Mean-field analysis of a multi-scale stochastic model for free-floating car sharing

Alessia Rigonat, *DI ENS, INRIA, Paris*

Joint work with Christine Fricker and Hanene Mohamed

December 4th, 2024

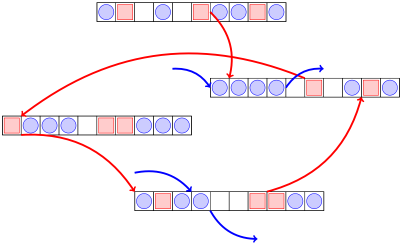
Université Paris Nanterre

informatics mathematics

ENS | PSL

Problem formulation

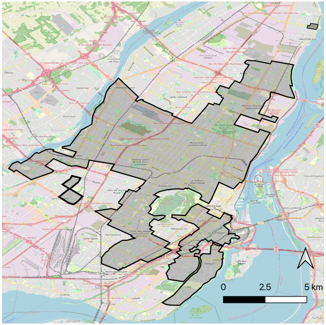
- N nodes, with **finite** but large capacity $C\,N$, $C > 0$.
- $M \sim sN$, particles moving between the nodes.
- *External particles*, **more numerous**, at each node.
- External particles leave the node after service.
- External particles **compete** with the M particles.



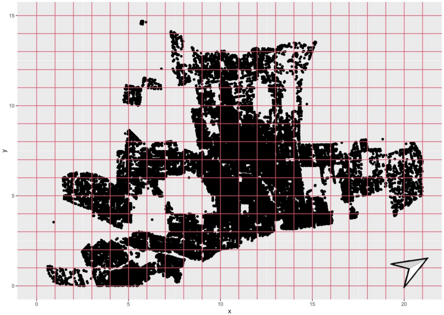
Q: interaction between both?
Aim: Long-time behavior of the system when it gets large ($N \to +\infty$).

Motivation: free-floating car sharing

- No physical stations
- Cars parked everywhere inside the service area



Free-floating service area of *Communauto* in Montreal



Communauto vehicles' arrivals from the dataset 2021

- In mesh of $1km^2$ $CN \sim 200$ parking spaces
- $N \sim 150$ zones
- $M \sim 2000$ free-floating cars = particles moving between the nodes
- private cars = external particles

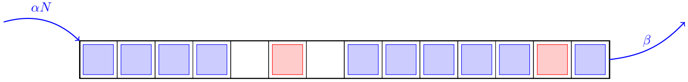
Probabilistic modeling of a car sharing system

- Free-floating has always been analyzed as a station-based system (Weikl, Bogenberg'2015)
- Once the service area is divided into small zones, from $0.25 km^2$ to $1\ km^2$ (Lippold et al.'2018), a fixed capacity is considered for each zone
- In this work: residual capacity = capacity seen by free-floating cars, it is random. (**Fricker, Mohamed, Rigonat.** *2024. Stochastic averaging and mean-field for a large system with fast varying environment with applications to free-floating car-sharing.* ⟨hal-04714886⟩)

The model

Dynamics 1: private cars

- Private cars are an open system. They outnumber free-floating cars
- They arrive in a station: Poisson Process of rate αN.
- If there is at least a free place they park there, otherwise they leave the system.
- The parking time of each private car is exponentially distributed with mean $1/\beta$.
- Without ff-cars, they behave in every station as an $M/M/CN/CN$ queue with arrival rate αN and service rate β



The Markov process

For a given zone i, and a given instant t, denote by

- $m_i^N(t)$: number of free places
- $V_i^N(t)$: number of available free-floating cars
- $R_i^N(t)$: number of reserved free-floating cars
- $X_i^N(t)$: number of private cars

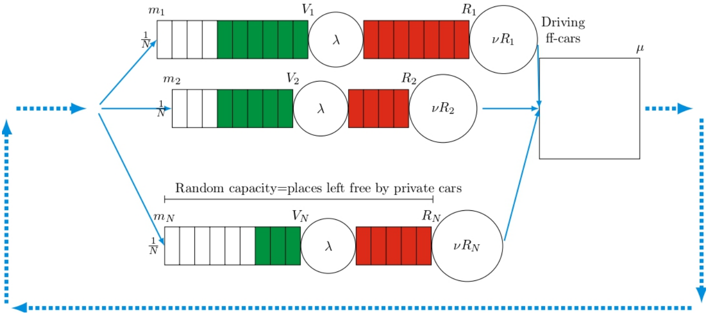
Mass conservation

$m_i^N(t) + V_i^N(t) + R_i^N(t) + X_i^N(t) = CN$

$\Longrightarrow$ the vector $(m_i^N(t), V_i^N(t), R_i^N(t))$ represents the state of station i.

