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Higher orders terms in fermion spin polarization

OUTLINE

- Introduction and theory summary
- Second order corrections in the expansion of local equilibrium
- Exact resummation at all orders in global equilibrium
- Conclusions

Chirality 2024, Timisoara

Motivations

To assess higher order terms in the gradient expansion of spin polarization at local thermodynamic equilibrium and its convergence as an asymptotic expansion

Introduction: a theory summary

F. B., Lecture Notes in Physics 987, 15 (2021) arXiv:2004.04050

Spin polarization vector for spin ½ particles:

$$S^\mu(p) = \frac{1}{2} \frac{\int \mathrm{d}\Sigma \cdot p \; \mathrm{tr}_4(\gamma^\mu \gamma^5 W_+(x,p))}{\int \mathrm{d}\Sigma \cdot p \; \mathrm{tr}_4 W_+(x,p)}$$

Wigner function:

$$\begin{aligned}\widehat{W}(x,k)_{AB} &= -\frac{1}{(2\pi)^4} \int \mathrm{d}^4 y \, \mathrm{e}^{-ik\cdot y} : \Psi_A(x-y/2)\overline{\Psi}_B(x+y/2) : \\ &= \frac{1}{(2\pi)^4} \int \mathrm{d}^4 y \, \mathrm{e}^{-ik\cdot y} : \overline{\Psi}_B(x+y/2)\Psi_A(x-y/2) :\end{aligned}$$

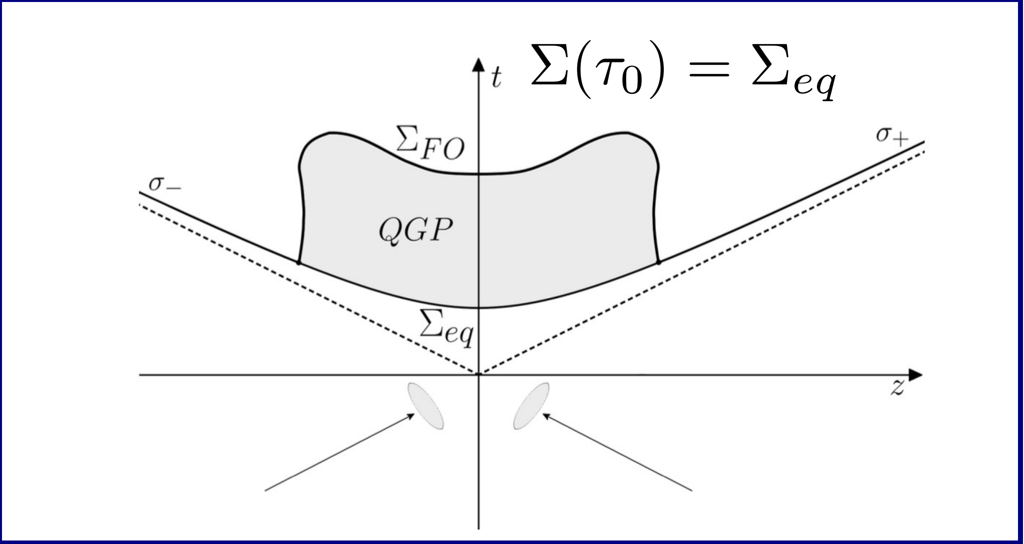
$$W(x,k) = \mathrm{Tr}(\widehat{\rho}\widehat{W}(x,k))$$

Density operator

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_{\Sigma(\tau_0)} \mathrm{d}\Sigma_\mu \left(\widehat{T}_B^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) \right] .$$

Density operator: local equilibrium at the initial time

$$\beta = \frac{1}{T} u \quad \zeta = \frac{\mu}{T}$$



With the Gauss theorem: calculate at Freeze-out

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu \left(\widehat{T}_B^{\mu\nu} \beta_\nu - \zeta \widehat{j}^\mu \right) + \int_\Theta \mathrm{d}\Theta \, \left(\widehat{T}_B^{\mu\nu} \nabla_\mu \beta_\nu - \widehat{j}^\mu \nabla_\mu \zeta \right) \right] ,$$

