Chiral restoration driven spin polarization

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S. Bhadury, A. Das, W. Florkowski, Gowthama K. K., R. R., *Phys. Lett.* B 849 (2024) 138464

Features of non-central collisions :

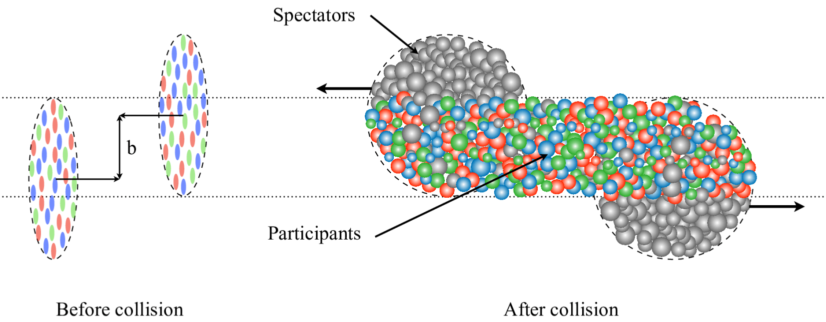

Figure 1: Heavy-ion collision experiments. [LHC Collaboration, JINST 17 (2022) 05, P05009]

- Special feature of Non-Central Collisions :
  - Large Magnetic Field. [A. Bzdak and, V. Skokov, Phys. Lett. B 710 (2012) 171–174]
  - Large Angular Momentum. [F. Becattini et. al. Phys. Rev. C 77 (2008) 204906]
  - Particle polarization at small $\sqrt{S_{NN}}$. [STAR Collaboration, Nature 548 62-65, 2017]

Particle polarization :

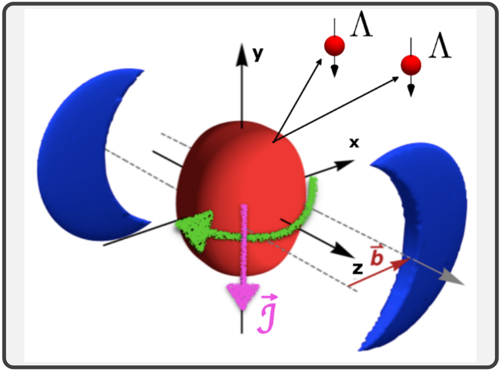

Figure 2: Origin of particle polarization. [W. Florkowski *et al*, PPNP 108 (2019) 103709]

- Large orbital angular momentum → local vorticity → spin alignment

  [Z.-T. Liang and X.-N. Wang, Phys. Rev. Lett. 94, 102301 (2005); Phys. Lett. B 629, 20 (2005)]

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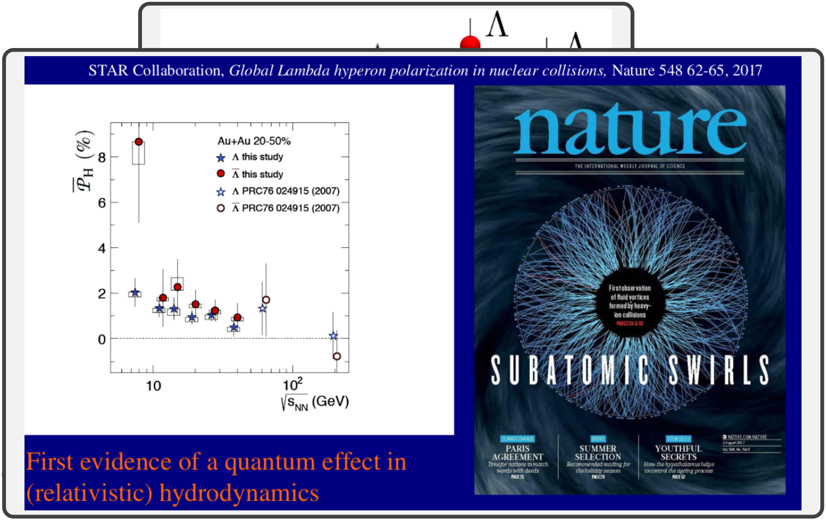

Experimental evidence, [STAR Collaboration, Nature 548, 62 (2017), Phys. Rev. Lett. 123, 132301 (2019), Phys. Rev. Lett. 126, 162301 (2021)]

- Theoretical models assuming equilibration of spin d.o.f. explain this data.

