Effective Lagrangian for the macroscopic motion of fermionic matter

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Zubarev statistical operator method-preliminaries

relativistically covariant formulation of the statistical operator:

- assumption 1: spacetime possesses foliation into a family of spacelike hypersurfaces Σ_σ parametrized by "time" σ
- assumption 2: continuous medium (hydrodynamical) approximation is valid
- assumption 3: local thermalization timescale $\Delta\tau \ll \Delta t$ with Δt a characteristic time scale of interest (LTE)
- assumption 4: global thermalization of the physical system of interest (GTE)

The Zubarev statistical operator is constructed from conserved currents which characterize the system macroscopically.

density operator from the maximum entropy principle with constraints:

$$n_\mu(x) Tr(\hat{\rho}\hat{T}^{\mu\nu}(x)) = n_\mu(x) T^{\mu\nu}_{cm}(x),\; n_\mu(x) Tr(\hat{\rho}\hat{j}^\mu(x)) = n_\mu(x) j^\mu_{cm}(x)$$

$\hat{T}^{\mu\nu} \equiv \hat{T}^{\mu\nu}_{BR}$ or $\hat{T}^{\mu\nu} \equiv \hat{T}^{\mu\nu}_{can}$ and include the Lorentz transformation operator $\hat{M}^{\mu\nu\rho}_{can}$

$$\hat{\rho}_{LE} = \frac{1}{Z_{LE}} exp(-\int_{\Sigma_\sigma} d\Sigma_\sigma n_\mu(\hat{T}^{\mu\nu}(x)\beta_\nu(x) - \hat{j}^\mu(x)\zeta(x)))$$

$$Tr(\hat{\rho}_{LE}) = 1, \text{ timelike } \beta_\mu = \beta u_\mu,\; u_\mu u^\mu = 1,\; \zeta = \beta\mu$$

$$n_\mu(x) T^{\mu\nu}_{LE}[\beta,\zeta,u](x) = n_\mu(x) T^{\mu\nu}_{cm}(x),\; n_\mu(x) j^\mu_{LE}[\beta,\zeta,u](x) = n_\mu(x) j^\mu_{cm}(x)$$

$$T^{\mu\nu}_{LE}[\beta,\zeta,u](x) = Tr(\hat{\rho}_{LE}\hat{T}^{\mu\nu}(x)),\;\; j^\mu_{LE}[\beta,\zeta,u](x) = Tr(\hat{\rho}_{LE}\hat{j}^\mu(x))$$

GTE condition:

$$\frac{d\hat{\rho}}{d\sigma} = 0,\ \log\hat{\rho} = -log(Z) - \int d\Sigma_\sigma \beta n^\mu g_{\mu\nu}(\hat{T}^{\nu\rho}u_\rho - \sum_i \mu_i \hat{j}_i^\nu)$$

$$\Rightarrow 0 = \partial_\nu(\hat{T}^{\nu\rho}\beta u_\rho - \sum_i \hat{j}_i^\nu \beta\mu_i) = \hat{T}^{\nu\rho}\partial_\nu(\beta u_\rho) - \sum_i \hat{j}_i^\nu \partial_\nu(\beta\mu_i)$$

(operators vanish at spatial infinity, currents are conserved)
solution:

$$\beta\mu_i = \zeta_i = \mathrm{const.},\ \beta_\rho = \beta u_\rho = b_\rho + \omega_{\rho\sigma}x^\sigma,\ b_\rho, \omega_{\rho\sigma} = \mathrm{const.}$$

A preliminary Zubarev statistical operator constructed out of a set of general currents is projected by the GTE condition onto the subspace of conserved currents (by constraining the current coefficients).