Local equilibrium, non-dissipative terms

Dissipative term

Hydrodynamic limit: Taylor expansion of LE

$$W(x,k)_{\rm LE} = \frac{1}{Z}{\rm Tr}\left(\exp\left[-\int_{\Sigma_{FO}} {\rm d}\Sigma_\mu(y)\widehat{T}^{\mu\nu}_B(y)\beta_\nu(y) - \zeta(y)\widehat{j}^\mu(y)\right]\widehat{W}(x,k)\right)$$

Expand the β and ζ fields from the point *x* where the Wigner operator is to be evaluated

$$\beta_\nu(y) = \beta_\nu(x) + \partial_\lambda\beta_\nu(x)(y-x)^\lambda + \ldots$$

$$\int_\Sigma {\rm d}\Sigma_\mu T^{\mu\nu}_B(y)\beta_\nu(x) = \beta_\nu(x)\int_\Sigma {\rm d}\Sigma_\mu T^{\mu\nu}_B(y) = \beta_\nu(x)\widehat{P}^\nu$$

$$\widehat{\rho}_{\rm LE} \simeq \frac{1}{Z}\exp\left[-\beta_\mu(x)\widehat{P}^\mu - \frac{1}{4}(\partial_\mu\beta_\nu(x) - \partial_\nu\beta_\mu(x))\widehat{J}^{\mu\nu}_x - \frac{1}{4}(\partial_\mu\beta_\nu(x) + \partial_\nu\beta_\mu(x))\widehat{Q}^{\mu\nu}_x + \ldots\right]$$

$$\widehat{J}^{\mu\nu}_x = \int {\rm d}\Sigma_\lambda(y-x)^\mu\widehat{T}^{\lambda\nu}_B(y) - (y-x)^\nu\widehat{T}^{\lambda\mu}_B(y)$$

$$\widehat{Q}^{\mu\nu}_x = \int {\rm d}\Sigma_\lambda(y-x)^\mu\widehat{T}^{\lambda\nu}_B(y) + (y-x)^\nu\widehat{T}^{\lambda\mu}_B(y)$$

$$\widehat{\rho}_{\mathrm{LE}} \simeq \frac{1}{Z} \exp[-\beta_\mu(x)\widehat{P}^\mu + \frac{1}{2}\varpi_{\mu\nu}(x)\widehat{J}^{\mu\nu}_x - \frac{1}{2}\xi_{\mu\nu}(x)\widehat{Q}^{\mu\nu}_x + \ldots]$$

$$\varpi_{\mu\nu} = -\frac{1}{2}(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu)$$

Thermal vorticity
Adimensional in natural units

$$\xi_{\mu\nu} = \frac{1}{2}(\partial_\mu\beta_\nu + \partial_\nu\beta_\mu)$$

Thermal shear
Adimensional in natural units

At global equilibrium the thermal shear vanishes because of the Killing equation

Linear response theory

$$\mathrm{e}^{\widehat{A}+\widehat{B}} = \mathrm{e}^{\widehat{A}} + \int_0^1 \mathrm{d}z\; \mathrm{e}^{z(\widehat{A}+\widehat{B})}\widehat{B}\mathrm{e}^{-z\widehat{A}}\mathrm{e}^{\widehat{A}} \simeq \mathrm{e}^{\widehat{A}} + \int_0^1 \mathrm{d}z\; \mathrm{e}^{z\widehat{A}}\widehat{B}\mathrm{e}^{-z\widehat{A}}\mathrm{e}^{\widehat{A}}$$

$$\widehat{A} = -\beta_\mu(x)\widehat{P}^\mu$$

$$\widehat{B} = \frac{1}{2}\varpi_{\mu\nu}(x)\widehat{J}_x^{\mu\nu} - \frac{1}{2}\xi_{\mu\nu}(x)\widehat{Q}_x^{\mu\nu} + \ldots]$$

$$W(x,k) \simeq \frac{1}{Z}\mathrm{Tr}(\mathrm{e}^{\widehat{A}+\widehat{B}}\widehat{W}(x,k)) \simeq \ldots$$

