Ideal-spin hydrodynamics on top of a rotating background

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I. Abstract and motivations

- We write the equations of **spin hydrodynamics** in a **covariant form** on top of an uncharged fluid in global equilibrium with a non-vanishing thermal vorticity. .
- Assuming that the spin degrees of freedom are not in equilibrium, we derive relaxation-type equations for the components of the spin potential.
- We aim to numerically solve these equations to understand better the **equilibration timescale of spin degrees of freedom**, which will help us answer the requirement of dynamically evolving spin equations in heavy-ion collisions.

II. Covariant spin hydrodynamics

- **Fundaments of spin hydrodynamics** for uncharged fluid are given by the conservation of energy-momentum tensor and the angular momentum

$$\partial_\mu T^{\mu\nu} = 0, \qquad \partial_\lambda J^{\lambda\mu\nu} = 0, \tag{1}$$

where the angular momentum can be decomposed into the orbital and the spin part as

$$J^{\lambda\mu\nu} = L^{\lambda\mu\nu} + \hbar S^{\lambda\mu\nu}. \tag{2}$$

The orbital part is expressed as $L^{\lambda\mu\nu} := T^{\lambda[\nu}x^{\mu]}$ in special relativity and Minkowski coordinates $x^\mu = (t, x, y, z)$. However, this definition is not covariant under coordinate transformations in flat space.

- **Covariant formulation** Since $T^{\mu\nu}$ is a proper tensor, the conservation of the energy-momentum tensor can be covariantly written as

$$\nabla_\mu T^{\mu\nu} = 0 \; . \tag{3}$$

Furthermore, the angular-momentum conservation law is also a tensorial equation. Hence, the RHS admits a covariant formulation as

$$\hbar \nabla_\lambda S^{\lambda\mu\nu} = 2T^{[\nu\mu]} \; . \tag{4}$$

Since the fluid is been assumed to be uncharged, the only conserved charges arise from spacetime symmetries and conservation of the energy-momentum tensor (3).
In flat spacetime, 10 independent Killing vectors ξ^h_μ exist, which are generators of the Poincaré algebra: 4 generators of translations, 3 of rotations and 3 of boosts. The generators of translations satisfy $\nabla_{[\mu}\xi_{\nu]} = 0$, giving rise to the conservation of energy and linear momentum even if the energy-momentum is not symmetric. On the other hand, for the generators of rotations, and boosts, the gradients are not symmetric. For example, let us consider the generator of rotations around z, which we label with $h = (yx)$,

$$\xi^{\mu(yx)} = (0, -y, x, 0) \;\; , \;\; \text{such that } \nabla_{[\mu}\xi^{(yx)}_{\nu]} = \delta^x_\mu \delta^y_\nu - \delta^y_\nu \delta^x_\mu \tag{5}$$

Contracting this with the energy-momentum tensor gives rise to

$$K^{\lambda(yx)} = T^{\lambda\mu}\xi_\mu = yT^{\lambda x} - xT^{\lambda y} \; , \tag{6}$$

which is equal to $L^{\lambda yx}$ and satisfies

$$\nabla_\lambda \left(K^{\lambda(yx)} + \hbar S^{\lambda yx} \right) = 0 \; . \tag{7}$$

Motivated by this example we can define the following form of indices for h

$$h \in \{(tt), (xx), (yy), (zz), (yx), (xz), (zy), (tx), (ty), (tz)\} \; .$$

Therefore the covariant definition of the orbital angular momentum is found by replacing $L^{\lambda\mu\nu}$ with

$$L^{\lambda\mu\nu} \longrightarrow \mathcal{L}^{\lambda\mu\nu} = T^{\lambda\rho}\xi^{(\mu\nu)}_\rho . \tag{8}$$

III. Ideal-spin hydrodynamics in a rotating background

An **ideal spin fluid** defined as a fluid for which, for a given energy-momentum tensor, the evolution of spin tensor is fully determinated by (4)

