Noncooperative Games with Prospect Theoretic Preferences

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Motivation

Motivation

- Game theory has been used in many fields to model and analyze the outcomes of conflicts involving strategic agents in competitive scenarios.
- Expected utility theory (EUT) is based on the assumption that agents are rational and systematically maximize the expectation of possible outcomes.
- However, empirical studies have shown that humans, subject to cognitive and emotional biases, often deviate from this ideal model, adopting unpredictable or biased behavior in the face of uncertainty (Shalev, 2000)

Motivation

Motivation

- Prospect theory (PT) models human decisions by taking into account cognitive biases, such as loss aversion or sensitivity to relative gains, thus departing from classical rationality assumptions.
- Preference integration based on prospect theory (PT) leads to non-convex and non-smooth problems, making it difficult to use classical tools to demonstrate the existence of equilibria, define their properties, and design efficient algorithms.

Contributions

Our contribution is threefold:

Motivation

1. Based on a new notion of equilibrium, we establish conditions for the existence and computation of equilibrium for PT-based games, relying on a recently introduced generalization of the Clarke Jacobian.
2. We introduce the concept of price of irrationality (PoI) to quantify how system efficiency degrades due to agents' irrational behavior.
3. We analytically characterize the impact of prospect-theoretic preferences on the class of aggregative games.

We corroborate our results on a case study of electricity market involving strategic end-users exposed to a two-part tariff.

Problem formulation

Problem formulation

We consider a stochastic noncooperative game between

- N self-interested agents, indexed by $i \in \mathcal{I} \stackrel{\text{def}}{=} \{1, ..., N\}$.
- Each agent $i \in \mathcal{I}$ selects a strategy x_i from a feasible set $\mathcal{X}_i \subseteq \mathbb{R}^{m_i}$, with $m_i \in \mathbb{N}^\star$.
- Let $\boldsymbol{x} = \mathrm{col}(x_1, x_2, \ldots, x_N)$ be the collection of all players' strategies and
- $\mathcal{X} \stackrel{\text{def}}{=} \prod_i \mathcal{X}_i \subseteq \mathbb{R}^m$ be the joint feasible strategy set, with $m \stackrel{\text{def}}{=} \sum_i m_i$.

EUT game

The goal of each agent is to maximize a profit determined by a function $f_i(x_i, \boldsymbol{x}_{-i}, \xi)$, $f_i : \mathbb{R}^m \times \Xi \to \mathbb{R}$ that depends on its own strategy, the other agents' strategies $\boldsymbol{x}_{-i} \in \mathcal{X}_{-i} \stackrel{\text{def}}{=} \prod_{j \in \mathcal{I} \setminus \{i\}} \mathcal{X}_j$, and a stochastic parameter $\xi \sim \mathbb{P} \in \mathcal{P}(\Xi)$ which is shared among all agents.

EUT game

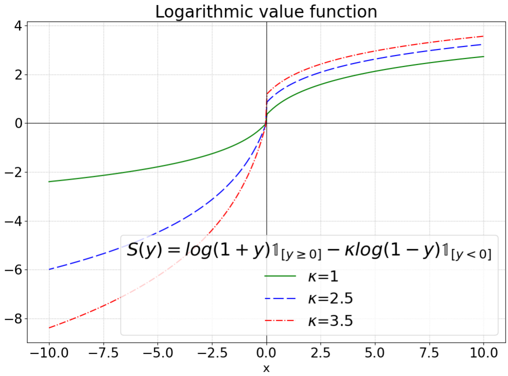
Let $\Gamma_{\text{EUT}} \stackrel{\text{def}}{=} (\mathcal{X}, \{f_i\}_{i \in \mathcal{I}}, \mathbb{P})$ be the EUT game, specifying a set of interdependent optimization problems where each player i maximizes the expected profit under the ground truth distribution

$$\forall i \in \mathcal{I}, \ \max_{x_i \in \mathcal{X}_i} \mathbb{E}_{\mathbb{P}}[f_i(x_i, \boldsymbol{x}_{-i}, \xi)] \tag{1}$$

A solution can be obtained with the classical concept of Nash Equilibrium.