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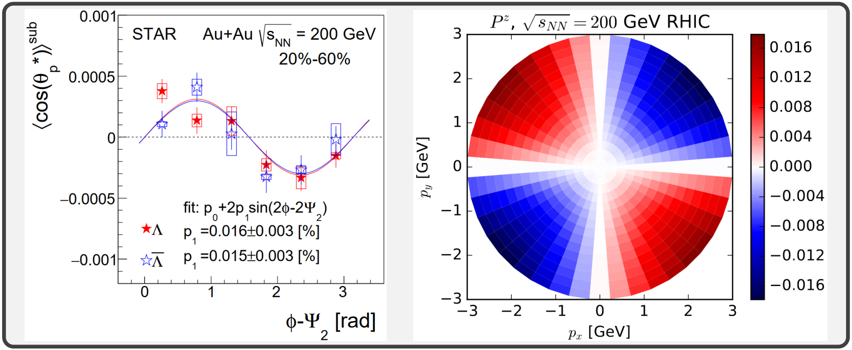

Figure 3: Observation (L) and prediction (R) of longitudinal polarization.
[Left: Phys. Rev. Lett. **123** 132301 (2019); Right: Phys. Rev. Lett. **120** 012302 (2018)]

- Theoretical models assuming equilibration of spin d.o.f. predict the opposite sign.

Recent developments :

- Non-local collisions have been considered.

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◦ We would like to examine the effect of mean scalar field.

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$$\boxed{\text{QFT} \xrightarrow{\text{WF}} \text{Kinetic Equation} \xrightarrow{\int_{\text{p}}} \text{Macroscopic theory.}}$$

NJL model in the mean field approximation :

- Let us consider the Lagrangian of the Nambu-Jona-Lasinio (NJL) type

[W. Florkowski, J. Hufner , S.P. Klevansky, L. Neise, Annals Phys. **245** 445-463 (1996)]

$$\mathscr{L} = \bar{\psi}\left(i\,\partial\!\!\!/ - m_0\right)\psi + G\Big[\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi}i\gamma_5\psi\right)^2\Big].$$

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- This leads to the equation of motion (we assume $m_0 = 0$)

$$\Big[i\,\partial\!\!\!/ - \sigma(x) - i\gamma_5\,\pi(x)\Big]\psi = 0,$$

where we restrict ourselves to the mean field approximation

$$\sigma = \langle\hat{\sigma}\rangle = -2G\,\left\langle\bar{\psi}\psi\right\rangle, \qquad \pi = \langle\hat{\pi}\rangle = -2G\,\left\langle\bar{\psi}\,i\gamma_5\psi\right\rangle.$$

Transport equation for the Wigner function :

- The Wigner function is defined as

$$\mathscr{W}_{\alpha\beta}(x,k) \equiv \int d^4y\, e^{ik\cdot y}\, G_{\alpha\beta}\left(x+\frac{y}{2}, x-\frac{y}{2}\right)$$

where $G_{\alpha\beta}(x,y) = \langle \bar{\psi}_\beta(y)\psi_\alpha(x) \rangle$.

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- The kinetic equation satisfied by the Wigner function is

$$\left[K^\mu\gamma_\mu - \sigma + \frac{i\hbar}{2}\left(\partial_\mu\sigma\right)\partial_k^\mu - i\gamma_5\pi - \frac{\hbar}{2}\gamma_5\left(\partial_\mu\pi\right)\partial_k^\mu\right]\mathscr{W}(x,k) = 0.$$

where $K^\mu = k^\mu + \frac{i\hbar}{2}\partial^\mu$.

Clifford-algebra decomposition :

- We can decompose the Wigner function in the Clifford-algebra basis as

$$\mathscr{W} = \mathscr{F} + i\gamma_5\mathscr{P} + \gamma_\mu\mathscr{V}^\mu + \gamma^\mu\gamma_5\mathscr{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathscr{S}_{\mu\nu},$$

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- The quantum kinetic equations for the components resulting from the kinetic equation for the Wigner function are

$$K^\mu\mathscr{V}_\mu - \sigma\mathscr{F} + \pi\mathscr{P} = -\frac{i\hbar}{2}\Big[(\partial_\nu\sigma)(\partial_k^\nu\mathscr{F}) - (\partial_\nu\pi)(\partial_k^\nu\mathscr{P})\Big]$$

$$-iK^\mu\mathscr{A}_\mu - \sigma\mathscr{P} - \pi\mathscr{F} = -\frac{i\hbar}{2}\Big[(\partial_\nu\sigma)(\partial_k^\nu\mathscr{P}) + (\partial_\nu\pi)(\partial_k^\nu\mathscr{F})\Big]$$