- The number of independent components of $S^{\lambda\mu\nu}$ must reduce to six, and can be encoded into the antisymmetric second-rank tensor $\Omega^{\mu\nu}$

$$S^{\lambda\mu\nu} = Au^\lambda\Omega^{\mu\nu} + 2Bu^\lambda u_\alpha \Omega^{\alpha[\mu}u^{\nu]} + 2Cu^\lambda \Omega^{\alpha[\mu}\Delta^{\nu]}{}_\alpha + 2Du_\alpha\Omega^{\alpha[\mu}\Delta^{\nu]\lambda} + 2E\Delta^\lambda{}_\alpha\Omega^{\alpha[\mu}u^{\nu]}, \tag{9}$$

where the coefficients $\{A,\, B,\, C,\, D,\, E\}$ are functions of the fluid variables.
$\Omega^{\mu\nu}$ is the *spin potential,* which in equilibrium satisfies $\Omega_{\mu\nu} = \varpi^\star_{\mu\nu}$. It can be decomposed into "electric" κ^μ and "magnetic" ω^μ components as

$$\Omega^{\mu\nu} = u^{[\mu}\kappa^{\nu]} + \varepsilon^{\mu\nu\rho\sigma}u_\rho\omega_\sigma . \tag{10}$$

- **Kinetic theory**, developed with the Wigner function formalism, is assumed as the *underlying microscopic theory*. According to the latter, the antisymmetic part of $T^{\mu\nu}$ can be expressed as [1]

$$T^{[\mu\nu]} = -\hbar^2\Gamma^{(\kappa)}u^{[\mu}\left(k^{\nu]} + \beta\dot{u}^{\nu]}\right) + \hbar^2\Gamma^{(\omega)}\varepsilon^{\mu\nu\rho\sigma}u_\rho\left(\omega_\sigma + \beta\Omega_\sigma\right) + \hbar^2\Pi^{\mu\nu} \; , \tag{11}$$

where Ω_σ is the vorticity vector and $\Pi^{\mu\nu}$ the dissipative corrections.

- The **rigidly rotating background** is imposed by assuming a vanishing expansion and shear tensor but keeping a non-zero thermal vorticity

$$\sigma_{\mu\nu} = 0 \; , \qquad \theta = 0 \; , \qquad \nabla_\mu\beta = -\beta a_\mu \; . \tag{12}$$

Consequently, using these relations in Eq. (4) we arrive at

$$\begin{aligned} \tau_\kappa\left(\dot\kappa^\mu + a\cdot\kappa u^\mu\right) &= \rho^{\mu\lambda}_\kappa\kappa_\lambda + \epsilon^{\mu\nu\rho\sigma}u_\nu\left[\lambda_\kappa a_\rho\omega_\sigma + \mu_\kappa\nabla_\rho\omega_\sigma\right] - \hbar\left(\kappa^\mu + \beta a^\mu\right) \; , \\ \tau_\omega\left(\dot\omega^\mu + a\cdot\omega u^\mu\right) &= \rho^{\mu\lambda}_\omega\kappa_\omega - \epsilon^{\mu\nu\rho\sigma}u_\nu\left[\lambda_\kappa a_\rho\kappa_\sigma + \mu_\omega\nabla_\rho\kappa_\sigma\right] - \hbar\left(\omega^\mu + \beta^\mu\right) \; , \end{aligned} \tag{13}$$

where we have defined

$$\begin{aligned} \tau_\kappa &:= -\tfrac{A-B-C}{\hbar\Gamma^{(\kappa)}} , & \mu_\kappa &:= -\tfrac{E}{\hbar\Gamma^{(\kappa)}} , & \rho_\kappa &:= -\tfrac{D}{\hbar\Gamma^{(\kappa)}} , & \lambda_\kappa &:= -\tfrac{E+2C-A-(\partial E\partial\beta)}{\hbar\Gamma^{(\kappa)}} \\ \tau_\omega &:= -\tfrac{A-B-C}{\hbar\Gamma^{(\omega)}} , & \mu_\omega &:= -\tfrac{D}{\hbar\Gamma^{(\omega)}} , & \rho_\omega &:= -\tfrac{E}{\hbar\Gamma^{(\omega)}} , & \lambda_\omega &:= -\tfrac{B+C-A-D-\beta(\partial D\partial\beta)}{\hbar\Gamma^{(\omega)}} . \end{aligned}$$