define the charge operators

$$\hat{P}^{\mu} = \int d\Sigma n_{\nu} \hat{T}^{\nu\mu}, \quad \hat{Q}_i = \int d\Sigma n_{\nu} \hat{j}_i^{\nu},$$
$$\hat{M}^{\nu\mu} = \int d\Sigma n_{\rho} (\hat{T}^{\rho\mu} x^{\nu} - \hat{T}^{\rho\nu} x^{\mu}) = \epsilon^{\nu\mu\rho\sigma} \hat{J}^{\rho} u^{\sigma} + \hat{K}^{\mu} u^{\nu} - \hat{K}^{\nu} u_{\mu}$$

introduce the linear velocity, acceleration and vorticity

$$v_{\mu} = \frac{1}{\beta} b_{\mu}, \quad a_{\mu} = \frac{1}{\beta} \omega_{\mu\nu} u^{\nu}, \quad \omega_{\mu} = -\frac{1}{2\beta} \epsilon_{\mu\nu\rho\sigma} u^{\nu} \omega^{\rho\sigma}$$
$$\Leftrightarrow \omega_{\mu\nu} = \beta(\epsilon_{\mu\nu\rho\sigma} \omega^{\rho} u^{\sigma} + a_{\mu} u_{\nu} - a_{\nu} u_{\mu})$$

the Zubarev statistical operator may be written as

$$\hat{\rho} = \frac{1}{Z} e^{-\beta(v_{\mu} \hat{P}^{\mu} + a_{\mu} \hat{K}^{\mu} - \omega_{\mu} \hat{J}^{\mu} - \sum_i \mu_i \hat{Q}_i)}$$

Lagrangian and Hamiltionian comprising macroscopic motion

define a Hamiltonian comprising macroscopic motion in the following way

$$\mathcal{H}^{mm} := n_\mu(\hat{T}^{\mu\nu}u_\nu - \sum_i \hat{j}_i^\mu \mu_i)$$

proceed to determine the corresponding Lagrangian $\mathcal{L}^{mm}$ and the partition function (in Minkowski spacetime) (the superscript will be implicit from now on)

$$\mathcal{Z}[n(x), u(x), \beta(x), \mu_i(x), h] = \int D\bar{\psi} D\psi DA_\mu \, e^{i\int d^4x \, \mathcal{L}(\bar{\psi},\psi,A)}$$

which is related to the statistical partition function by

$$Z[n(x), u(x), \beta(x), \mu_i(x)] = \mathcal{Z}[n(x), u(x), \beta(x), \mu_i(x), -i]$$

$$\mathcal{Z}[n(x), u(x), \beta(x), \mu_i(x), h] = \int D\bar{\psi} D\psi DA_\mu \, e^{i \int d^4x \, \mathcal{L}(\overline{\psi}, \psi, A)}$$

integration region: $\Sigma_0(hyperplane) \to \Sigma_h = \{(h\mathfrak{B}(\vec{x}), \vec{x}) | \vec{x} \in \Sigma\}$
introduce the new variable $\mathfrak{U}^\mu(\vec{x}) = \frac{\beta(0,\vec{x})}{\mathfrak{B}(\vec{x})} u^\mu(0, \vec{x})$
two convenient choices for the scaling function $\mathfrak{B}(\vec{x})$:

- $\mathfrak{B}(\vec{x}) = \beta(0, \vec{x}) \Rightarrow \mathfrak{U}^\mu(\vec{x}) = u^\mu(0, \vec{x})$
- $\mathfrak{B}(\vec{x}) = \beta(0, \vec{x}) u^0(0, \vec{x}) \Rightarrow \mathfrak{U}(\vec{x}) = u^\mu(0, \vec{x}) / u^0(0, \vec{x})$

boundary conditions:

fermions: $\Psi(\mathfrak{B}(\vec{x})h, \vec{x}) = -\Psi(0, \vec{x}), \;\; \overline{\Psi}(\mathfrak{B}(\vec{x})h, \vec{x}) = -\overline{\Psi}(0, \vec{x})$
gauge bosons: $A_\mu(\mathfrak{B}(\vec{x})h, \vec{x}) = A_\mu(0, \vec{x})$

