Relativistic stochastic advection-diffusion equation using Metropolis

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Thank you for listening!

Outline

Why we need stochastic relativistic hydrodynamics?

Relativistic advection-diffusion equation from density frame

Importance of Metropolis for stochastic dynamics

Standard hydro success

$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$

→ "Standard" hydro model is amazingly successful in large collision systems

$\partial_\mu T^{\mu\nu} = 0$ with $T^{\mu\nu} = e u^\mu u^\nu + P(e)\Delta^{\mu\nu} + \text{dissipation}$

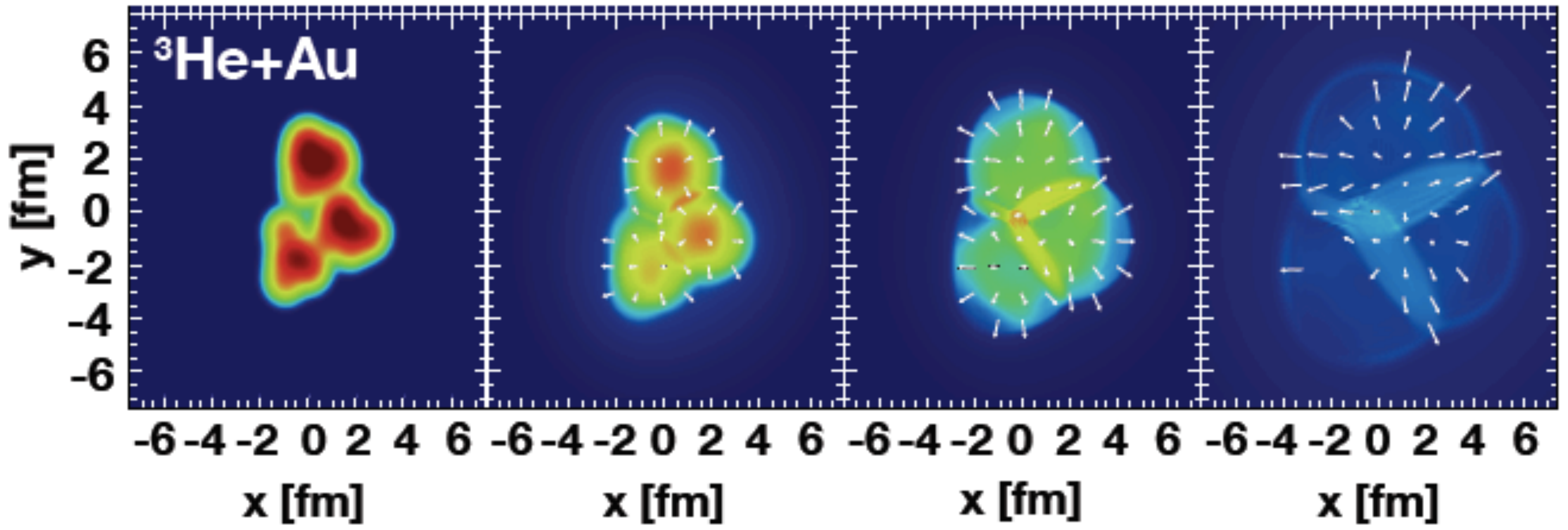
Standard hydro success

$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$

- "Standard" hydro model is amazingly successful in large collision systems

$$\partial_\mu T^{\mu\nu} = 0 \quad \text{with} \quad T^{\mu\nu} = e u^\mu u^\nu + P(e)\Delta^{\mu\nu} + \text{dissipation} + \text{noise}$$

- What about small collision systems: d + Au , He + Au



It is difficult to define flow pattern with small no. of particles

- Collective behavior in small systems

In this case: fluctuations become important

Recap: Stochastic Viscous Hydro

Issues?

$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$

$$T^{\mu\nu} = e u^\mu u^\nu + P(e)\Delta^{\mu\nu} - \eta\sigma^{\mu\nu} + \xi^{\mu\nu}$$

$$\langle \xi^{\mu\nu}(x)\, \xi^{\rho\sigma}(y) \rangle \equiv 2T\eta\delta^4_{xy} \left[\Delta^{\mu\rho}\Delta^{\nu\sigma} + \Delta^{\mu\sigma}\Delta^{\nu\rho} - \frac{2}{d}\Delta^{\mu\nu}\Delta^{\rho\sigma} \right]$$

→ Choice of hydrodynamic frame: $u^\nu T^\mu{}_\nu = - e(T) u^\mu$ to all orders

- Different hydro frames (Landau/Eckart) will give different answers!
- But, will agree at first order after using ideal EoMs

→ Unstable $j^i_D = - D\partial^i n$

- Equations are second order in time with runaway solutions

→ Israel-Stewart formulation

→ BDNK

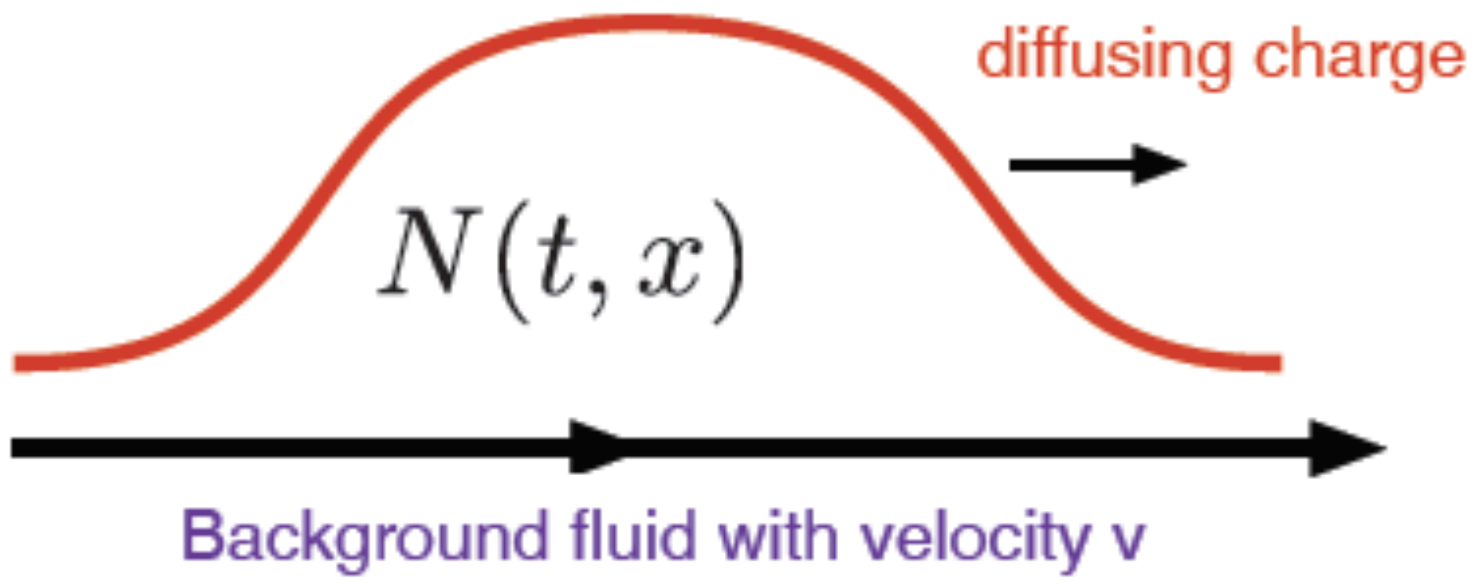
Alternative?