$$K_\mu\mathscr{F} + iK^\nu\mathscr{S}_{\nu\mu} - \sigma\mathscr{V}_\mu + i\pi\mathscr{A}_\mu = -\frac{i\hbar}{2}\Big[(\partial_\nu\sigma)(\partial_k^\nu\mathscr{V}_\mu) - (\partial_\nu\pi)(\partial_k^\nu\mathscr{A}_\mu)\Big]$$

$$iK^\mu\mathscr{P} - K_\nu\tilde{\mathscr{S}}^{\nu\mu} - \sigma\mathscr{A}^\mu + i\pi\mathscr{V}^\mu = -\frac{i\hbar}{2}\Big[(\partial_\nu\sigma)(\partial_k^\nu\mathscr{A}^\mu) - (\partial_\nu\pi)(\partial_k^\nu\mathscr{V}^\mu)\Big]$$

$$2iK^{[\mu}\mathscr{V}^{\nu]} - \varepsilon^{\mu\nu\alpha\beta}K_\alpha\mathscr{A}_\beta - \pi\tilde{\mathscr{S}}^{\mu\nu} + \sigma\mathscr{S}^{\mu\nu} = \frac{i\hbar}{2}\Big[(\partial_\gamma\sigma)(\partial_k^\gamma\mathscr{S}^{\mu\nu}) - (\partial_\gamma\pi)(\partial_k^\gamma\tilde{\mathscr{S}}^{\mu\nu})\Big]$$

where $\tilde{\mathscr{S}}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\mathscr{S}_{\alpha\beta}$.

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- In the classical limit the spin dynamics is described by the behavior of the axial current density $\mathcal{A}^{\mu}_{(0)}$ whose evolution equation is determined by considering the transport equations up to the first order in $\hbar$.

Kinetic equation for axial current

- In the following we will set $\pi = 0$ and $\sigma_{(0)}(x) = M(x)$.
- $M(x)$ is the in-medium mass of particles, which is treated as externally given and plays the role of a background scalar field.

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$$k^\alpha \left(\partial_\alpha \mathcal{A}^\mu\right) + M \left(\partial_\alpha M\right) \left(\partial^\alpha_{(k)} \mathcal{A}^\mu\right) + \left(\partial_\alpha \ln M\right) \left(k^\mu \mathcal{A}^\alpha - k^\alpha \mathcal{A}^\mu\right) = 0.$$

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- In the leading order of the semiclassical expansion, one can use the ansatz

[S. Bhadury, W. Florkowski, A. Jaiswal, A. Kumar, R. R., Phys. Lett. B **814** (2021) 136096]

$$\mathcal{A}^\mu(x,k) = 2M \int dP\, dS\, s^\mu \left[f^+(x,p,s)\delta^{(4)}(k-p) + f^-(x,p,s)\delta^{(4)}(k+p)\right].$$

where $f^\pm(x,p,s)$ are the distribution functions for particles $(+)$ and antiparticles $(-)$ in the extended phase-space of position x, on-shell momentum $p^\mu = (p^0, \mathbf{p}) \quad (p^2 = M^2(x))$, and spin $s^\mu = (s^0, \mathbf{s})$.
- Integration measures are: $dP = \frac{d^3p}{E_p}$, $dS = \left(\frac{M}{\pi \mathfrak{s}}\right) d^4s\, \delta(s\cdot s + \mathfrak{s}^2)\, \delta(p\cdot s)$.

- This ansatz satisfies the condition: $k \cdot \mathcal{A}(x,k) = 0$.

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- $S^{\lambda\mu\nu}(x)$ and $S^{\lambda\mu\nu}_{\text{can}}(x)$ are related by

[W. Florkowski, A. Kumar, R.R., Prog. Part. Nucl. Phys. **108** 103709 (2019)]

$$S^{\lambda\mu\nu}_{\text{can}} = S^{\lambda,\mu\nu} + S^{\mu,\nu\lambda} + S^{\nu,\lambda\mu}. \qquad (\star)$$

Evolution of Spin Tensor :

- Recall the kinetic equation for the axial current

$$k^\alpha \left(\partial_\alpha \mathscr{A}^\mu\right) + M \left(\partial_\alpha M\right) \left(\partial^\alpha_{(k)} \mathscr{A}^\mu\right) + \left(\partial_\alpha \ln M\right) \left(k^\mu \mathscr{A}^\alpha - k^\alpha \mathscr{A}^\mu\right) = 0.$$

- Multiplying this by $k_\beta \varepsilon_\mu^{\ \beta\gamma\delta}$ and integrating over k we get evolution equation for the GLW spin tensor

[S. Bhadury, A. Das, W. Florkowski, Gowthama K. K., R. R., Phys. Lett. B 849 (2024) 138464]

$$\boxed{\partial_\alpha S^{\alpha,\gamma\delta} = \left(\partial_\alpha \ln M\right) \left(S^{\gamma,\delta\alpha} - S^{\delta,\gamma\alpha}\right)} \neq 0$$

- As expected, the spin tensor is conserved when M is constant.
- However, if M varies the spin tensor is sourced through its derivative.