CORRELATORS

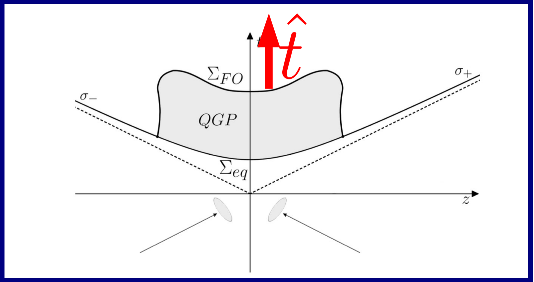
$$\langle \widehat{Q}_x^{\mu\nu}\widehat{W}(x,p)\rangle \qquad \langle \widehat{J}_x^{\mu\nu}\widehat{W}(x,p)\rangle$$

Spin mean vector at leading order

$$\widehat{\rho}_{\rm LE} \simeq \frac{1}{Z}\exp\left[-\beta_\mu(x)\widehat{P}^\mu + \frac{1}{2}\varpi_{\mu\nu}(x)\widehat{J}^{\mu\nu}_x - \frac{1}{2}\xi_{\mu\nu}(x)\widehat{Q}^{\mu\nu}_x + \dots\right]$$

$$S^\mu(p) = \frac{1}{8m}\epsilon^{\mu\nu\rho\sigma}p_\sigma \frac{\int_\Sigma {\rm d}\Sigma_\tau p^\tau n_F(1-n_F)\partial_\nu\beta_\rho}{\int_\Sigma {\rm d}\Sigma_\tau p^\tau \; n_F}$$

Neglected by “prejudice” until 2021



See also

R. H. Fang, L. G. Pang, Q. Wang, X. N. Wang, Phys. Rev. C 94 (2016) 024904
W. Florkowski, A. Kumar and R. Ryblewski, Phys. Rev. C 98 (2018) 044906
Y. C. Liu, L. L. Gao, K. Mameda and X. G. Huang, Phys. Rev. D 99 (2019) 085014
N. Weickgenannt, X. L. Sheng, E. Speranza, Q. Wang and D. H. Rischke, Phys. Rev. D 100 (2019) 056018

$$S^\mu_\xi(p) = -\frac{1}{4m}\epsilon^{\mu\nu\sigma\tau}\frac{p_\tau p^\rho}{\varepsilon}\frac{\int_\Sigma {\rm d}\Sigma\cdot p\; n_F(1-n_F)\hat{t}_\nu\xi_{\sigma\rho}}{\int_\Sigma {\rm d}\Sigma\cdot p\; n_F},$$

$$\xi_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\beta_\nu + \partial_\nu\beta_\mu\right).$$

F. B., M. Buzzegoli, A. Palermo, Phys. Lett. B 820 (2021) 136519
S. Liu, Y. Yin, JHEP 07 (2021) 188
Confirmed by C. Yi, S. Pu, D. L. Yang, Phys.Rev.C 104 (2021) 6, 064901
Y. C. Liu, X. G. Huang, Sci.China Phys.Mech.Astron. 65 (2022) 7, 272011

Why do we have a dependence on Σ ?

$$\widehat{J}_x^{\mu\nu} = \int \mathrm{d}\Sigma_\lambda (y-x)^\mu \widehat{T}_B^{\lambda\nu}(y) - (y-x)^\nu \widehat{T}_B^{\lambda\mu}(y)$$

$$\widehat{Q}_x^{\mu\nu} = \int_{\Sigma_{FO}} \mathrm{d}\Sigma_\lambda (y-x)^\mu \widehat{T}_B^{\lambda\nu}(y) + (y-x)^\nu \widehat{T}_B^{\lambda\mu}(y)$$

