# Quantum-statistical formulation of spin hydrodynamics

## Entropy production and Dissipation

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# Outline

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2. **Quantum-statistical Framework for Relativistic Fluid**
3. **Entropy Current and Entropy Production Rate**
4. **Dissipative currents: Method and results (Ongoing work)**

# Motivation: Progress Made So Far in Entropy Current Analysis

- There is a growing interest in spin hydrodynamics,


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[2] K. Fukushima and S. Pu, "Spin hydrodynamics and symmetric energy-momentum tensors – A current induced by the spin vorticity –," *Phys. Lett. B* **817** (2021) 136346, arXiv:2010.01608 [hep-th].

[3] A. D. Gallegos, U. Gürsoy, and A. Yarom, "Hydrodynamics of spin currents," *SciPost Phys.* **11** (2021) 041, arXiv:2101.04759 [hep-th].

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[5] M. Hongo, X.-G. Huang, M. Kaminski, M. Stephanov, and H.-U. Yee, "Relativistic spin hydrodynamics with torsion and linear response theory for spin relaxation," *JHEP* **11** (2021) 150, arXiv:2107.14231 [hep-th].

[6] N. Weickgenannt, D. Wagner, E. Speranza, and D. H. Rischke, "Relativistic second-order dissipative spin hydrodynamics from the method of moments," *Phys. Rev. D* **106** no. 9, (2022) 096014, arXiv:2203.04766 [nucl-th].

[7] R. Biswas, A. Daher, A. Das, W. Florkowski, and R. Ryblewski, "Relativistic second-order spin hydrodynamics: An entropy-current analysis," *Phys. Rev. D* **108** no. 1, (2023) 014024, arXiv:2304.01009 [nucl-th].


- Spin hydrodynamics involves spin. Therefore quantum methods cannot be avoided.

- The main goal of spin hydro is to determine the dissipative currents:

$$\delta T_A^{\mu\nu} \ , \quad \delta\mathcal{S}^{\lambda\mu\nu}$$

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$$\delta T_A^{\mu\nu} \ , \quad \delta \mathcal{S}^{\lambda\mu\nu}$$

- Using entropy-current analysis, we can fix the number and expressions of their corresponding transport coefficients [Phys. Lett. B 795 (2019) 100–106].
- Several inspiring works done so far starts from assuming a generalized form of the local thermodynamic relations:

$$\epsilon + p = Ts + \mu n + \frac{1}{2}\omega_{\mu\nu} S^{\mu\nu}$$

- Their results show that the spin transport coefficients can only be obtained at the second-order of the hydrodynamic gradient expansion.

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- Other related works were able to obtain the spin transport at the first-order, yet they loose all the information about the antisymmetric part of the energy-momentum tensor [Sci. Bull. 67 (2022) 2265–2268].
- The goal is to just use quantum density operators to derive the entropy current, the entropy production rate, whence the constitutive equations.

# Quantum-statistical Framework for Relativistic Fluid

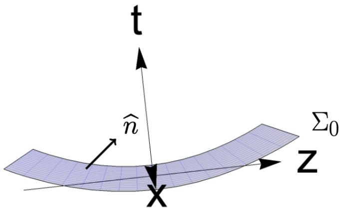
Local equilibrium is achieved at initial hypersurface $\Sigma_0$, where entropy is maximum provided that the mean vales of energy, momentum, particle number, and spin densities are their actual values:

$$S = -\operatorname{Tr}(\widehat{\rho}\log\widehat{\rho})$$

$$F[\widehat{\rho}] = -Tr[\widehat{\rho}\log\widehat{\rho}] - \int d\Sigma_0\, n_\mu\,(T^{\mu\nu}_{\rm LE} - T^{\mu\nu})\,\beta_\nu(x) - \int d\Sigma_0\, n_\mu\,(j^\mu_{\rm LE} - j^\mu)\,\zeta(x) - \int d\Sigma_0\, n_\mu \left(S^{\mu\lambda\nu}_{\rm LE} - S^{\mu\lambda\nu}\right)\Omega_{\lambda\nu}(x)$$

$$T^{\mu\nu}_{\rm LE} \sim Tr\left[\widehat{\rho}\,\widehat{T}^{\mu\nu}\right]$$

$$T^{\mu\nu} \equiv \text{Actual Value}$$



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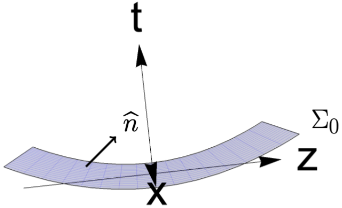