Alternative: Density frame hydro

J. Armas and A. Jain, SciPost Phys. 11 no. 3, (2021) 054

- Hydro without boosts
- Easy to implement Metropolis updates
- No non-hydro modes and no additional parameters
- Density frame EoMs can be derived from Landau ones if ideal EoMs are used to replace lab frame time derivatives
- Equations are first order in time and stable. Numerically easy to implement
- Results obtained in different Lorentz frames will vary, but the variation is beyond the accuracy of the diffusion equation

Density frame & relativistic diffusion



- Consider a charge in a moving fluid which is not Lorentz invariant

$$\partial_t N + \nabla \cdot J = 0$$

- Hydrodynamics will still be valid

$$J^i = \underbrace{N v^i}_{\text{ideal advection}} + \overbrace{J^i_D}^{\text{diffusion}}$$

- The diffusive current is expanded in gradients of the charge

$$J^i_D = -\underbrace{D_{\|}(v)\, \hat{v}^i \hat{v}^j \partial_j N}_{\text{diffusion parallel to v}} - \underbrace{D_{\perp}(v)(\delta^{ij} - \hat{v}^i \hat{v}^j)\partial_j N}_{\text{diffusion perpendicular to v}}$$

See poster by R. Singh to obtain density frame equations from Landau frame and kinetic theory

J. Armas and A. Jain, SciPost Phys. 11 no. 3, (2021) 054

For Lorentz covariant fluid

- In the Landau frame

$$J^\mu = nu^\mu + j_D^\mu \qquad \text{with} \qquad j_D^\mu = -D\Delta^{\mu\nu}\partial_\nu n$$

- Define $N \equiv J^0$ and use lowest order equation of motion $\partial_t n + v^i \partial_i n = 0$

- to rewrite time derivatives, leads to the density frame form

$$J^i = Nv^i + \frac{D}{\gamma^3}\hat{v}^i\hat{v}^j\partial_j N + \frac{D}{\gamma}(\delta^{ij} - \hat{v}^i\hat{v}^j)\partial_j N$$

where

$$D_{||}(v) = \frac{D}{\gamma^3} \qquad D_\perp(v) = \frac{D}{\gamma}$$

diffusion parallel to v — diffusion perpendicular to v

- Form of the advection-diffusion equation in the density frame is $\partial_t N + \partial_i (N v^i) = \partial_i (D^{ij} \partial_j N)$ with $D^{ij} = \frac{D}{\gamma}(\delta^{ij} - v^i v^j)$

- Equations are strictly first order in time and stable

- Each Lorentz observer has his own hydrodynamics frame where $\mathrm{d}S^0 = -\beta^\mu \mathrm{d}T^0{}_\mu + \frac{\mu}{T}\mathrm{d}J^0$

- Chemical potential is defined to all orders by the charge J^0

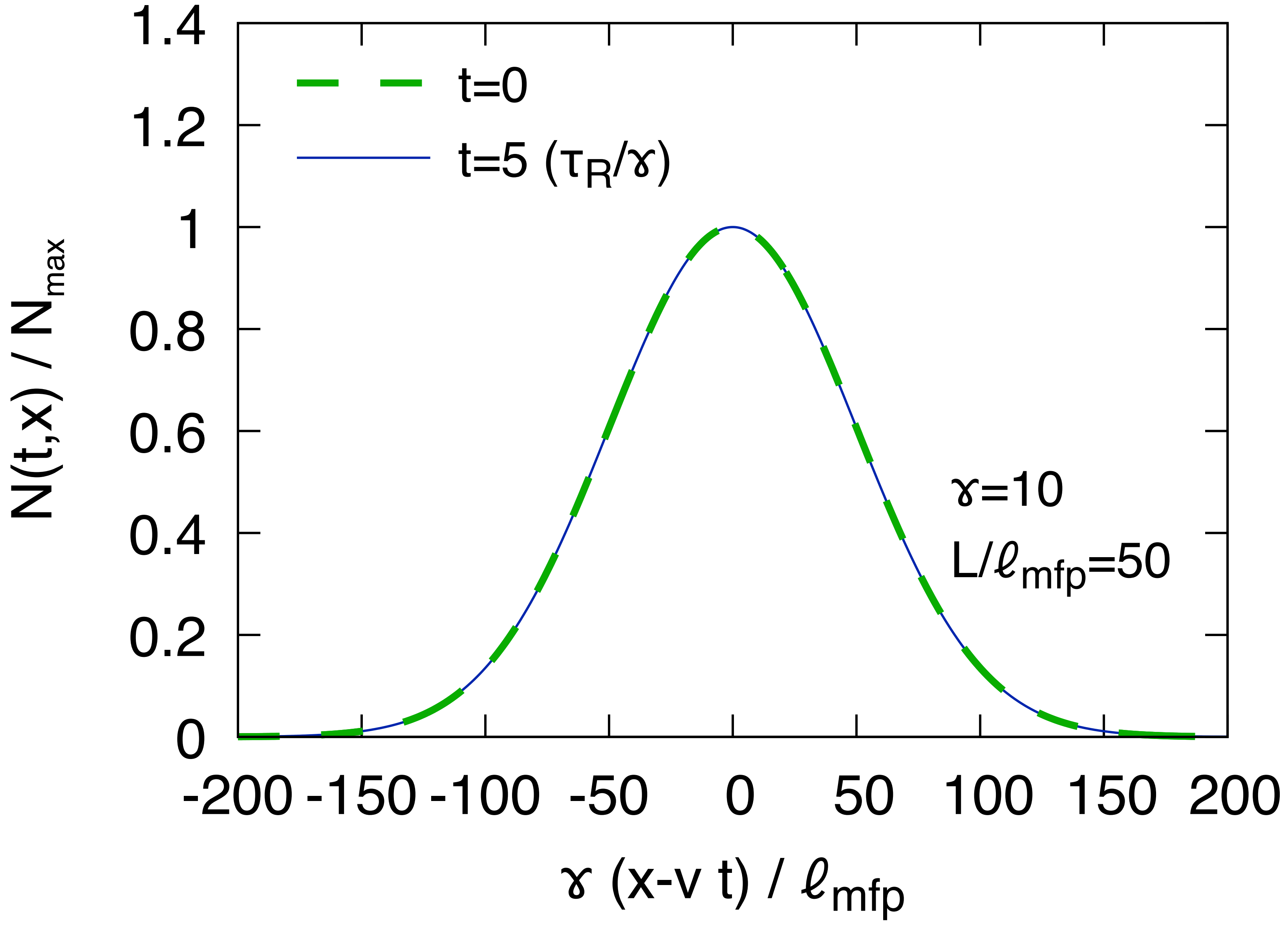
$$\underbrace{\mu = \frac{J^0}{\chi u^0}}_{\text{Density Frame}} \quad \text{vs.} \quad \underbrace{\mu = -\frac{u_\mu J^\mu}{\chi}}_{\text{Landau Frame}}$$

Approach to dissipative steady state

$$J = Nv + J_D$$

Start at $t = 0$ with $N(x)$ as a Gaussian and $J_D = 0$ in lab frame with

$L/l_{\text{mfp}} =$ width in fluid frame in units of $l_{\text{mfp}} = 2\,c\tau_R$



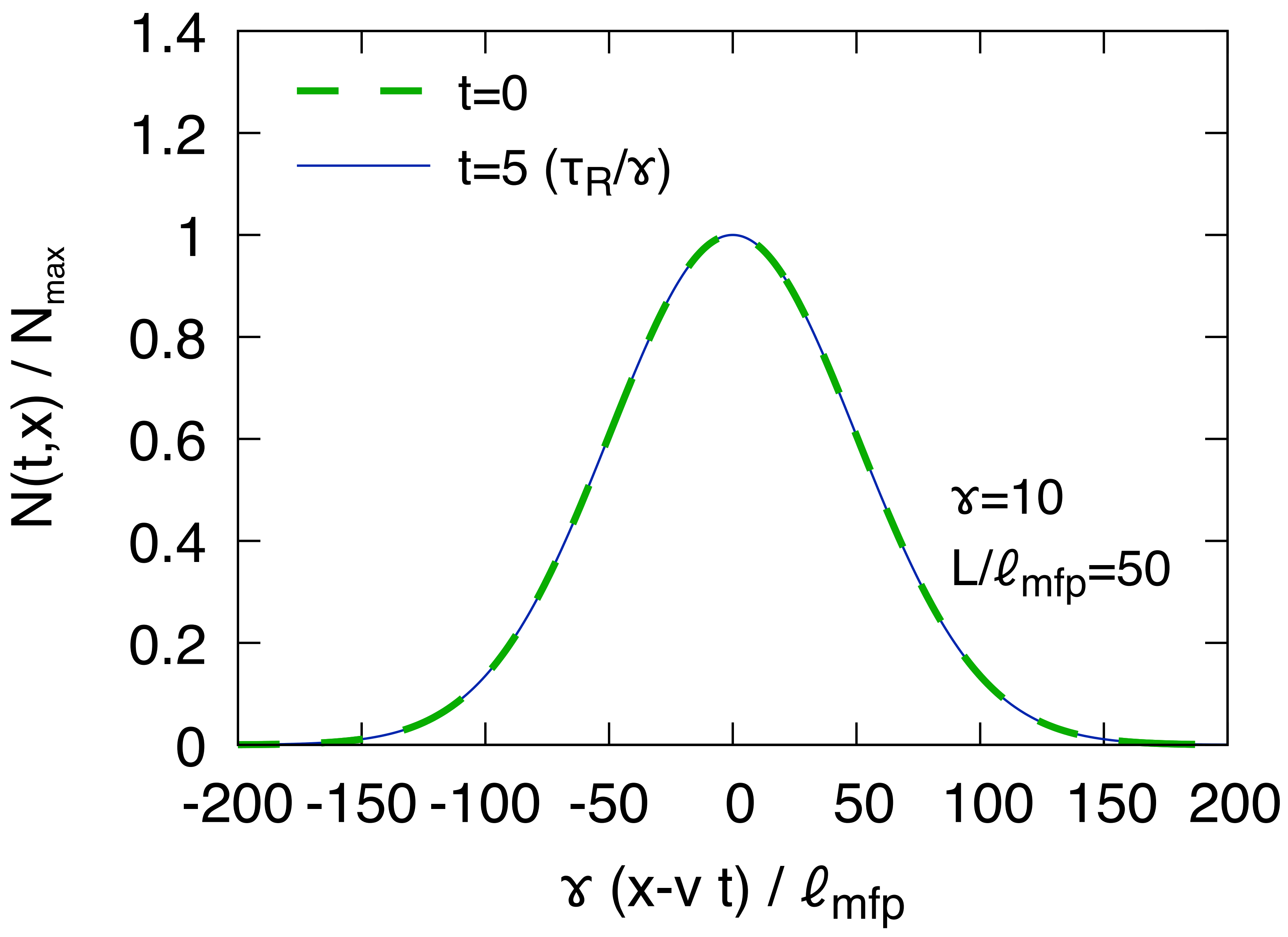
The dissipative steady state is approached on a timescale τ_R/γ