The divergence of the integrand of $J^{I\ K}$ vanishes, therefore it does not depend on the integration hypersurface (it is a constant of motion) and

$$\widehat{\Lambda} \widehat{J}_x^{\mu\nu} \widehat{\Lambda}^{-1} = \Lambda_\alpha^{-1\mu} \Lambda_\beta^{-1\nu} \widehat{J}_x^{\alpha\beta}$$

The divergence of the integrand of $Q^{I\ K}$ does not vanish, therefore it does depend on the integration hypersurface and

$$\widehat{\Lambda} \widehat{Q}^{\mu\nu} \widehat{\Lambda}^{-1} \neq \Lambda_\alpha^{-1\mu} \Lambda_\beta^{-1\nu} \widehat{Q}^{\alpha\beta}$$

Second order terms in LE expansion

X. L. Sheng, F. B., X. G. Huang, Z. H. Zhang, arXiv:2407.12130

Quadratic in the gradients

$$\widehat{A}_x = -\beta(x) \cdot \widehat{P}$$
$$\widehat{B}_x = \frac{1}{2}\varpi(x) : \widehat{J}_x + \frac{1}{2}\xi : \widehat{Q}_x$$

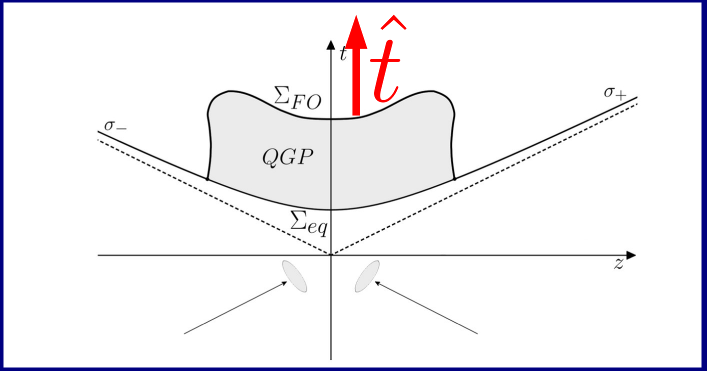
$$\begin{aligned} e^{\widehat{A}_x+\widehat{B}_x} \approx\ & e^{\widehat{A}_x} + \int_0^1 dz\, e^{z\widehat{A}_x}\widehat{B}_x e^{-z\widehat{A}_x} \\ & + \int_0^1 dz_1 \int_0^{z_1} dz_2\, e^{z_1\widehat{A}_x}\widehat{B}_x e^{(z_1-z_2)\widehat{A}_x}\widehat{B}_x e^{-z_1\widehat{A}_x} + \mathcal{O}(\widehat{B}^3)\,. \end{aligned}$$

Linear in the second-order derivatives

$$\widehat{\rho}_{\rm LE} \simeq \frac{1}{Z}\exp\left[-\beta_\mu(x)\widehat{P}^\mu + \frac{1}{2}\varpi_{\mu\nu}(x)\widehat{J}_x^{\mu\nu} - \frac{1}{2}\xi_{\mu\nu}(x)\widehat{Q}_x^{\mu\nu} + \partial_\lambda\partial_\mu\beta^\nu(x)\widehat{\Theta}_x^{\lambda\mu\nu} + \ldots\right]$$