$$F[\widehat{\rho}] = -Tr[\widehat{\rho}\log\widehat{\rho}] - \int d\Sigma_0\, n_\mu\, (T^{\mu\nu}_{\rm LE} - T^{\mu\nu})\,\beta_\nu(x) - \int d\Sigma_0\, n_\mu\, (j^\mu_{\rm LE} - j^\mu)\,\zeta(x)$$
$$- \int d\Sigma_0\, n_\mu \left(S^{\mu\lambda\nu}_{\rm LE} - S^{\mu\lambda\nu}\right)\Omega_{\lambda\nu}(x)$$

$$T^{\mu\nu}_{\rm LE} \sim Tr\left[\widehat{\rho}\,\widehat{T}^{\mu\nu}\right]$$
$$T^{\mu\nu} \equiv \text{Actual Value}$$

$$\widehat{\rho}_{\rm LE} = \frac{1}{Z}\exp\left[-\int_{\Sigma_0} \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \zeta\widehat{j}^\mu - \frac{1}{2}\Omega_{\lambda\nu}\widehat{S}^{\mu\lambda\nu}\right)\right]$$

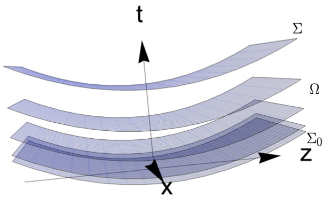
The Lagrange multipliers are obtained by solving the constraint equations at $\Sigma_0$. Their evolution is determined by solving the conservation equations:

- $\beta^\mu \to u^\mu = \beta^\mu/\sqrt{\beta^2} \quad T = 1/\sqrt{\beta^2}$
- $\zeta = \mu/T$
- $\Omega_{\mu\nu} = \omega_{\mu\nu}/T$
- Thermal Shear: $\xi_{\mu\nu} = \frac{1}{2}\left(\nabla_\mu\beta_\nu + \nabla_\nu\beta_\mu\right)$ Thermal Vorticity: $\varpi_{\mu\nu} = \frac{1}{2}\left(\nabla_\nu\beta_\mu - \nabla_\mu\beta_\nu\right)$

At global equilibrium,

$$\beta_\mu = b_\mu + \varpi_{\mu\nu}x^\nu\,, \quad \text{with} \;\; b, \varpi = \text{const}\,, \qquad \Omega = \varpi\,, \qquad \zeta = \text{const}$$

Using Gauss and Divergence theorems:




$$\widehat{\rho} = \frac{1}{Z}\exp\left[\underbrace{-\int_{\Sigma} \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \zeta\widehat{j}^\mu - \frac{1}{2}\Omega_{\lambda\nu}\widehat{S}^{\mu\lambda\nu}\right)}_{\widehat{\rho}_{\mathrm{LE}}(t)\ at\ \Sigma} + \underbrace{\int_{\Omega} d\Omega\ \widehat{T}_S^{\mu\nu}\xi_{\mu\nu} + \widehat{T}_A^{\mu\nu}(\Omega_{\mu\nu} - \varpi_{\mu\nu}) - \frac{1}{2}\widehat{S}^{\mu\lambda\nu}\nabla_\mu\Omega_{\lambda\nu}}_{\text{Dissipative Corrections}}\right]$$

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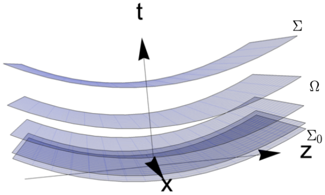



$$\widehat{\rho} = \frac{1}{Z}\exp\left[\underbrace{-\int_\Sigma \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \zeta\widehat{j}^\mu - \frac{1}{2}\Omega_{\lambda\nu}\widehat{S}^{\mu\lambda\nu}\right)}_{\widehat{\rho}_{\mathrm{LE}}(t)\ at\ \Sigma} + \underbrace{\int_\Omega d\Omega\ \widehat{T}_S^{\mu\nu}\xi_{\mu\nu} + \widehat{T}_A^{\mu\nu}(\Omega_{\mu\nu} - \varpi_{\mu\nu}) - \frac{1}{2}\widehat{S}^{\mu\lambda\nu}\nabla_\mu\Omega_{\lambda\nu}}_{\text{Dissipative Corrections}}\right]$$