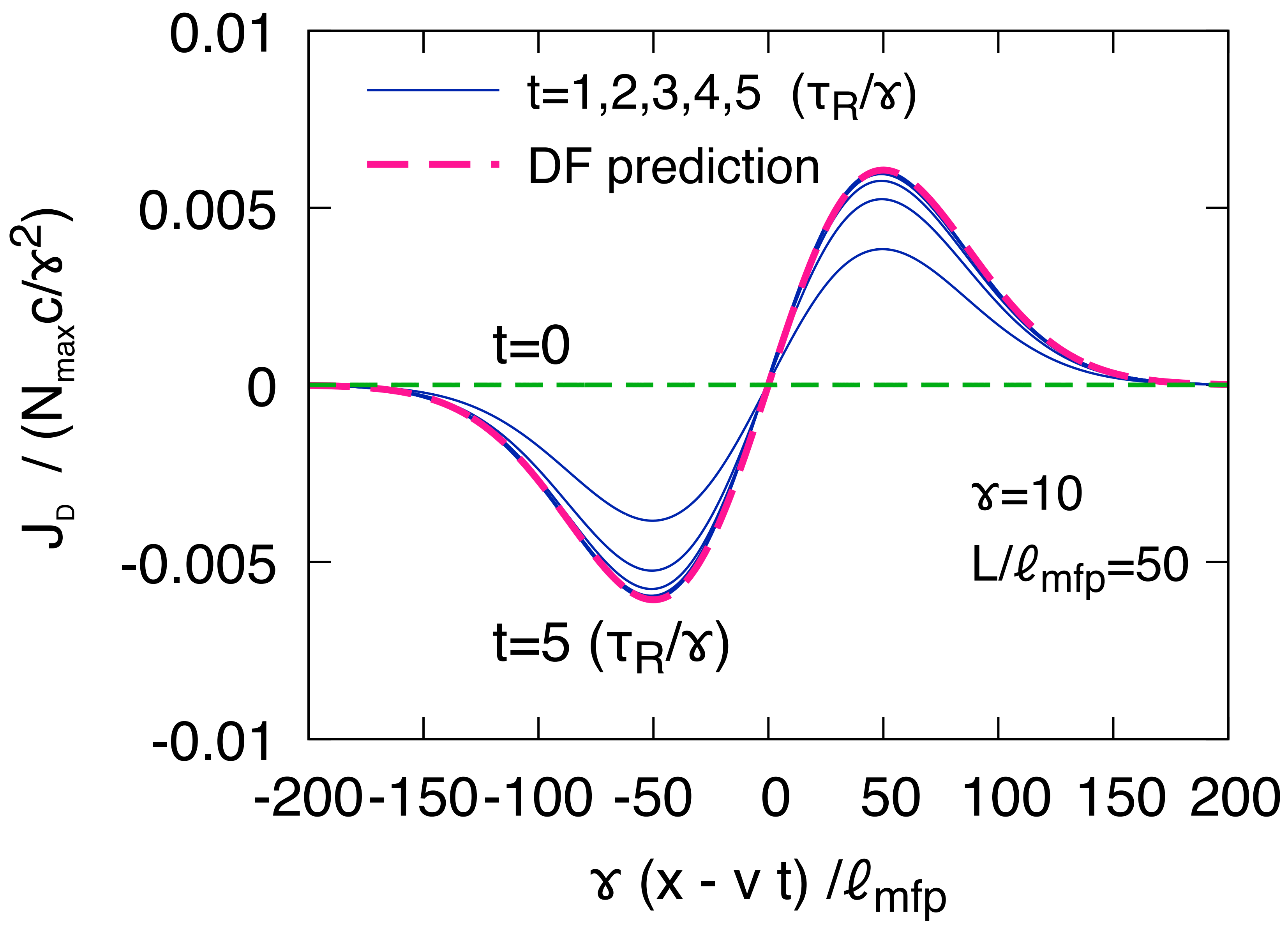
Approach to Density frame

$$J = Nv + J_D$$ ← with prediction $J_D = \frac{D}{\gamma^3}\partial_x N$

Start at $t = 0$ with $N(x)$ as a Gaussian and $J_D = 0$ in lab frame with

$L/l_{\rm mfp} =$ width in fluid frame in units of $l_{\rm mfp} = 2\,c\tau_R$



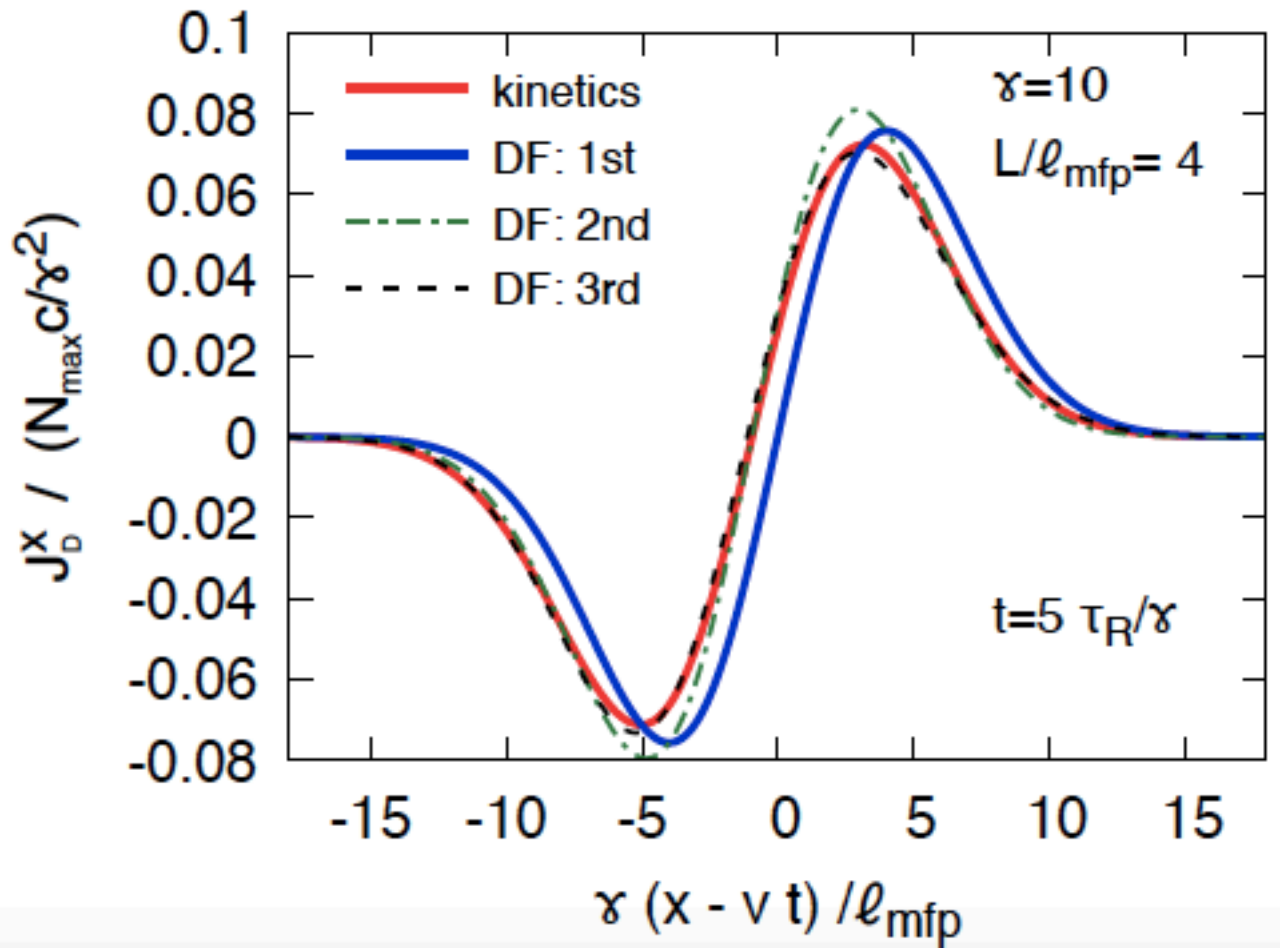


The dissipative steady state is approached on