Fermions coupled to non-Abelian gauge bosons 1: QCD

QCD: fermions (quarks) coupled to $SU(3)$ gauge bosons (gluons) in the fundamental representation
→derivation of the effective Lagrangian for general macroscopic motion in GTE for quarks and gluons
Dirac Lagrangian + gauge field Lagrangian:

$$\mathcal{L} = \overline{\Psi}(x)(\frac{i}{2}\gamma^{\mu}\overleftrightarrow{D}_{\mu} - m)\Psi(x) - \frac{1}{2}Tr(F_{\mu\nu}F^{\mu\nu})$$

$$D_{\mu} = \partial_{\mu} - igA_{\mu}, \; F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} - ig[A_{\mu}, A_{\nu}]$$

Dirac field and gauge field BR energy momentum tensors:

$$T_F^{\mu\nu}(x) = \frac{i}{4}\overline{\Psi}(x)(\gamma^{\mu}\overleftrightarrow{D}^{\nu} + \gamma^{\nu}\overleftrightarrow{D}^{\mu})\Psi(x), \; \overleftrightarrow{D}_{\mu} = \overrightarrow{D}_{\mu} - \overleftarrow{D}_{\mu}$$

$$T_B^{\mu\nu}(x) = 2\,Tr(F^{\rho\mu}(x)F^{\nu}{}_{\rho}(x) + \frac{1}{4}g^{\mu\nu}(x)F_{\rho\sigma}(x)F^{\rho\sigma}(x))$$

effective Lagrangian comprising macroscopic motion:

$$\begin{aligned}\mathcal{L}(\overline{\psi},\psi,A) =&\overline{\psi}(\gamma^\mu \frac{i}{2}\overset{\leftrightarrow}{D}_\mu - m)\psi + \sum_i \mu_i j_i^0 \\ &+\mathfrak{U}_k\overline{\psi}\gamma^0(\frac{i}{8}[\gamma^j,\gamma^k](\overset{\leftarrow}{D}_j + D_j) - \frac{i}{2}\overset{\leftrightarrow}{D}^k)\psi \\ &+ \frac{1}{2}\Big(E^{ai}E^{ai} - B^{ai}B^{ai}\Big) \\ &+ \frac{1}{2}(\mathfrak{U}^k\mathfrak{U}_k\delta^{ij} - \mathfrak{U}^i\mathfrak{U}^j)B_i^a B_j^a - \epsilon_{ijk}E_i^a B_j^a \mathfrak{U}^k\end{aligned}$$

$$\mathfrak{B}(\vec{x}) = \beta(t_0,\vec{x})u^0(t_0,\vec{x}) \Leftrightarrow \mathfrak{U}^\mu(\vec{x}) = \frac{u^\mu(t_0,\vec{x})}{u^0(t_0,\vec{x})}$$

$$E_i^a = F^a_{\mu i}n^\mu, \quad B_i^a = -\frac{1}{2}\epsilon_{\mu ijk}F^{ajk}n^\mu, \quad V^0 = V^\mu n_\mu,\ i,j \text{ spacelike}$$