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- Using the results for the semiclassical expansion of the Wigner function (for spin-1/2 particles) one finds

  [W. Florkowski, A. Kumar, R. Ryblewski, Phys. Rev. C 98 (4) (2018) 044906]

$$M\partial_\lambda S^{\lambda,\mu\nu}_{\text{can}} = \partial_\lambda \left(MS^{\nu,\lambda\mu}\right) - \partial_\lambda \left(MS^{\mu,\lambda\nu}\right)$$

  which using $\star$ can be shown to reproduce equation for GLW spin tensor.

- Thus our approach is consistent with *conservation of total angular momentum*.

Analytic Solutions

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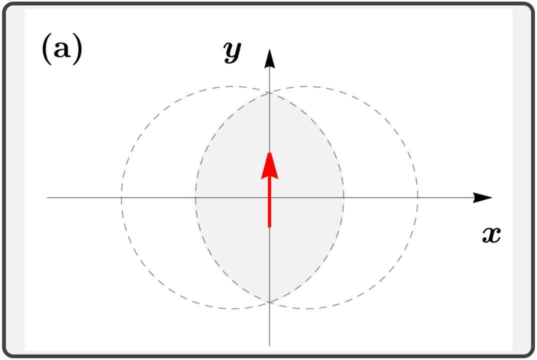
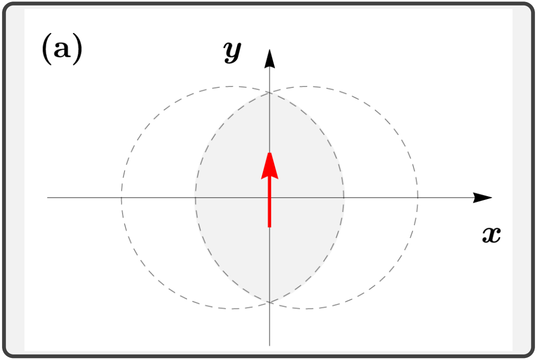


Figure 4: Transverse view of non-central collisions. [SB et. al. PLB 849 (2024) 138464]



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$$f(x, p, s) = g(x, p, s)\delta(p_x)\delta(p_y)$$

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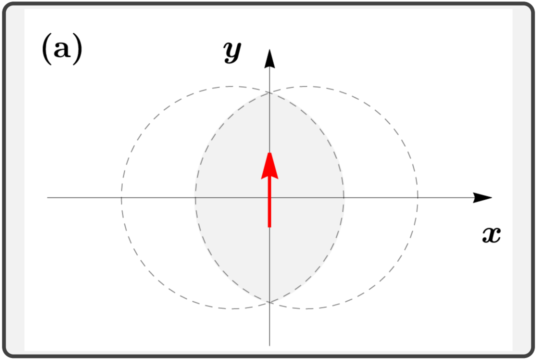

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$$f(x,p,s) = g(x,p,s)\delta(p_x)\delta(p_y)$$

- Hence: $S^{\lambda,\mu\nu}(x) = \int dP\, dS\, p^\lambda s^{\mu\nu} g(x,p,s)\delta(p_x)\delta(p_y) \implies S^{1,\mu\nu} = S^{2,\mu\nu} = 0.$

Analytic Solutions - I (Transverse Polarization):

- Transverse polarization implies:

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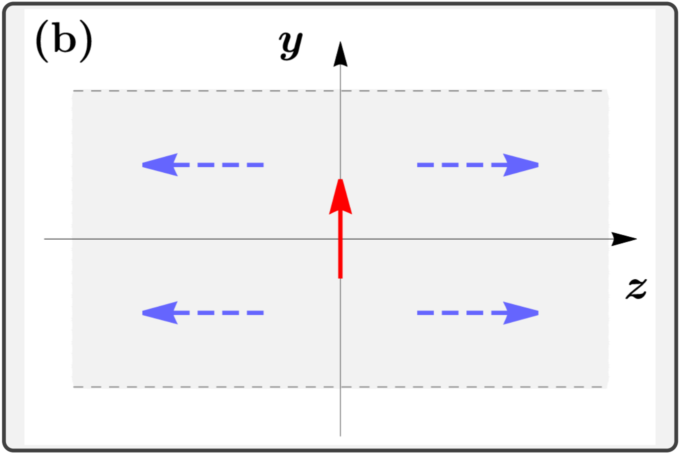