a timescale τ_R/γ

Convergence of gradient expansion

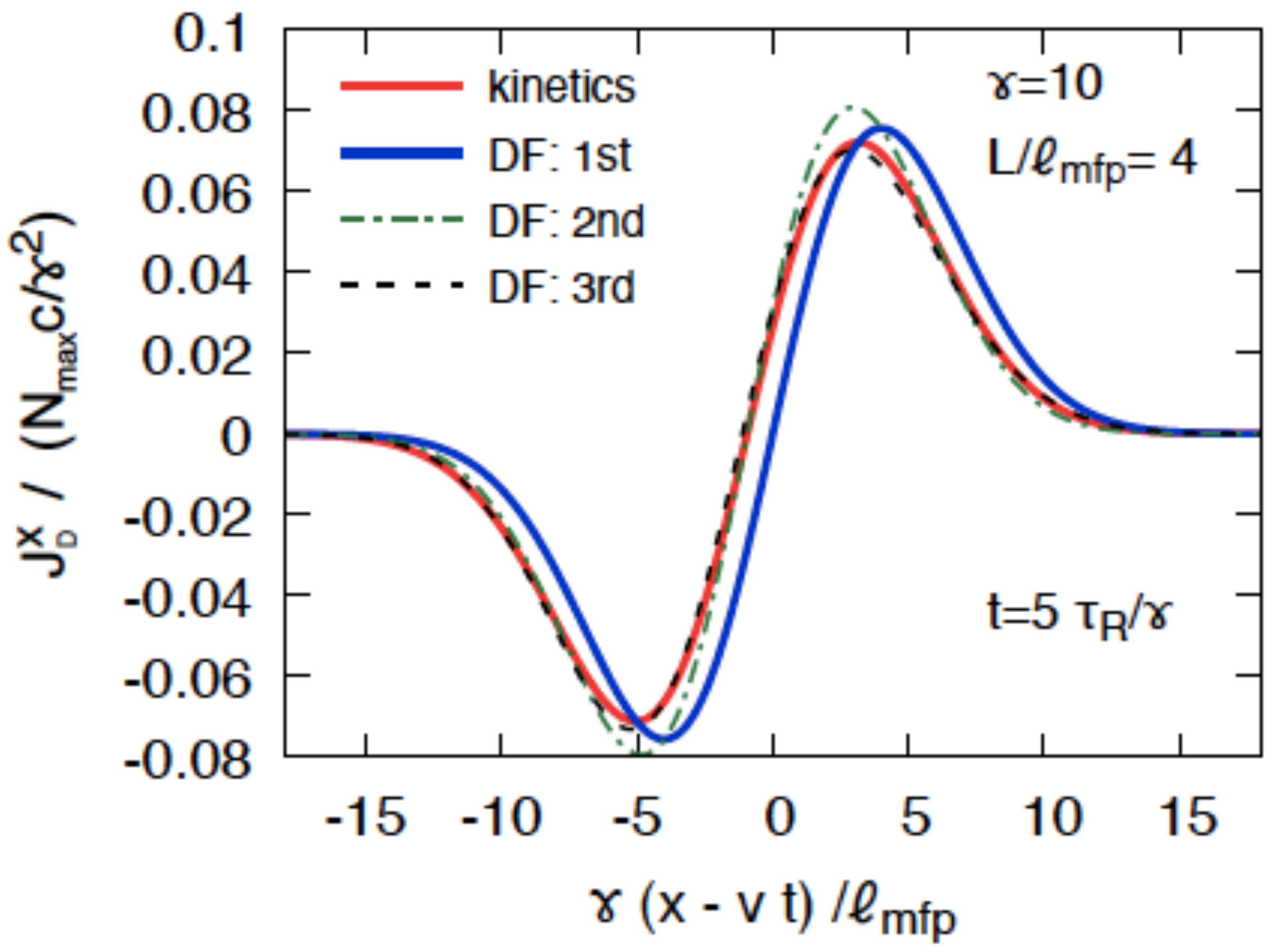
$$\gamma^2 J_D = \frac{\tau_R}{\gamma}\partial_x N + c_2 \left(\frac{\tau_R}{\gamma}\right)^2 v\,\partial_x^2 N + c_3 \left(\frac{\tau_R}{\gamma}\right)^3 v\,\partial_x^3 N + \ldots$$

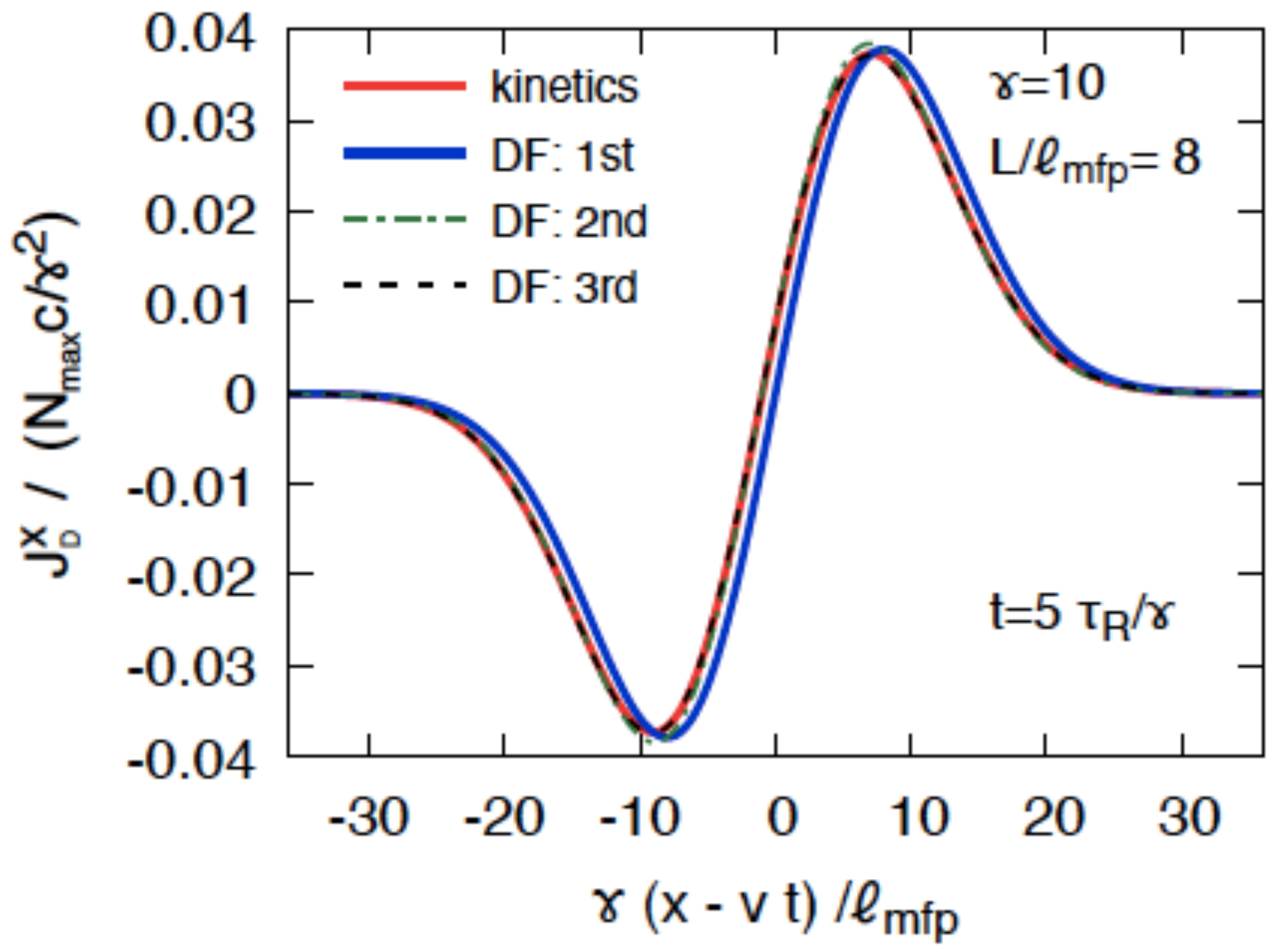
Start with narrow Gaussian $N(x)$ with rest frame width $L/l_{\rm mfp} = 4$