This implies that dissipation in spin hydrodynamics occurs when:

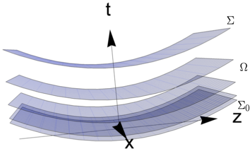
$$\xi \neq 0 \quad \Omega \neq \varpi \quad \nabla\Omega \neq 0$$

# Entropy Current and Entropy Production Rate

# Entropy Current

Near local equilibrium at the hypersuface $\Sigma$, the entropy is defined as:

$$\begin{aligned} S &= -\operatorname{Tr}\left[\widehat{\rho}_{\mathrm{LE}}(t) \log \widehat{\rho}_{\mathrm{LE}}(t)\right] \\ &= \log Z_{\mathrm{LE}} + \int_{\Sigma} \mathrm{d}\Sigma_{\mu} \left[\operatorname{Tr}\left(\widehat{\rho}_{\mathrm{LE}} \widehat{T}^{\mu\nu}\right)\beta_{\nu} - \zeta \operatorname{Tr}\left(\widehat{\rho}_{\mathrm{LE}} \widehat{j}^{\mu}\right) - \frac{1}{2}\Omega_{\lambda\nu} \operatorname{Tr}\left(\widehat{\rho}_{\mathrm{LE}} \widehat{S}^{\mu\lambda\nu}\right)\right] \end{aligned}$$




Can we define an entropy current out of $S$ ? In other words, is it possible to show that $\log Z_{\mathrm{LE}}$ is an extensive quantity?

$$\log Z_{\rm LE} \sim \int_\Sigma {\rm d}\Sigma_\mu \, \phi^\mu$$

[F. Becattini, D. Rindori PhysRevD.99.125011]

where $\phi^\mu$ is defined as thermodynamic potential vector field:

$$\phi^\mu(x) = \int_0^{T(x)} \frac{{\rm d}T'}{T'^2} \left( T^{\mu\nu}_{\rm LE}(x)[T',\mu,\omega] u_\nu(x) - \mu(x) j^\mu_{\rm LE}(x)[T',\mu,\omega] - \frac{1}{2}\omega_{\lambda\nu}(x) \mathcal{S}^{\mu\lambda\nu}_{\rm LE}(x)[T',\mu,\omega] \right)$$

For a fluid at global equilibrium with vanishing thermal vorticity $\varpi_{\mu\nu} = 0$:

$$\phi^\mu = p\, \beta^\mu$$

where "$p$" is the hydrostatic pressure.

Therefore, entropy current exists:

$$S = \int_{\Sigma} \mathrm{d}\Sigma_\mu \, \phi^\mu + T^{\mu\nu}_{\mathrm{LE}}\beta_\nu - \zeta j^\mu_{\mathrm{LE}} - \frac{1}{2}\Omega_{\lambda\nu}\mathcal{S}^{\mu\lambda\nu}_{\mathrm{LE}}$$

$$s^\mu_{LE} = \phi^\mu + T^{\mu\nu}_{\mathrm{LE}}\beta_\nu - \zeta j^\mu_{\mathrm{LE}} - \frac{1}{2}\Omega_{\lambda\nu}\mathcal{S}^{\mu\lambda\nu}_{\mathrm{LE}} .$$

In quantum theory, we only have the total entropy, not the entropy current. We need to construct an entropy current through an integral. However, this introduces ambiguities, as several fields can lead to the same integral. However if $s^\mu - s^\mu_{LE} \perp n^\mu$,

$$s^\mu = \phi^\mu + T^{\mu\nu}\beta_\nu - \zeta j^\mu - \frac{1}{2}\Omega_{\lambda\nu}\mathcal{S}^{\mu\lambda\nu} \qquad \phi^\mu = \int_0^T \frac{\mathrm{d}T'}{T'^2}\left(T^{\mu\nu}[T']u_\nu - \mu j^\mu[T'] - \frac{1}{2}\omega_{\lambda\nu}\mathcal{S}^{\mu\lambda\nu}[T']\right)$$