IV. Relaxation-type equations

For a rigidly rotating background, it is also convenient to decompose the equations along an orthonormal tetrad $\{u^\mu, \ell^\mu, \psi^\mu, \zeta^\mu\}$. In the **co-rotating frame**, the dynamical equations for spin can be written as

$$\frac{\partial\mathbf{U}}{\partial t} + \frac{\partial\mathbf{F}_\phi}{\partial\phi} + \frac{\partial\mathbf{F}_\rho}{\partial\rho} + \frac{\partial\mathbf{F}_z}{\partial z} = \mathbf{S}(\mathbf{V}) \tag{14}$$

i.e., in a *conservative form* where

$$\mathbf{U} = \begin{bmatrix} \tau_\kappa\gamma\kappa_\ell - \mu_\kappa\gamma\rho\Omega_0\omega_\psi \\ \tau_\kappa\gamma\kappa_\psi - \mu_\kappa\gamma\rho\Omega_0\omega_\ell \\ \tau_\kappa\gamma\kappa_\zeta \\ \tau_\omega\gamma\omega_\ell - \mu_\omega\gamma\rho\Omega_0\kappa_\psi \\ \tau_\omega\gamma\omega_\psi - \mu_\omega\gamma\rho\Omega_0\kappa_\ell \\ \tau_\omega\gamma\omega_\zeta \end{bmatrix} , \quad \mathbf{F}_\phi = \begin{bmatrix} \frac{\mu_\kappa}{\gamma\rho}\omega_\psi \\ \frac{\mu_\kappa}{\gamma\rho}\omega_\ell \\ 0 \\ \frac{\mu_\omega}{\gamma\rho}\kappa_\psi \\ \frac{\mu_\omega}{\gamma\rho}\kappa_\ell \\ 0 \end{bmatrix} , \quad \mathbf{F}_\rho = \begin{bmatrix} 0 \\ \mu_\kappa\omega_\zeta \\ \mu_\kappa\omega_\psi \\ 0 \\ \mu_\omega\kappa_\zeta \\ \mu_\omega\kappa_\psi \end{bmatrix} , \quad \mathbf{F}_z = \begin{bmatrix} \mu_\kappa\omega_\zeta \\ 0 \\ -\mu_\kappa\omega_\ell \\ -\mu_\omega\kappa_\zeta \\ 0 \\ \mu_\omega\kappa_\ell \end{bmatrix} \tag{15}$$

and the sources of the primitive variables $\mathbf{V} = (\kappa_\ell, \kappa_\psi, \kappa_\zeta, \omega_\ell, \omega_\psi, \omega_\zeta)^T$

$$\mathbf{S}(\mathbf{V}) = \begin{bmatrix} -\kappa_\ell + (\tau_\kappa + \rho_\kappa)\gamma^2\Omega_0\kappa_\zeta - \frac{\gamma\rho\Omega_0^2}{T_0} \\ -\kappa_\psi - (\lambda_\kappa\gamma^2\rho\Omega_0 + \mu_\kappa\frac{\gamma^2}{\rho^2})\omega_\zeta \\ -\kappa_\zeta - (\tau_\kappa + \rho_\kappa)\gamma^2\Omega_0\kappa_\ell - \lambda_\kappa\gamma^2\rho\Omega_0^2\omega_\psi \\ -\omega_\ell + (\tau_\omega + \rho_\omega)\gamma^2\Omega_0\omega_\zeta \\ -\omega_\psi - (\lambda_\omega\gamma^2\rho\Omega_0 + \mu_\omega\frac{\gamma^2}{\rho^2})\kappa_\zeta - \frac{\gamma\Omega_0}{T_0} \\ -\omega_\zeta - (\tau_\omega + \rho_\omega)\gamma^2\Omega_0\omega_\ell - \lambda_\omega\gamma^2\rho\Omega_0^2\kappa_\psi . \end{bmatrix} . \tag{16}$$

References

- [1] D. Wagner, M. Shokri, and D. H. Rischke, *On the damping of spin waves*, arXiv:2405.00533 .

V. Outlook

- Numerically solving the equations to find the timescales τ_κ and τ_ω on which the spin potential relaxes to thermal vorticity.
- Solving spin hydrodynamics in a **realistic setup**.