Convergence of gradient expansion

$$\gamma^2 J_D = \frac{\tau_R}{\gamma}\partial_x N + c_2\left(\frac{\tau_R}{\gamma}\right)^2 v\,\partial_x^2 N + c_3\left(\frac{\tau_R}{\gamma}\right)^3 v\,\partial_x^3 N + \ldots$$

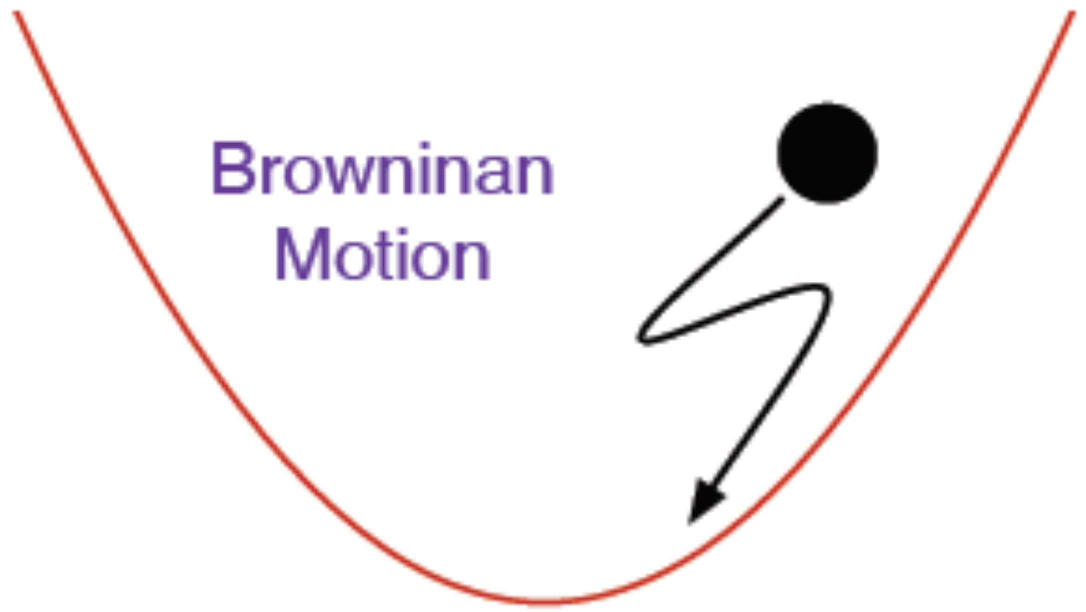




The first term in the gradient expansion is always well behaved
in contrast to higher orders

Enough! Let's add some "noise"

Brownian Motion



→ The stochastic equations of motion:

$$\partial_t q + \{x, \mathscr{H}\} = 0$$

$$\partial_t p + \{p, \mathscr{H}\} = -\eta\left(\frac{\partial \mathscr{H}}{\partial p}\right) + \xi$$

$$= -\eta \frac{p}{m} + \xi$$

velocity

noise: $\langle \xi(t)\xi(t')\rangle = 2\,T\eta\,\delta_{tt'}$

Free energy: $\mathscr{H} = \dfrac{p^2}{2m} + V(q)$

→ The probability $P(t,q,p)$ distribution evolves to equilibrium: $P_{\mathrm{eq}} = e^{-\beta\mathscr{H}}$

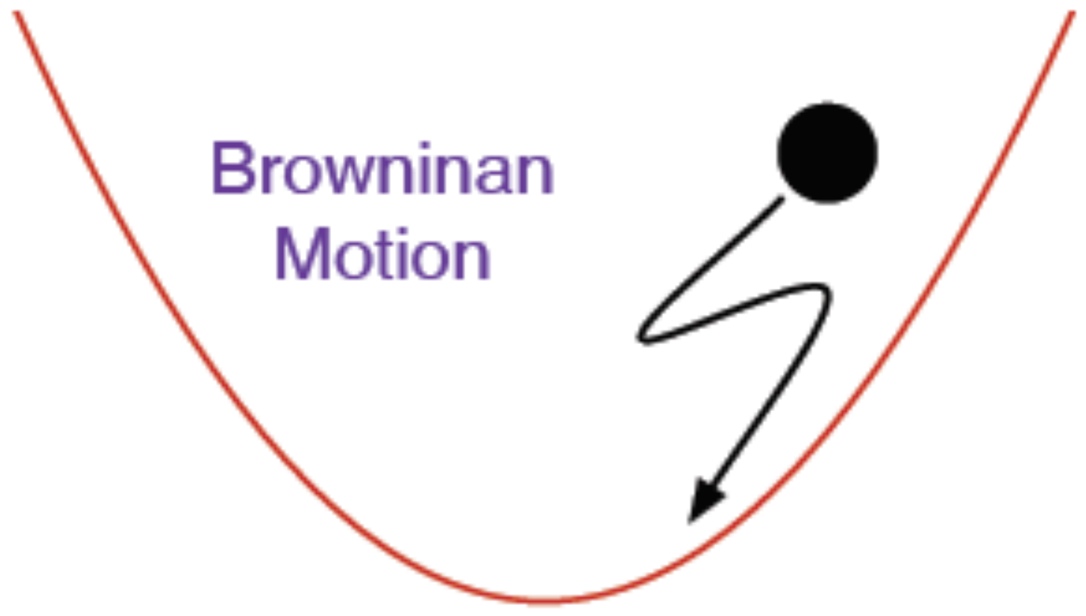
$$\partial_t P + \{\mathscr{H}, P\} = T\eta\nabla_p\left(\underbrace{\beta\nabla_p\mathscr{H}}P + \nabla_p P\right)$$

$-\nabla_p\left(e^{-\beta\mathscr{H}}\right)$

A unique mathematical structure which reaches equilibrium

Dissipative dynamics from Metropolis updates

$$\partial_t p = -\eta \left(\frac{\partial \mathscr{H}}{\partial p} \right) + \xi \qquad \text{with} \qquad \langle \xi(t)\xi(t') \rangle = 2\,T\eta\,\delta_{tt'}$$



Free energy: $\mathscr{H} = \frac{p^2}{2m} + V(q)$

- Make a proposal with the right variance: $p \to p + \Delta p$ with $\langle \Delta p^2 \rangle = 2\,T\eta\,\Delta t$

- Find the change in free energy: $$\Delta\mathscr{H} = \mathscr{H}(p+\Delta p) - \mathscr{H}(p) \simeq \left(\frac{\partial \mathscr{H}}{\partial p} \right) \Delta p$$

- Proposal is accepted if $\Delta\mathscr{H} < 0$. If $\Delta\mathscr{H} > 0$, accept with probability $$P_{\text{up}} = e^{-\beta\Delta\mathscr{H}}$$

- The accepted proposals reproduce the dissipation and variance

$$\langle \Delta p \rangle = -\eta \left(\frac{\partial \mathscr{H}}{\partial p} \right) \Delta t \qquad \text{and} \qquad \langle (\Delta p)^2 \rangle = 2\,T\eta\,\Delta t$$

Advantages of Metropolis approach

- Metropolis steps are guaranteed to converge to the required equilibrium distribution

- For Δt the Metropolis updates naturally reproduce the Langevin dynamics of the diffusion equation

- Detailed balance and the Fluctuation Dissipation Theorem are automatically preserved, independently of Δt