Linear terms

Calculation with canonical stress-energy tensor and spin potential



$$\widehat{\rho}_{\rm LE} = \frac{1}{Z}\exp\left[-\int_{\Sigma_{FO}} {\rm d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}_C \beta_\nu - \zeta \widehat{j}^\mu(y) + \frac{1}{2}\Omega_{\lambda\nu}\widehat{\mathcal{S}}^{\mu\lambda\nu}\right)\right]$$

$$S^{(1)\mu}(p) = -\frac{1}{8mN}\int {\rm d}\Sigma\cdot p_+ \, n_F(x,p)\left[1-n_F(x,p)\right] \times \left\{\epsilon^{\mu\nu\lambda\sigma}\Omega_{\nu\lambda}p_\sigma - \frac{2}{p\cdot\hat{t}}\hat{t}_\nu \epsilon^{\mu\nu\lambda\sigma}p_\lambda\left[\left(\xi_{\sigma\rho}+\Omega_{\sigma\rho}-\varpi_{\sigma\rho}\right)p^\rho - \partial_\sigma\zeta\right]\right\},$$

Confirms previous findings, with an extension to the canonical spin potential (M. Buzzegoli PRC 105 (2022) 4, 044907)

$$S^{(2)\mu}_{\rm lin}(p) = \frac{1}{4m(p\cdot\hat{t})^2N}\int {\rm d}\Sigma\cdot p_+ \, n_F(x,p)\left[1-n_F(x,p)\right](y_\Sigma(0)-x)\cdot\hat{t} \times \hat{t}_\alpha p_\rho\left[\epsilon^{\mu\sigma\alpha\rho}p^\lambda p^\nu \partial_\sigma \xi_{\nu\lambda} + \left(\frac{1}{2}p^\alpha \epsilon^{\mu\nu\lambda\rho} - \epsilon^{\mu\alpha\lambda\rho}p^\nu\right)p^\sigma\partial_\sigma\varpi_{\nu\lambda} - \epsilon^{\mu\sigma\alpha\rho}p^\lambda\partial_\sigma\partial_\lambda\zeta + \frac{1}{2}\epsilon^{\alpha\nu\lambda\sigma}\partial^\rho(\Omega_{\nu\lambda}-\varpi_{\nu\lambda})(p^\mu p_\sigma - m^2 g^\mu_\sigma)\right],$$

NEW!

