} := \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell} k^{\nu_1}\cdots k^{\nu_\ell}$

Nonlocal collisions

DW, NW, ES, 2306.05936 (2023)

Spacetime shifts

$$\Delta^\mu \;\; := \;\; -\frac{i\hbar}{8m}\frac{m^4}{\mathcal{W}} M^{\gamma_1\gamma_2\delta_1\delta_2} M^{\zeta_1\zeta_2\eta_1\eta_2} h_{1,\gamma_1\eta_1} h_{2,\gamma_2\eta_2} h'_{\zeta_2\delta_2} [h, \gamma^\mu]_{\zeta_1\delta_1}$$

- Depend on the transfer-matrix elements

$$\langle 11' | \,\widehat{t}\, | 22' \rangle = \bar{u}_{1,\alpha} \bar{u}_{1',\beta} u_{2,\gamma} u_{2',\delta} M^{\alpha\beta\gamma\delta}$$

- Manifestly covariant
  - $\rightarrow$ no "no-jump" frame

$h := \frac{1}{4}(\mathbb{1} + \gamma_5 \not{s})(\not{k} + m)$

Moment equations: Spin-rank 1

- Same procedure as for the moments of spin-rank 0

Moment equation for $\ell = 2$

$$\dot{\tau}_r^{\langle\mu\rangle,\nu\lambda} - \mathfrak{C}_{r-1}^{\langle\mu\rangle,\nu\lambda} = \cdots$$

- Navier-Stokes limit: $\mathfrak{C}_{r-1}^{\langle\mu\rangle,\nu\lambda} = 0$
- Contains local and nonlocal contributions
  - $\mathfrak{C}_{r,\text{local}}^{\langle\mu\rangle,\nu\lambda} \sim \tau_r^{\mu,\nu\lambda}$
  - $\mathfrak{C}_{r,\text{nonlocal}}^{\langle\mu\rangle,\nu\lambda} \sim \sigma_\rho{}^{\langle\nu}\epsilon^{\lambda\rangle\mu\alpha\rho}u_\alpha$
- Leads to shear-induced polarization, coefficient independent of total cross-section
- Magnitude not yet clear

N. Weickgenannt, DW, E. Speranza, D. H. Rischke, Phys. Rev. D 106, L091901 (2022)
N. Weickgenannt, DW, E. Speranza, D. H. Rischke, Phys. Rev. D 106, 096014 (2022)

Moment equations: Spin-rank 2

Moment equation for $\ell = 0$

$$\dot{\psi}_r^{\langle\mu\nu\rangle} - \mathfrak{C}_{r-1}^{\langle\mu\nu\rangle} = -\frac{\theta}{3}\left[(r+2)\psi_r^{\langle\mu\nu\rangle} - (r-1)m^2\psi_{r-2}^{\langle\mu\nu\rangle}\right]\theta + r\psi_{r-1}^{\langle\mu\nu\rangle,\alpha}\dot{u}_\alpha - \Delta^{\mu\nu}_{\alpha\beta}\nabla_\gamma\psi_{r-1}^{\alpha\beta,\gamma} + (r-1)\psi_{r-2}^{\langle\mu\nu\rangle,\alpha\beta}\sigma_{\alpha\beta}$$

- No dependence on equilibrium quantities appears because moments of spin-rank 2 do not appear in any conserved current
- Nonetheless, they are important to e.g. describe **tensor polarization** of spin-1 particles

  DW, NW, ES, 2207.01111 (2022)

  - Main argument: $\mathfrak{C}_{r-1}^{\langle\mu\nu\rangle} \simeq A_r\psi_r^{\mu\nu} + B_r\pi^{\mu\nu}$ leads to $\psi_r^{\mu\nu} \sim \pi^{\mu\nu}$ in the Navier-Stokes limit

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$\Delta^{\mu\nu}_{\alpha\beta} := (\Delta^{(\mu}_{\alpha}\Delta^{\nu)}_{\beta})/2 - (1/3)\Delta^{\mu\nu}\Delta_{\alpha\beta}$