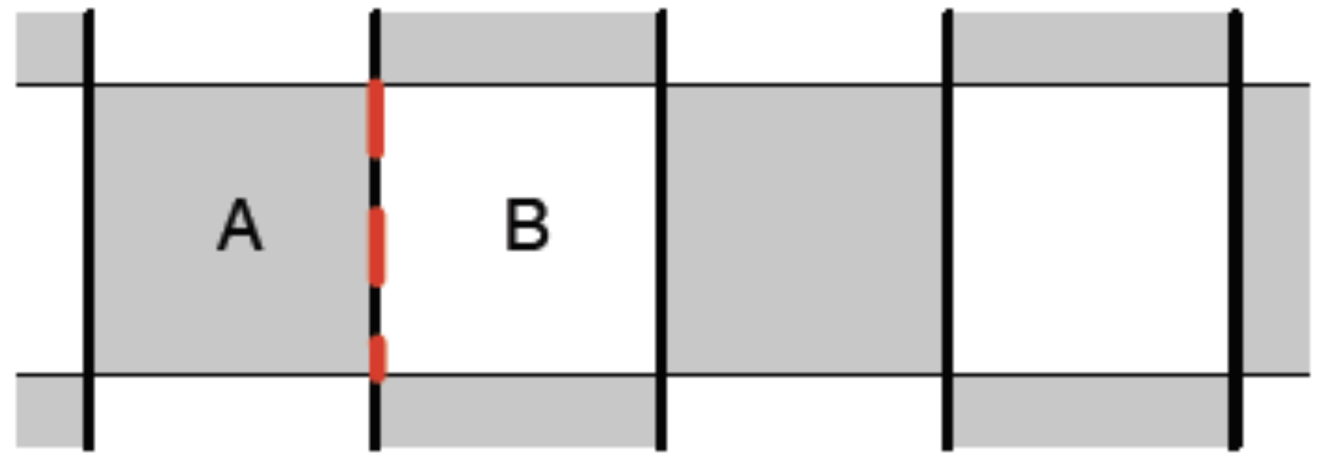
- Simplifies the renormalization of kinetic coefficients

  P. B. Arnold, Phys. Rev. E 61 (2000) 6091-6098

- Used for other problems: Sphaleron rate, O(4) critical point, Model B

  G. D. Moore Nucl. Phys. B 568 (2000),
  Florio, Grossi, Soloviev, Teaney, Phys. Rev. D 105 no. 5, (2022) 054512
  Florio, Grossi, Teaney, Phys. Rev. D 109 no. 5, (2024) 054037
  Chattopadhyay, Ott, Schaefer, Skokov, Phys. Rev. D 108 no. 7, (2023) 074004

Simple diffusion equation



$n_A \to n_A - q$ and $n_B \to n_B + q$

→ The stochastic equation of motion with noise $\langle \xi(x)\xi(x') \rangle = 2\,T\sigma\,\delta(x - x')$

$$\partial_t n + \nabla \cdot j_D = 0 \quad \text{with} \quad \vec{j}_D = -\sigma\nabla\frac{\delta\mathcal{H}}{\delta n} + \xi$$

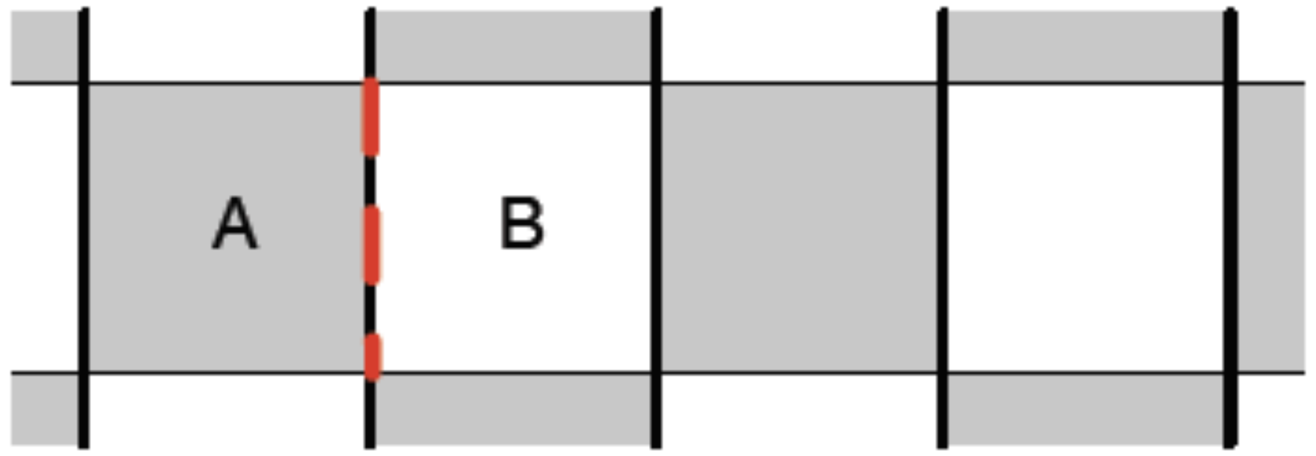
→ The free energy describing the fluctuations is $\mathcal{H}[n] = \int d^3x \frac{n^2}{2\chi}$ with $\delta\mathcal{H} = \mu(x)\delta n(x)$

→ Make a proposal for a charge transfer between cells: $\langle q^2 \rangle \approx 2\,T\sigma\,\Delta t$

Florio, Grossi, Soloviev, Teaney, Phys. Rev. D 105 no. 5, (2022) 054512

Simple diffusion equation

randomly transfer charge
between cell A and B



$n_A \to n_A - q$ and $n_B \to n_B + q$

→ The proposals have the correct variance: $\langle q^2 \rangle \approx 2\, T\, \sigma\, \Delta t$

→ The change in entropy is:

$$\Delta \mathscr{H} = \left(\frac{\delta \mathscr{H}[n]}{\delta n_B} - \frac{\delta \mathscr{H}[n]}{\delta n_A} \right) q \simeq (\mu_B - \mu_A)\, q \simeq q\, \partial_x \mu$$

→ The accepted proposals reproduce the dissipation and variance $\langle q \rangle = -\sigma \partial_x \mu \Delta t$ ← expected diffuse current

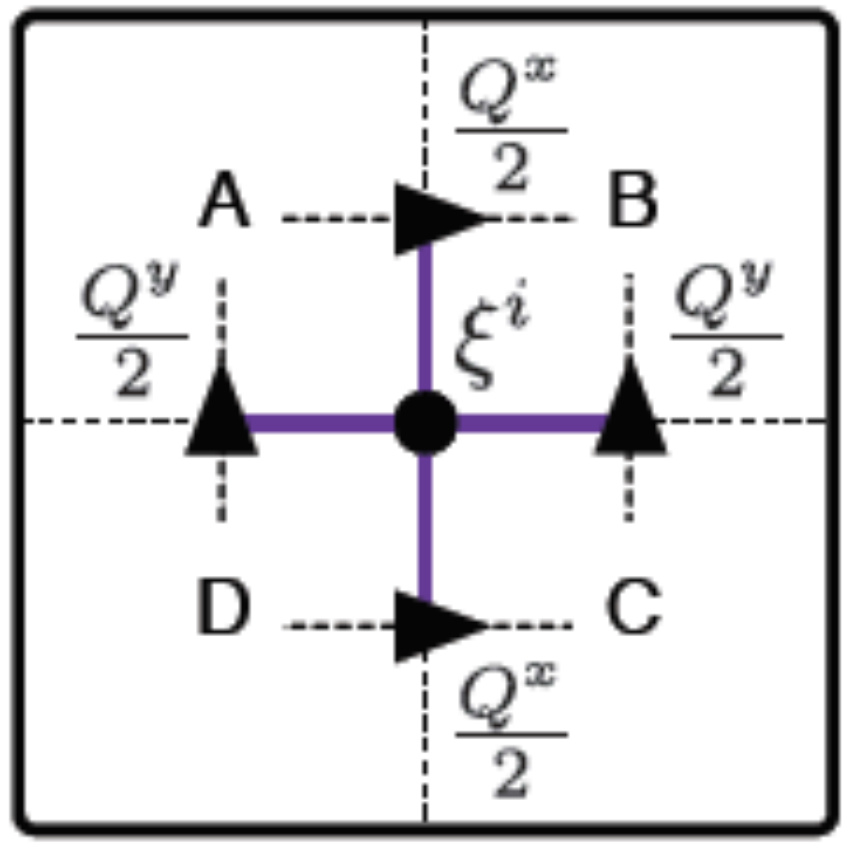
Florio, Grossi, Soloviev, Teaney, Phys. Rev. D 105 no. 5, (2022) 054512

Stochastic diffusion equation

→ Form of the advection-diffusion equation in the density frame with noise is $\partial_t N + \partial_i (N v^i) = \partial_i (D^{ij} \partial_j N + \xi^i)$

with dissipative matrix $D^{ij} = \dfrac{D}{\gamma}(\delta^{ij} - v^i v^j)$ and variance $\langle \xi^i(x) \xi^j(x') \rangle = 2\, T \chi D^{ij} \delta_{xx'}$