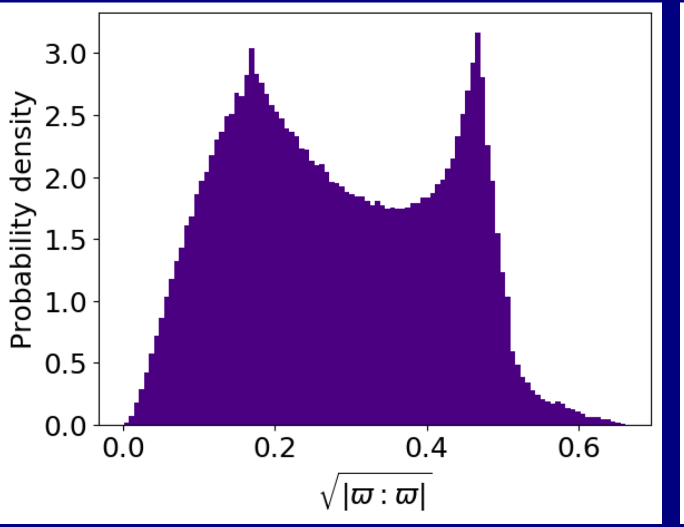
Quadratic terms

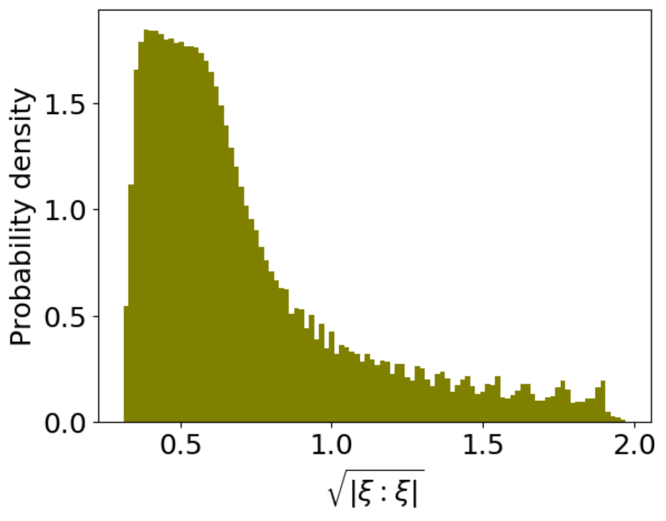
$$S^{(2)\mu}_{\rm quad}(p) = \frac{1}{2}\frac{\int \mathrm{d}\Sigma\cdot p_+ \,\mathrm{tr}\left[\gamma^\mu\gamma^5 W_{\rm quad}(x,p)\right]}{\int \mathrm{d}\Sigma\cdot p_+ \,\mathrm{tr}[W^{(0)}(x,p)]} - S^{(1)\mu}(p)\frac{\int \mathrm{d}\Sigma\cdot p_+ \,\mathrm{tr}[W^{(1)}(x,p)]}{\int \mathrm{d}\Sigma\cdot p_+ \,\mathrm{tr}[W^{(0)}(x,p)]}\,,$$

$$\mathrm{tr}\left[\gamma^\mu\gamma^5 W_{\rm quad}(x,p)\right] = \frac{[1-2n_F(x,p)]\,\mathrm{tr}\left[\gamma^\mu\gamma^5 W^{(1)}(x,p)\right]\mathrm{tr}\left[W^{(1)}(x,p)\right]}{[1-n_F(x,p)]\,\mathrm{tr}\left[W^{(0)}(x,p)\right]}\,.$$

$$W_0(x,p) = \frac{\delta(p^2-m^2)\mathrm{sgn}(p^0)}{(2\pi)^3}(\not p + m)n_F(x,p)\,,$$

$$\mathrm{tr}\left[W^{(1)}(x,p)\right] \;=\; -\frac{\delta(p^2-m^2)}{(2\pi)^3|p^0|}n_F(x,p)\left[1-n_F(x,p)\right](y^0_\Sigma - x^0)4mp^\lambda\partial_\lambda[p^\sigma\beta_\sigma(x)-\zeta(x)]\,,$$





Expectation: should be small

Amplitude distribution (one entry per cell) at the freeze-out

Exact formulae at global equilibrium

Resummation of the power series expansion in (constant) thermal vorticity
A. Palermo, F.B., Eur. Phys. J. Plus 138 (2023) 6, 547

Global equilibrium density operator

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-b_\mu \widehat{P}^\mu + \frac{1}{2}\varpi_{\mu\nu}\widehat{J}^{\mu\nu}\right]$$

$$\langle \widehat{O} \rangle = \mathrm{Tr}\left(\widehat{\rho}\widehat{O}\right)$$

$$\beta^\mu(x) = b^\mu + \varpi^{\mu\nu}x_\nu \equiv \frac{u^\mu}{T}$$

1) Analytic continuation to imaginary thermal vorticity

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-b_\mu \widehat{P}^\mu - \frac{i}{2}\phi_{\mu\nu}\widehat{J}^{\mu\nu}\right]$$

3) Factorization of the density operator

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-\widetilde{b}_\mu(\phi)\widehat{P}^\mu\right] \exp\left[-i\frac{\phi_{\mu\nu}}{2}\widehat{J}^{\mu\nu}\right] \equiv \frac{1}{Z} \exp\left[-\widetilde{b}_\mu(\phi)\widehat{P}^\mu\right] \widehat{\Lambda}$$

$$\widetilde{b}^\mu(\phi) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \underbrace{(\phi^\mu{}_{\alpha_1}\phi^{\alpha_1}{}_{\alpha_2} \ldots \phi^{\alpha_{k-1}}{}_{\alpha_k})}_{k \text{ times}} b^{\alpha_k}$$