$$
\begin{aligned}
\mathcal{L}(\overline{\psi}, \psi, A) =& (\mathfrak{U} n)\overline{\psi}(\gamma^\mu \frac{i}{2}\overleftrightarrow{D}_\mu - m)\psi \\
&+ (1 - (\mathfrak{U} n))(\overline{\psi}(\gamma n)\frac{i}{2}(n\overleftrightarrow{D})\psi) + \sum_i \mu_i (j_i n) \\
&+ \mathfrak{U}^\mu \Delta_{\mu\rho}\Delta_{\nu\sigma}(\overline{\psi}(\gamma n)\frac{i}{8}[\gamma^\sigma, \gamma^\rho](\overleftarrow{D}^\nu + D^\nu)\psi) \\
&- \mathfrak{U}^\mu \Delta_{\mu\rho}\overline{\psi}(\gamma n)\frac{i}{2}\overleftrightarrow{D}^\rho \psi \\
&- \frac{1}{4\mathfrak{U} n} F^{a\mu\nu} F^a_{\mu\nu} - \frac{\mathfrak{U}^2 - 1}{4\mathfrak{U} n} F^{a\rho\sigma} F^{a\mu\nu} \Delta_{\mu\rho}\Delta_{\nu\sigma} \\
&- \frac{1}{8\mathfrak{U} n}(n^\mu \epsilon_{\mu\nu\rho\sigma} F^{a\rho\sigma}\mathfrak{U}^\nu)(n^{\bar{\mu}} \epsilon_{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}} F^{a\bar{\rho}\bar{\sigma}}\mathfrak{U}^{\bar{\nu}}) \\
&- \frac{1}{\mathfrak{U} n} F^a_{\nu\mu} F^{a\nu\rho}\mathfrak{U}_\rho n^\mu + F^a_{\nu\mu} F^{a\nu\rho} n_\rho n^\mu \\
\mathfrak{U}^\mu(\vec{x}) =& (\beta(0, \vec{x})/\mathfrak{B}(\vec{x})) u^\mu(0, \vec{x}), \quad \Delta_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu
\end{aligned}
$$

Consistency check for the case of rotation

1. without macroscopic motion but relative rotation:

$$S(g^R_{\mu\nu}, A_\mu) = \int d^4x \mathcal{L}(g^R_{\mu\nu}, A_\mu),$$

$$\mathcal{L}(g^R_{\mu\nu}, A_\mu) = -\frac{1}{4} F^a_{\mu\rho} g^{\mu\nu}_R g^{\rho\sigma}_R F^a_{\nu\sigma}$$

2. with rotational macroscopic motion:

$$S^{mm}(\eta_{\mu\nu}, A_\mu, u^\mu) = \int d^4x \mathcal{L}^{mm}(\eta_{\mu\nu}, A_\mu, u^\mu),$$

$$\begin{aligned}\mathcal{L}^{mm}(\eta_{\mu\nu}, A_\mu, u^\mu) = &- \frac{1}{4\mathfrak{U}n} F^{a\mu\nu} F^a_{\mu\nu} - \frac{\mathfrak{U}^2 - 1}{4\mathfrak{U}n} F^{a\rho\sigma} F^{a\mu\nu} \Delta_{\mu\rho} \Delta_{\nu\sigma} \\ &- \frac{1}{8\mathfrak{U}n} (n^\mu \epsilon_{\mu\nu\rho\sigma} F^{a\rho\sigma} \mathfrak{U}^\nu)(n^{\bar\mu} \epsilon_{\bar\mu\bar\nu\bar\rho\bar\sigma} F^{a\bar\rho\bar\sigma} \mathfrak{U}^{\bar\nu}) \\ &- \frac{1}{\mathfrak{U}n} F^a_{\nu\mu} F^{a\nu\rho} \mathfrak{U}_\rho n^\mu + F^a_{\nu\mu} F^{a\nu\rho} n_\rho n^\mu\end{aligned}$$

insert the following background data:

1. without macroscopic motion but relative rotation:

$$\begin{aligned} ds^2 =& g^R_{\mu\nu} dx^\mu dx^\nu \\ =& (1-\omega^2(x^2+y^2))dt^2 + 2\omega y dt dx - 2\omega x dt dy - dx^2 - dy^2 - dz^2 \end{aligned}$$

2. with rotational macroscopic motion:

$$\begin{aligned} ds^2 =& \eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - dx^2 - dy^2 - dz^2 \\ \mathfrak{U}^\mu(\vec{x}) =& \frac{\beta(0,\vec{x})}{\mathfrak{B}(\vec{x})} u^\mu(0,\vec{x}), \quad \mathfrak{B}(\vec{x}) = \beta(0,\vec{x}) u^0(0,\vec{x}) \\ u^\mu =& \frac{1}{\sqrt{1-\omega^2(x^2+y^2)}}(1,-\omega y,\omega x,0) \;\Rightarrow\; \mathfrak{U}^\mu = (1,-\omega y,\omega x,0) \end{aligned}$$