→ The framework of Metropolis applies



Propose a charge transfer with transverse and longitudinal variances

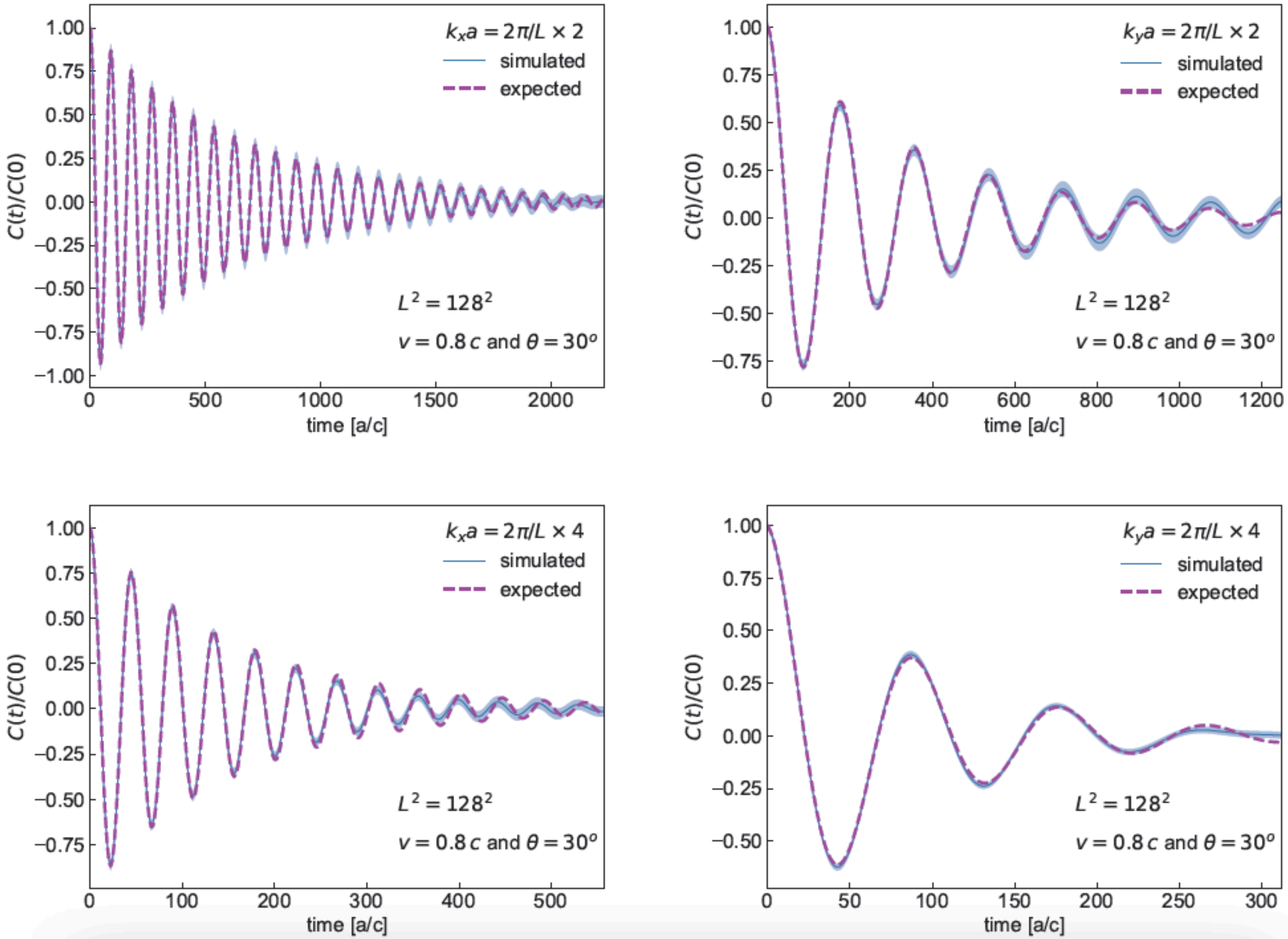
$$N_A \to N_A - \frac{Q^x}{2} + \frac{Q^y}{2}$$

Then accept/reject according to ΔS yields the mean diffusive current

Correlation functions

→ Expected form: $C_{NN}(t-t',\mathbf{k}) = T\chi u^0 \cos(\mathbf{v}\cdot\mathbf{k}\,(t-t'))\,\exp(-D^{ij}k_ik_j\,|\,t-t'\,|\,)$

$$D^{ij} = \frac{D}{\gamma^3}\hat{v}^i\hat{v}^j + \frac{D}{\gamma}\left(\delta^{ij} - \hat{v}^i\hat{v}^j\right)$$