4) TEV of creation/annihilation quadratic combination obtained by iterations

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle = 2\varepsilon' \sum_{n=1}^{\infty} (-1)^{2S(n+1)} \delta^3(\Lambda^n \boldsymbol{p} - \boldsymbol{p}') D^S(W(\Lambda^n, p))_{ts} \mathrm{e}^{-\widetilde{b}\cdot\sum_{k=1}^{n}\Lambda^k p}$$

$$W(\Lambda, p) = [\Lambda p]^{-1}\Lambda[p]$$

5) Calculate the Wigner function

$$W(x,k) = \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3 p}{2\varepsilon} \sum_{n=1}^{\infty} (-1)^{n+1} \mathrm{e}^{-n\tilde{\beta}(in\phi)\cdot p} \times$$
$$\left[\mathrm{e}^{-in\frac{\phi:\Sigma}{2}}(m + \not{p})\delta^4\left(k - (\Lambda^n p + p)/2\right) + (m - \not{p})\mathrm{e}^{in\frac{\phi:\Sigma}{2}}\delta^4\left(k + (\Lambda^n p + p)/2\right)\right]$$

Spin vector

$$S^\mu(p) = \frac{1}{2}\frac{\int \mathrm{d}\Sigma\cdot p\ \mathrm{tr}\left(\gamma^\mu\gamma_5 W_+(x,p)\right)}{\int \mathrm{d}\Sigma\cdot p\ \mathrm{tr}\left(W_+(x,p)\right)}$$

$$S^\mu(p) = \frac{1}{2m}\frac{\sum_{n=1}^\infty (-1)^{n+1}\mathrm{e}^{-n\widetilde{b}(in\phi)\cdot p}\mathrm{tr}\left(\gamma^\mu\gamma_5\mathrm{e}^{-in\frac{\phi:\Sigma}{2}}\not{p}\right)\delta^3(\Lambda^n p - p)}{\sum_{n=1}^\infty (-1)^{n+1}\mathrm{e}^{-n\widetilde{b}(in\phi)\cdot p}\mathrm{tr}\left(\mathrm{e}^{-in\frac{\phi:\Sigma}{2}}\right)\delta^3(\Lambda^n p - p)}$$

The series can be resummed:

$$S^\mu(p) = \frac{-i\xi^\mu}{2\sqrt{-\xi^2}}\frac{\sin\left(\sqrt{-\xi^2}/2\right)}{\cos\left(\sqrt{-\xi^2}/2\right)+\mathrm{e}^{-b\cdot p+\zeta}}$$