Dynamics 2: free-floating cars

- The model is **homogeneous**
- service area divided into N **zones**
- **total capacity** of each zone is CN
- $M \sim Ns$ **is the total number of free-floating cars** in the system



Closed network of *pseudo-stations* with spots occupied by free-floating cars. Capacity of the pseudo-stations is random and determined by spots left free by private cars.

Process $((m_i^N(t), V_i^N(t), R_i^N(t)),\ 1 \leq i \leq N)$ is a **multi-scale** Markov process on the state space

$$\mathcal{S}^N = \left\{(m_i, v_i, r_i)_{1\leq i\leq N} \in \mathbb{N}^{3N}, m_i + v_i + r_i \leq CN,\ \sum_{k=1}^{N}(v_k + r_k) \leq M\right\}$$

with transitions which modify just the state of one station. For station i,

$$(m_i, v_i, r_i) \to \begin{cases} (m_i + 1, v_i, r_i) & \beta(CN - m_i - v_i - r_i) \\ (m_i - 1, v_i, r_i) & \alpha N \mathbb{1}_{m_i>0} \\ (m_i - 1, v_i + 1, r_i) & \mu(s_N - \frac{1}{N}\sum_{l=1}^{N}(v_l + r_l))\mathbb{1}_{m_i>0} \\ (m_i, v_i - 1, r_i + 1) & \lambda \mathbb{1}_{v_i>0} \\ (m_i + 1, v_i, r_i - 1) & \nu r_i. \end{cases}$$

Stochastic averaging

1. Process $(m_i^N(t))$ evolves quickly
   - its speed is **O(N)**
   - its dynamics **ignore** the presence of free-floating cars $(V_i^N(t))$
   - it shows a **phase transition**
2. Process $(V_i^N(t), R_i^N(t))$ evolves slowly
   - its speed is **O(1)**
   - Stochastic averaging: it sees the fast process $(m_i^N(t))$ at **equilibrium**

Related works:

- **V. Fromion, P. Robert, J. Zaherddine,** *Stochastic Models of Regulation of Transcription in Biological Cells*, Journal of Mathematical Biology 87(5) (2023);
- **C. Bordenave, D. McDonald, A. Proutiere,** *A particle system in interaction with a rapidly varying environment: Mean field limits and applications*, ACM Sigmetrics, 2008

Results

Phase transition

For large values of N, if m_i, v_i and r_i are $o(N)$, $(m_i^N(t))$ behaves like a birth and death process (L_{m_i}) with jumps:

$$L_{m_i} \to \begin{cases} L_{m_i} - 1, & \alpha N \\ L_{m_i} + 1, & \beta C N \end{cases}$$

There are two regimes in the limit for $N \to \infty$ $\begin{cases} \text{geometric}(\beta C/\alpha) & \text{if } \alpha > \beta C \\ +\infty & \text{if } \alpha < \beta C. \end{cases}$

- Overloaded regime: $\alpha/\beta > C$
  - the number of free places is limited

$$\frac{\beta C}{\alpha} = \text{acceptance probability} < 1$$

- Underloaded regime: $\alpha/\beta < C$
  - there is always a free parking place

Mean-field analysis

Result at the normal timescale: $t \mapsto t$

Mean-field limit

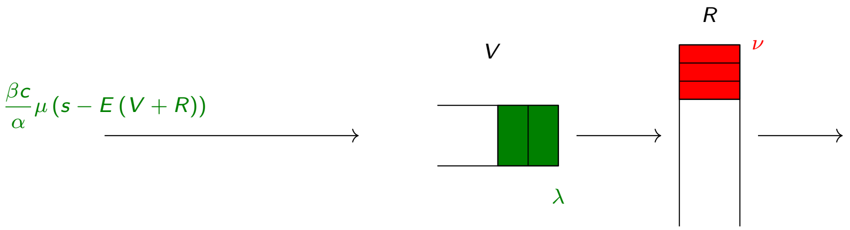
Assume $\alpha > \beta C$. In the limit $N \to \infty$ any pseudo-station has the distribution of the tandem of two queues:

The first is a $M/M/1$ queue with

- arrival rate $(\beta C/\alpha)\mu(s - \mathbb{E}(V(t)) - \mathbb{E}(R(t)))$
- service rate λ

The second is a $M/M/\infty$ queue with service rate ν

$$\frac{\beta C}{\alpha} = \text{acceptance probability} < 1$$



Dynamics of available and reserved ff cars as a tandem of two queues. The *horizontal* is a $M/M/1$ queue, the *vertical* queue is $M/M/\infty$ queue.