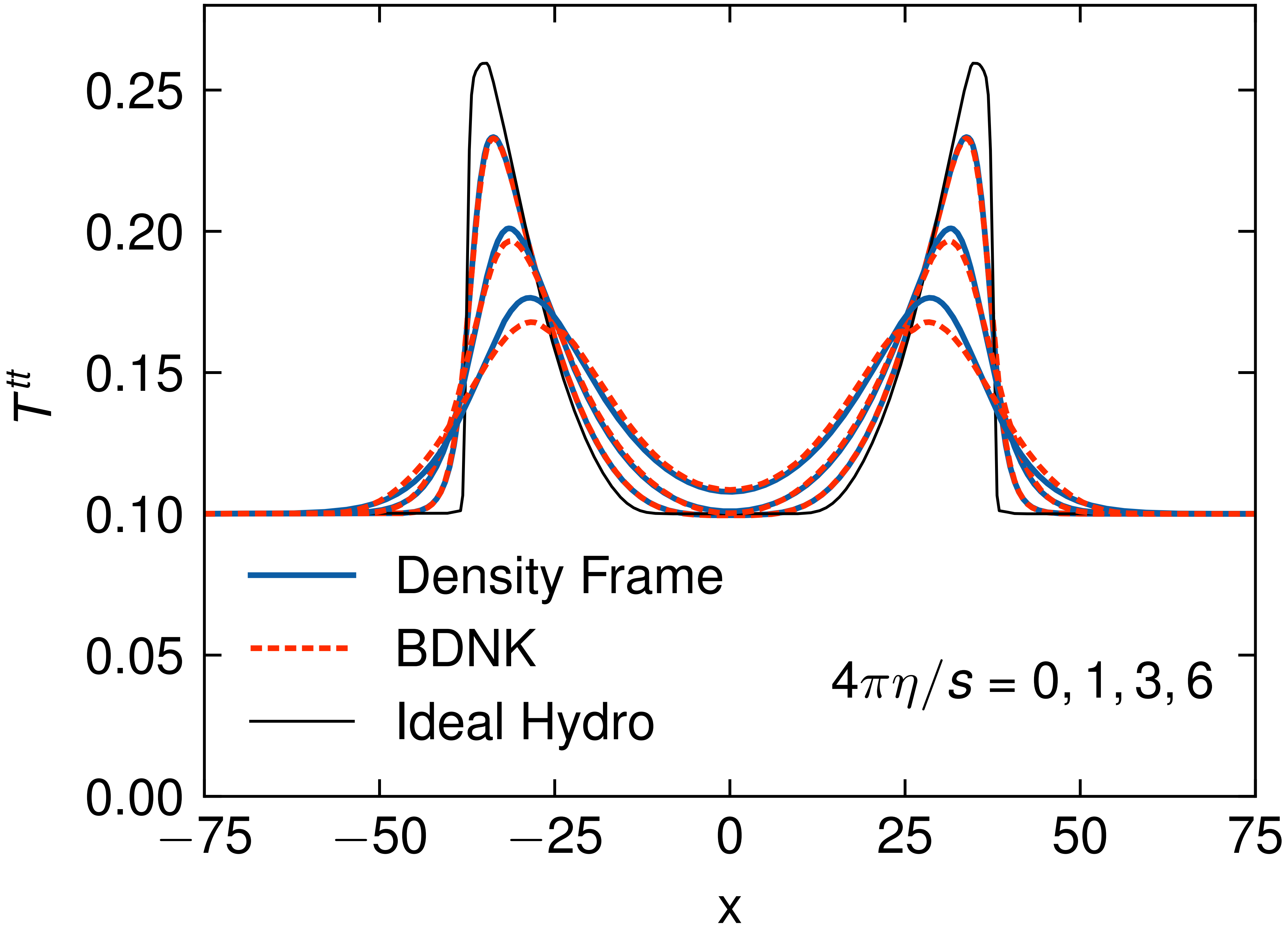


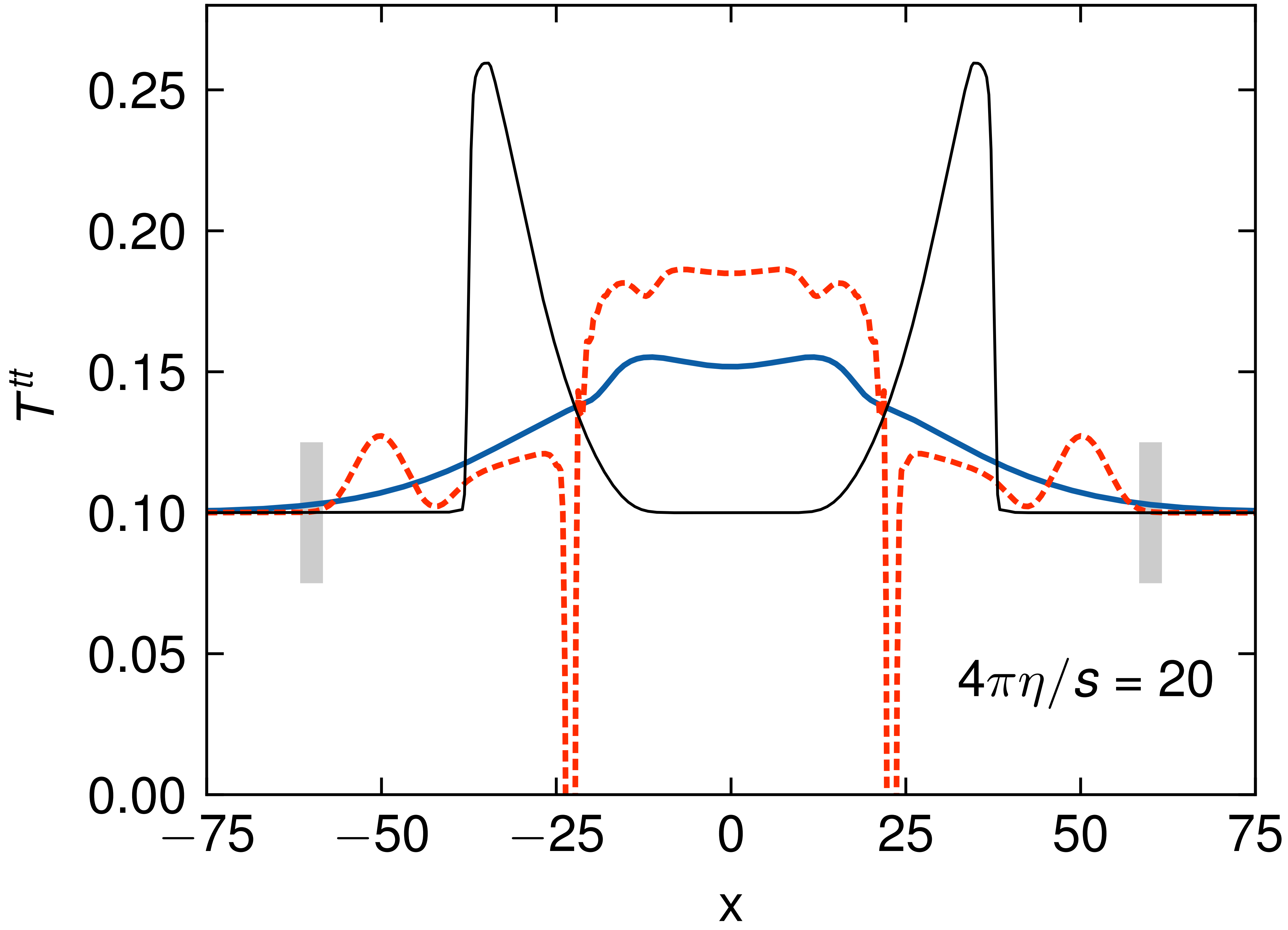
We have generalized to full viscous hydrodynamics

It’s boring, show some video!

Density frame vs BDNK

ongoing work
stay tuned!





A. Pandya and F. Pretorius, Phys. Rev. D 104, 023015 (2021)

Density frame vs BDNK

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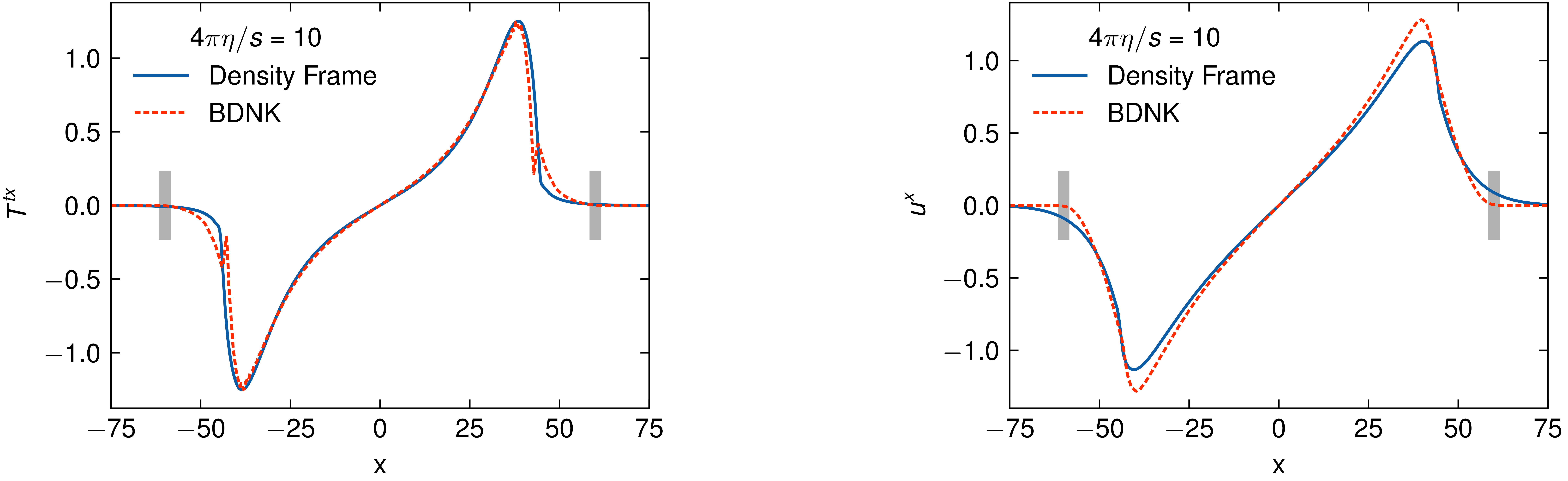

A. Pandya and F. Pretorius, Phys. Rev. D 104, 023015 (2021)

Summary

The mathematical structure follows the particle in a potential example

Stable first order and has no non-hydrodynamic modes

Noise comes first and then dissipation

Procedure is to take an ideal step and make a random momentum transfers with specific variances

We have generalized to full viscous hydrodynamics in Bjorken and General coordinates

Have very good agreement with relativistic MIS and BDNK for small viscosity and works better than BDNK (and similar to MIS) for larger viscosities

The momentum proposal is parallel transported from cell-face to cell-centers for the accept/reject

The parallel transport reproduces the covariant derivatives in the dissipative strain

It is hoped that the Metropolis algorithm for stochastic hydrodynamics will be robust and effective, yielding a significant advance in the modeling of the quark-gluon plasma created in heavy ion collisions

Thank you for listening!

Mulțumesc pentru atenție!

See you in next Chirality!

Metropolis-Hastings algorithm

Let $f(x)$ be the (possibly unnormalized) target density, $x^{(j)}$ be a current value, and $q(x|x^{(j)})$ be a proposal distribution, then

- Sample $x^* \sim q(x|x^{(j)})$.
- Calculate the acceptance probability

$$\rho(x^{(j)}, x^*) = \min\left\{1, \frac{f(x^*)}{f(x^{(j)})}\frac{q(x^{(j)}|x^*)}{q(x^*|x^{(j)})}\right\}.$$

- Set $x^{(j+1)} = x^*$ with probability $\rho(x^{(j)}, x^*)$, otherwise set $x^{(j+1)} = x^{(j)}$.

Notes:

- $x^{(j)} \xrightarrow{d} X$ where $X \sim f(x)$.
- The sequence $x^{(j)}$ is not independent.
-

$$\frac{1}{J}\sum_{j=1}^{J} h\left(x^{(j)}\right) \rightarrow E_f[h(X)] = \int_{\mathcal{X}} h(x)f(x)dx$$