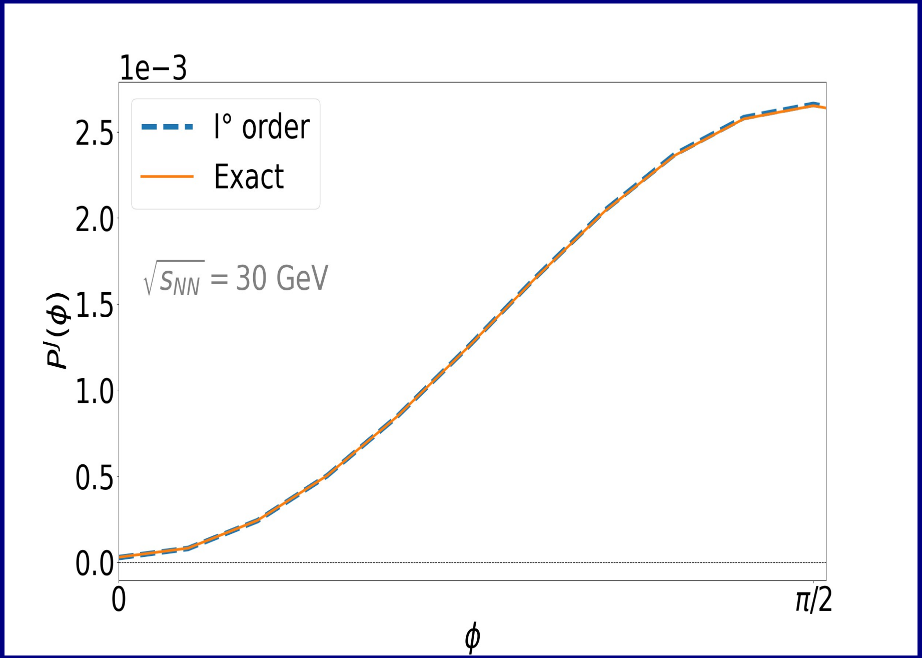
and analytically continued

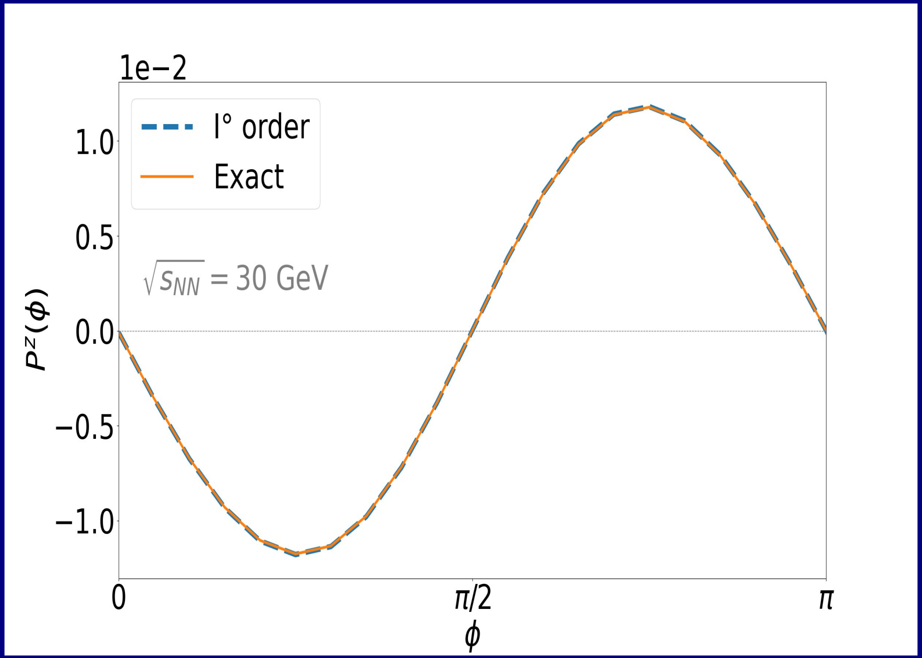
$$S^\mu(p) = \frac{1}{2}\frac{\theta^\mu}{\sqrt{-\theta^2}}\frac{\sinh\left(\frac{\sqrt{-\theta^2}}{2}\right)}{\cosh\left(\frac{\sqrt{-\theta^2}}{2}\right)+e^{-b\cdot p+\zeta}}$$

$$\xi^\mu \mapsto \theta^\mu = -\frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}\varpi_{\nu\rho}p_\sigma$$

Extending the formula to local equilibrium with ϖ(x)

$$S^\mu(p) = -\frac{1}{4m}\epsilon^{\mu\nu\rho\sigma}p_\sigma \frac{\int \mathrm{d}\Sigma\cdot p\, n_F \frac{\varpi_{\nu\rho}}{\sqrt{-\theta^2}} \frac{\sinh\left(\sqrt{-\theta^2}/2\right)}{\cosh\left(\sqrt{-\theta^2}/2\right)+\mathrm{e}^{-b\cdot p+\zeta}}}{\int \mathrm{d}\Sigma\cdot p\, n_F}$$





Summary and outlook

- Calculation of spin polarization corrections at local equilibrium (non-dissipative contribution) at the second-order in the gradient expansion

- Resummation of thermal vorticity series at all orders

- Next step: numerical implementation in hydrodynamic simulation to check their magnitude with respect to 1^{st} order terms