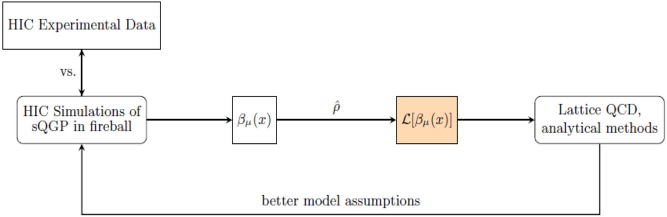
$$\Rightarrow S(g^R_{\mu\nu}, A_\mu) \equiv S^{mm}(\eta_{\mu\nu}, A_\mu, u^\mu)$$

The case of acceleration: boost in the x-direction

$$\begin{aligned} ds^2 =& \eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - dx^2 - dy^2 - dz^2 \\ \mathfrak{U}^\mu(\vec{x}) =& \frac{\beta(0,\vec{x})}{\mathfrak{B}(\vec{x})} u^\mu(0,\vec{x}), \quad \mathfrak{B}(\vec{x}) = \beta(0,\vec{x}) u^0(0,\vec{x}) \\ u^\mu =& \frac{1}{\sqrt{(1+ax)^2 - at^2}} (1+ax, at, 0)\Big|_{t=0} \Rightarrow \mathfrak{U}^\mu = (1,0,0,0) \end{aligned}$$

notice: in the Lagrangian comprising macroscopic motion only the spatial components of $\mathfrak{U}$ enter non-trivially! (for this choice of $\mathfrak{B}$) $\Rightarrow$ the only effect if acceleration is manifested through the scaling function ($\leftrightarrow$ spacetime dependent temperature)

$$\mathfrak{B}(\vec{x}) = \beta(0,\vec{x}) u^0(0,\vec{x}) = \beta(0,\vec{x}) = \beta_0(1+ax)$$



Schematic block diagram for proposed investigation of the strongly coupled quark - gluon plasma ((s)QGP) in Heavy Ion Collisions (HIC).

- optimize all model parameters: compare experimental data from the accelerators with simulations of various phenomenological models of sQGP for fireball dynamics
- consistency check/feedback: extract frigidity vector field $\beta_\mu(x) \rightarrow$ build Zubarev statistical operator $\hat{\rho}$ for the sufficiently small cell in thermal equilibrium and compare predictions with HIC simulations

Fermions coupled to non-Abelian gauge bosons 2: $^3He - A$

consider superfluid $^3He - A$ and repeat the just mentioned steps within the Zubarev statistical operator approach
preliminaries:

1. "normal" liquid ^3He is a Fermi liquid featuring a Fermi surface
2. it undergoes a phase transition (in case of appropriate external conditions - pressure and temperature) by spontaneous symmmetry breaking
3. the superfluid component of $^3He - A$ provides the vacuum background which is coupled to the normal component of excitations
4. difficulty: space and time dependent matrix-valued vierbein, the order parameter of the phase transition, in an emergent relativistic theory of chiral fermions

- starting point: effective Lagrangian of superfluid ^3He-A without macroscopic motion, slowly varying vierbein and thereby small values of the superfluid velocity

^3He without spin orbit interaction ($G=U(1)\times SO(3)^L\times SO(3)^S$):

$$S=\sum_{p,s}\bar{a}_s(p)\epsilon(p)a_s(p)-\frac{g}{\beta V}\sum_{p;i,\alpha=1,2,3}\bar{J}_{i\alpha}(p)J_{i\alpha}(p)$$

$$p=(\omega,k),\hat{k}=\frac{k}{|k|},\ \epsilon(p)=i\omega-(\frac{k^2}{2M_3}-\mu)\approx i\omega-v_F(|k|-k_F),$$