A nice time-scale: $t \mapsto tN$

If f is a function on $\mathbb{N}^3$ the occupation measure in $[0, t]$ is

$$\langle \mu_N, g \rangle = \frac{1}{N} \sum_{i=1}^{N} \int_0^t f(m_i^N(Nu), V_i^N(Nu), R_i^N(Nu))du = \int_0^t \Lambda^N(f)(Nu)du,$$

where $\Lambda^N(f)(t) = 1/N \sum_{i=1}^N f(m_i^N(t), V_i^N(t), R_i^N(t))$ is the empirical measure.

Stationary mean-field limit

The sequence $\left(\int_0^t \Lambda^N(f)(Nu)du\right)$ is tight and, for any converging subsequence μ_{N_k},

$$\lim_{k\to\infty} \left(\int_0^t \Lambda^N(f)(N_k u)du\right) = \left(\int_0^t \int_{\mathbb{N}^3} f(m, v, r)\pi_u(dm, dv, dr)du\right)$$

for any f with finite support on $\mathbb{N}^3$. Where

$$\pi_u(dm, dv, dr) = \text{geom}(\beta C/\alpha) \otimes \text{geom}(\rho) \otimes \text{Poi}(\rho\lambda/\nu)$$

and $\rho = \frac{A+s+1-\sqrt{(A+s+1)^2-4sA}}{2A}$ with $A = \lambda\left(\frac{1}{\nu} + \frac{1}{\mu}\left(1 \wedge \frac{\alpha}{\beta C}\right)\right)$.

Proof:
- T. Kurtz Lemma (see T. Kurtz *Averaging for martingale problems and stochastic approximation*, Applied Stochastic Analysis, pp. 186–209. Springer (1992))
- good scaling for the evolution equations

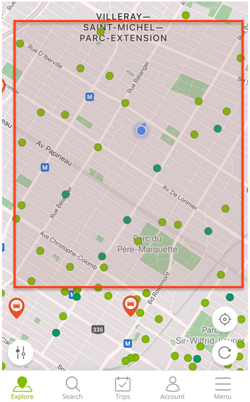
Performance evaluation

Goal: solving the unbalance problem

Problems:

- not finding a parking space

- not finding a car



Aim: reduce the number of zones without parking spaces or cars
Tools: incentive policies

Recall:

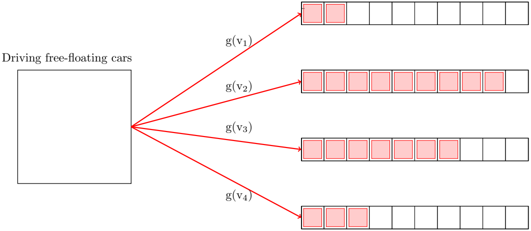
Limiting proportion of zones without available parking spaces is determined by the environment:

$$\begin{cases} 1 - \beta C/\alpha & \text{overloaded regime} \\ 0 & \text{underloaded regime} \end{cases}$$

Incentive policies

Transitions rate for station i are

$$(m_i, v_i, r_i) \to \begin{cases} (m_i+1, v_i) & \beta(CN - m_i - v_i) \\ (m_i-1, v_i) & \alpha N \mathbb{1}_{m_i>0} \\ (m_i-1, v_i+1) & \mu(M_N - \sum_{l=1}^N v_l)\mathbb{1}_{m_i>0} \frac{g(v_i)}{\sum_{l=1}^N g(v_l)} \\ (m_i+1, v_i-1) & \lambda \mathbb{1}_{v_i>0}. \end{cases}$$



The marginal distribution π_V has the following expression. For $k \in \mathbb{N}^+$

$$\pi_V(k) = \pi_V(0) \left(\frac{\mu}{\lambda} \frac{\beta C}{\alpha} \frac{s - \int_{\mathbb{N}} v \; \pi_V(dv)}{\int_{\mathbb{N}} g(v) \; \pi_V(dv)} \right)^k g!(k-1).$$