Figure 5: Transverse polarization schematic diagram. [SB et. al. PLB 849 (2024) 138464]

Analytic Solutions - I (Transverse Polarization):

- The spin tensor under transverse polarization becomes :

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- The dynamics of spin is described by:

$$\partial_0 S^{0,01} + \partial_3 S^{3,01} = \frac{\partial_0 M}{M} S^{0,10} + \frac{\partial_3 M}{M} S^{0,13},$$
$$\partial_0 S^{0,31} + \partial_3 S^{3,31} = \frac{\partial_0 M}{M} S^{3,10} + \frac{\partial_3 M}{M} S^{3,13}.$$

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- Let us consider the following vector basis :

$$u^\mu = \begin{pmatrix} \frac{t}{\tau} \\ 0 \\ 0 \\ \frac{z}{\tau} \end{pmatrix}, \quad S_x^\mu = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad S_y^\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad S_z^\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

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- The solution is equivalent to conservation law in Bjorken model.

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- Longitudinal polarization implies:

$$g(x, p, s) = h(x, p, s)\delta(s_x)\delta(s_y)$$

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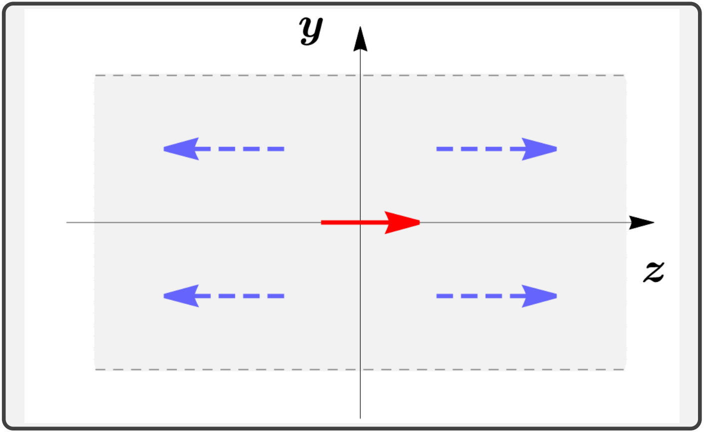

Figure 6: Longitudinal polarization schematic diagram. [SB et. al. PLB 849 (2024) 138464]

Analytic Solutions - II (Longitudinal Polarization):

- The spin tensor under longitudinal polarization becomes :

$$S^{\lambda,\mu\nu}(x) = \int dP\, dS\, p^{\lambda} s^{\mu\nu} h(x,p,s)\delta(s_x)\delta(s_y)\delta(p_x)\delta(p_y).$$

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- Similar to transverse case, the spin decouples from the gradient of $M(x)$ and we have a similar solution.

Analytic Solutions - III (Beoynd Boost Invariance):

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$$M = M(t), \qquad S^{3,01}(t,z) = \nu\, S^{0,01}(t,z) = \nu\, \sigma(t,z).$$

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$$\left(\frac{\partial}{\partial t} + \nu \frac{\partial}{\partial z}\right) \sigma(t,z) = -\left(\frac{\partial \ln M(t)}{\partial t}\right) \sigma(t,z).$$

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- This solution implies the increase of spin density with decreasing mass, indicating a connection between chiral restoration and spin polarization.

Summary and Outlook :

- Gradients of effective mass can act like a source of spin polarization.
- Spin evolution decouples from the source term in a highly symmetric system.
- By giving up boost-invariance, we find a connection between spin polarization and chiral restoration.

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- Gradients of effective mass can act like a source of spin polarization.
- Spin evolution decouples from the source term in a highly symmetric system.
- By giving up boost-invariance, we find a connection between spin polarization and chiral restoration.
- A self-consistently determined $M(x)$ should be used to study the evolution.
- Consequence of non-zero π should be explored.

Thank you for your attention!

This work was supported in part by NCN grant No. 2018/30/E/ST2/00432.

Other Aspects

- Let us recall the equation of motion,

$$\left[i \, \partial\!\!\!/ - \sigma(x) - i\gamma_5 \, \pi(x) \right] \psi = 0.$$

Chiral Spiral:

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- A particularly interesting case of $\sigma = \phi\,\cos\left(\mathbf{q}\cdot\mathbf{r}\right)$ and, $\pi = \phi\,\sin\left(\mathbf{q}\cdot\mathbf{r}\right)$, known as "*chiral spiral*", can be investigated.