$$J_{i\alpha}(p)=\frac{1}{2}\sum_{p_1+p_2}(\hat{k}_1^i-\hat{k}_2^i)a_A(p_2)[\sigma_\alpha]_B^C a_C(p_1)\epsilon^{AB},\ \ \epsilon^{-+}=-\epsilon^{+-}=1$$

requirement for validity of Taylor expansion around Fermi surface:

$$\frac{(\pm|k| \mp k_F)^2}{2M_3} \ll v_\perp |(\pm|k| \mp k_F)|$$

$\Rightarrow$ typical length scales a and time scales τ:

$$a \sim (|k| - k_F)^{-1} \gg \frac{v_F}{v_\perp k_F}, \quad \tau \gg \frac{1}{v_\perp k_F}$$

- carry out bosonization with complex bosonic fields $A_{i\alpha}$
- apply spontaneous symmetry breaking prescription for $^3He - A$ (specification of the form of $A_{i\alpha}$)

superfluid $^3He - A$-phase:

$$A_{i\alpha} = \sqrt{\beta V}\Delta_0(\mathbf{m}_i - i\mathbf{n}_i)d_\alpha = \sqrt{\beta V} k_F v_\perp(\mathbf{m}_i - i\mathbf{n}_i)d_\alpha, \quad i, \alpha = 1, 2, 3$$

with frame fields

$$\mathbf{d}\cdot\mathbf{d} = \mathbf{m}\cdot\mathbf{m} = \mathbf{n}\cdot\mathbf{n} = 1, \quad \mathbf{m}\cdot\mathbf{n} = 0, \quad \mathbf{l} = \mathbf{m}\times\mathbf{n}$$

introduce Nambu Gorkov spinors:

$$\Psi(p) = \begin{pmatrix} \chi^A(p) \\ \epsilon^{BA}\overline{\chi}^B(-p) \end{pmatrix} = \begin{pmatrix} a_+(p) \\ a_-(p) \\ \overline{a}_-(-p) \\ -\overline{a}_+(-p) \end{pmatrix}$$

- expansion around the two Fermi points yields an effective Lagrangian which is relativistically invariant near the Fermi points $K^i_{R,L} = K^i_{\pm} = \pm k_F l^i$ we define

$$\psi_R(p) = \Psi(K_+ + p) = \begin{pmatrix} \chi(K_+ + p) \\ -\chi^C(K_- - p) \end{pmatrix},$$

$$\psi_L(p) = \tau^3\Psi(K_- + p) = \begin{pmatrix} \chi(K_- + p) \\ \chi^C(K_+ - p) \end{pmatrix},\ \mathbf{e}^\mu_a = \mathbf{e}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & v_\perp \mathbf{m}(\mathbf{d}\boldsymbol{\sigma}) \\ 0 & v_\perp \mathbf{n}(\mathbf{d}\boldsymbol{\sigma}) \\ 0 & v_\parallel \mathbf{l} \end{pmatrix}$$

with $\chi^C = -i\sigma^2\chi^* \Leftrightarrow \psi_R(p) = i\tau^1\sigma^2\psi^*_L(-p)$

$$S_{eff} = \frac{1}{4}\int d^4x\mathbf{e}[\overline{\psi}_L i\mathbf{e}^\mu_b(x)\overline{\tau}^b\nabla_\mu\psi_L - [\nabla_\mu\overline{\psi}_L]i\mathbf{e}^\mu_b(x)\overline{\tau}^b\psi_L + \overline{\psi}_R i\mathbf{e}^\mu_b(x)\tau^b\nabla_\mu\psi_R - [\nabla_\mu\overline{\psi}_R]i\mathbf{e}^\mu_b(x)\tau^b\psi_R]$$

split off the spin space dependence and introduce Dirac spinors:

$$\mathbb{P}^s = \frac{1 + s(\mathbf{d}\boldsymbol{\sigma})}{2} \text{ with eigenspinors } \eta^s = \frac{1}{\sqrt{2(1 - sd_3)}} \begin{pmatrix} -s(d_1 - id_2) \\ sd_3 - 1 \end{pmatrix}$$