Example: a random weighted routing policy

We consider the routing rule

$$g_r(k) = 1 + \frac{r-1}{k+1},$$

with

$$\frac{r-1}{k+1} \sim \text{discouragement to join a pseudo-station with k cars parked},$$

and

$$r \sim \text{level of the discouragement.}$$

Proposition

Fix $r \in \mathbb{N}^+$. The equilibrium distribution of the number of FF cars in a zone is Negative Binomial. For $k \in \mathbb{N}$,

$$\pi_{r,\rho}(k) = (1-\rho)^r \rho^k \frac{(k+r-1)!}{k!(r-1)!}$$

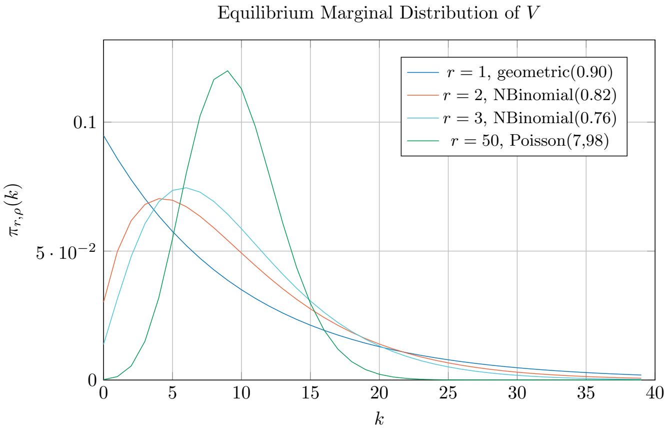
where $\rho \in (0, 1]$ is the solution of the fixed point equation

$$(1-\rho)^{r+1} + \rho(1 - As - Ar) = 1 - As.$$

Remark: when $r = 1$, no incentives, Geometric(ρ) with explicit. ρ

Performance evaluation 1

- $r = 1$, uniform routing, $\pi_{1,\rho} =$ Geometric(ρ)
- $1 < r < \infty$, $\pi_{r,\rho} =$ Negative Binomial(r, ρ)
- $r \to +\infty$, $\pi_{r,\rho(r)} \to$ Poisson(γ) in distribution, where $\gamma = \lim_{r\to\infty} r\rho(r)$



Marginal invariant distribution of the number of available free-floating cars in a zone: it becomes more centered around the mean for increasing values of the parameter r measuring the strengthness of the incentive policy.

Performance evaluation 2

Recall: $M \sim sN$, $s =$ fleet size

- limiting proportion of zones without available parking spaces:
  $1 - \beta C/\alpha \longleftarrow$ determined by the environment

Theorem (Performance)

The limiting probability of failure coincides with the stationary probability of absence of free-floating cars in any station which is given, for a fixed $r \in \mathbb{N}^+$, by

$$\pi_{r,\rho}(0) = (1-\rho)^r,$$

where $\rho \in (0,1]$ is the solution of the fixed point equation

$$(1-\rho)^{r+1} + \rho(1 - As - Ar) = 1 - As,$$

with

$$A = \frac{\mu\beta C}{\alpha\lambda}.$$

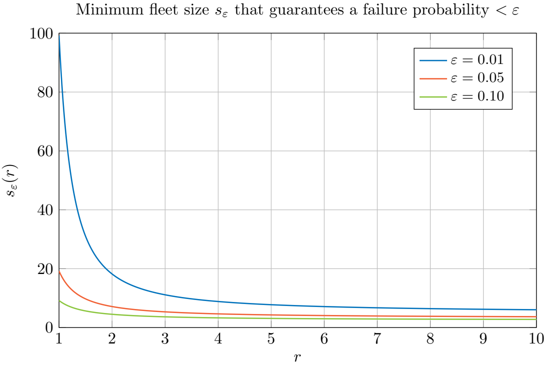
Performance evaluation 3

Given a threshold, say ε, for the probability of failure, one can compute the minimum fleet size s_ε that assures

$$\pi_{r,\rho}(0) \leq \varepsilon.$$

This is given by

$$s_\varepsilon(r) = \frac{1-\varepsilon}{A} + r\,\frac{1-\varepsilon^{1/r}}{\varepsilon^{1/r}}.$$



Minimum value of the fleet size that assures that the failure probability $\pi_{r,\rho}(0) \leq \varepsilon$, for three different values of the threshold ε.

Future work

- Heterogeneity: we showed that the heterogeneous model for a fixed valued of N has product form invariant distribution. How to pass to the limit for $N \to \infty$ in the multi-scale case? (talk of C. Fricker)
- other routing policies
- more realistic routing policies power-of-d, fixed neighborhoods

Thank you for your attention