# Entropy Production Rate

$$\partial_\mu s^\mu = \left(T_S^{\mu\nu} - T_{S(\mathrm{LE})}^{\mu\nu}\right)\xi_{\mu\nu} - \left(j^\mu - j_{\mathrm{LE}}^\mu\right)\partial_\mu\zeta + \left(T_A^{\mu\nu} - T_{A(\mathrm{LE})}^{\mu\nu}\right)\left(\Omega_{\mu\nu} - \varpi_{\mu\nu}\right) - \frac{1}{2}\left(\mathcal{S}^{\mu\lambda\nu} - \mathcal{S}_{\mathrm{LE}}^{\mu\lambda\nu}\right)\partial_\mu\Omega_{\lambda\nu}$$

$\varpi_{\mu\nu}$: is the thermal vorticity

1. This formula is a generalization of what was obtained *C. Van Weert* without spin:

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2. We stress that the formula is exact and not an approximation at some order of a gradient expansion.

3. A novel feature is apparently the simultaneous appearance of the last two terms of the right hand side.

# Dissipative currents: Method and results (Ongoing work)

- The goal is to find:

$$\delta T_s^{\mu\nu} = \left(T_s^{\mu\nu} - T_{s(LE)}^{\mu\nu}\right), \quad \delta T_A^{\mu\nu}, \quad \delta j^{\mu}, \quad \delta S^{\lambda\mu\nu}$$

- Hence we expand the above interms of all gradients in the system. For example:

$$\delta T_S^{\mu\nu} = H^{\mu\nu\rho\sigma}\xi_{\rho\sigma} + K^{\mu\nu\rho}\partial_\rho\zeta + L^{\mu\nu\rho\sigma}\left(\Omega_{\rho\sigma} - \varpi_{\rho\sigma}\right) + M^{\mu\nu\rho\sigma\tau}\partial_\rho\Omega_{\sigma\tau}$$

- The goal thus is to determine:

$$H^{\mu\nu\rho\sigma},\ K^{\mu\nu\rho},\ L^{\mu\nu\rho\sigma},\ M^{\mu\nu\rho\sigma\tau}$$

- We expand from $SO(3)$ invariant global equilibrium

$$u^{\mu}\ , \quad \Delta^{\mu\nu}\ , \quad \epsilon^{\lambda\mu\nu\gamma}$$

- Using the irreducible representation of $SO(3)$, a vector, symmetric rank-2 tensor, and antisymmetric tensor can be written as:

$$V^{\mu} = (0 \oplus 1) = (u^{\mu} \oplus \Delta^{\mu}_{\alpha}),$$
$$S^{\mu\nu} = (0 \oplus 0 \oplus 1 \oplus 2) = (u^{\mu}u^{\nu} \oplus \Delta^{\mu\nu} \oplus u^{\mu}\Delta^{\nu}_{\alpha} + u^{\nu}\Delta^{\mu}_{\alpha} \oplus \Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta} + \Delta^{\nu}_{\alpha}\Delta^{\mu}_{\beta}),$$
$$A^{\mu\nu} = (1 \oplus 1) = (u^{\mu}\Delta^{\nu}_{\alpha} - u^{\nu}\Delta^{\mu}_{\alpha} \oplus \epsilon^{\mu\nu\tau\alpha}u_{\tau}).$$

- Matching Condition

$$n_\mu(\delta T_S^{\mu\nu} + \delta T_A^{\mu\nu}) = 0, \quad n_\mu \delta j^\mu = 0, \quad n_\mu \delta S^{\mu\lambda\nu} = 0$$

- Entropy positivity

$$\partial_\mu s^\mu \geq 0$$

- We were able to reproduce (Work in progress):

$$\delta T_S^{\mu\nu} = \bar{h}_2 \frac{\Delta^{\mu\nu}}{T}\theta + \frac{2h_3}{T}\sigma^{\mu\nu} + .....$$

$$\delta T_A^{\mu\nu} = q_4 \Delta^{[\mu[\sigma}\Delta^{\nu]\rho]}(\Omega_{\rho\sigma} - \omega_{\rho\sigma}) + ......$$

$$\delta j^\mu = i_2 \nabla^\mu \zeta + ......$$

$$\delta S^{\lambda\mu\nu} = ......$$

# Conclusions and Outlooks

- We used first-principle density operator method to derive the entropy current and the entropy production rate.
- Established a method based on $SO(3)$ irreducible representations to drive the dissipative currents.

The first next step is to finish the:

- Spin dissipative current

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