$$\psi_{L/R} = \sum_{s=\pm} \psi^s_{L/R}, \quad \mathbb{P}^s \psi_{L/R} = \psi^s_{L/R} = \Psi^s_{L/R} \otimes \eta^s$$

$$\psi^s = (\psi^s_L, \psi^s_R)^T, \quad \Psi^s = (\Psi^s_L, \Psi^s_R)^T, \quad \mathbb{P}^s \psi = \psi^s = \Psi^s \otimes \eta^s$$

$$\Psi = \left(\Psi^+, \quad e^{\frac{\pi}{4}[\gamma^1, \gamma^2]}\Psi^-\right)^T, \quad e^\mu_a = e^{-1} \begin{pmatrix} 1 & 0 \\ 0 & v_\perp \mathbf{m} \\ 0 & v_\perp \mathbf{n} \\ 0 & v_\parallel \mathbf{l} \end{pmatrix}$$

$$S_{eff} = \frac{1}{4}\int d^4xe[\overline{\Psi}i\gamma^0\gamma^b e_b^\mu D_\mu\Psi - [\overline{\Psi}\overleftarrow{D}_\mu]i\gamma^0\gamma^b e_b^\mu\Psi]$$

$$D_\mu = \nabla_\mu - i\mathcal{B}_\mu, \quad \mathcal{B}_\mu^{rs} = i(\overline{\eta}^r\nabla_\mu\eta^s)(\delta^{rs}\mathbb{1} + i\epsilon^{rs}\frac{\pi}{8}[\gamma^1,\gamma^2])$$

$$\mathcal{B}_\mu = \begin{pmatrix} b_\mu^+ & \frac{1}{8}\omega_{\mu12}[\gamma^1,\gamma^2] \\ \frac{1}{8}\omega_{\mu12}^*[\gamma^1,\gamma^2] & b_\mu^- \end{pmatrix}$$

Abelian Berry connections and non-Abelian spin connection:

$$b_\mu^s = i\overline{\eta}^s\nabla_\mu\eta^s, \quad \omega_{\mu12} = 2\pi i\overline{\eta}^+\nabla_\mu\eta^-$$

- construct the Zubarev statistical operator from (conserved) current densities

$$\log\hat{\rho} = -\alpha - \int d\Sigma n_\mu(\hat{T}^\mu_a B^a - \frac{1}{2}\hat{M}^\mu_{ab}\Omega^{ab} - \zeta_A \hat{j}^\mu_A - \sum_i \zeta_i \hat{j}^\mu_i)$$
$$= -\alpha - \int d\Sigma n_\mu(\hat{T}^\mu_a \beta^a - \frac{1}{2}\hat{S}^\mu_{ab}\Omega^{ab} - \zeta_A \hat{j}^\mu_A - \sum_i \zeta_i \hat{j}^\mu_i)$$

for vanishing gauge field $\mathcal{B} \equiv 0$:

$$\beta^\mu = b^\mu + \omega^\mu{}_\nu x^\nu$$
$$\Omega_{\mu\nu} = \omega_{\mu\nu} + e^a_\mu(\nabla_\lambda e_{a\nu})\beta^\lambda = e^a_\mu(\mathcal{L}_\beta e)_{a\nu} = e^a_\mu[\beta^\lambda\nabla_\lambda e_{a\nu} + e_{a\lambda}\nabla_\nu\beta^\lambda]$$
$$\epsilon^{\mu\nu\rho\sigma}(\mathcal{L}_\beta T)_{\nu\rho\sigma} = 0, \quad T_{\mu\nu\rho} = e_{a\mu}T^a_{\nu\rho} = e_{a\mu}(\nabla_\nu e^a_\rho - \nabla_\rho e^a_\nu)$$

additionally for non-vanishing gauge field $\mathcal{B} \not\equiv 0$:

$$\beta^a\hat{G}_a(\mathcal{B}) = 0, \quad \Omega^{ab}\hat{P}_{ab}(\mathcal{B}) = 0$$