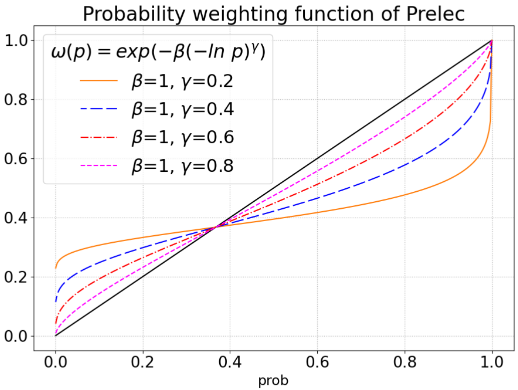
Prospect theory

Cumulative Prospect Theory (CPT) was proposed by (Kahneman and Tversky, 1979) as an advanced version of Prospect Theory



- κ is the risk aversion parameter: higher its value, greater is the appetite for risk.

Prospect theory



- $\beta < 1$: We overestimate low probabilities and underestimate high ones.
- $\beta \approx 1$: We bring the perception of probabilities closer to objective reality.
- $\gamma < 1$: Non-linear distortion is accentuated
- $\gamma \approx 1$: non-linear distortion is reduced, bringing the perception of low and high probabilities closer to their objective values

PT game

We formulate a PT-based game by exploiting a distortion function S_i and a continuous weighting function ω_i resulting in a distorted probability distribution $\mathbb{Q}_i = \omega_{i\#}\mathbb{P} \in \mathcal{P}(\Omega_i)$, where $\Omega_i \stackrel{\text{def}}{=} \{\omega_i(\xi) \mid \xi \in \Xi\}$.

PT game

Let $\Gamma_{\text{PT}} \stackrel{\text{def}}{=} (\mathcal{X}, \{S_i \circ f_i\}_{i\in\mathcal{I}}, \mathbb{Q}_i)$ be the PT game, specifying a set of interdependent optimization problems of the form

$$\forall i \in \mathcal{I},\ \max_{x_i\in\mathcal{X}_i} \underbrace{\mathbb{E}_{\mathbb{Q}_i}[\tilde{f}_i(x_i, \boldsymbol{x}_{-i}, \omega_i(\xi))]}_{\equiv \mathcal{O}_i} \tag{2}$$

where $\tilde{f}_i \stackrel{\text{def}}{=} S_i \circ f_i$ and $\zeta_i \stackrel{\text{def}}{=} \omega_i(\xi)$.

The composition $S_i \circ f_i$ yields a nonconvex non-smooth problem, the classical concept of Nash Equilibrium is not practically applicable.

PT game

- *Clarke Jacobian* is a set-valued mapping $\mathbf{J}_f^c : \mathbb{R}^m \rightrightarrows \mathbb{R}^{n\times m}$ defined as the convex hull of limits of jacobians at nearby differentiable points, e.g., $\mathbf{J}_f^c(x) = \mathrm{conv}\left\{ \lim_{k\to\infty} \mathbf{J}_f(x^k) \,|\, x^k \to x,\, x^k \in \Lambda \right\}$, where $\Lambda \subseteq \mathbb{R}^m$ is the full-measure set of points where f is differentiable (Clarke, 1990).
- And for a set $\mathcal{A} \subset \mathbb{R}^n$, $\mathcal{N}_{\mathcal{A}} : \mathcal{A} \rightrightarrows \mathbb{R}^n \;:\; x \mapsto \{v \in \mathbb{R}^n \,|\, \sup_{z\in\mathcal{A}} v^\top (z-x) \le 0\}$ denotes its normal cone.

Definition

A collective strategy $\boldsymbol{x}^\star$ is a Clarke's Local-Nash Equilibrium (CL-NE) for (2) if it satisfies

$$0 \in J^c_{\mathcal{O}_i, x_i}(x_i^\star, \boldsymbol{x}^\star_{-i}, \zeta) + \mathcal{N}_{\mathcal{X}_i}(x_i^\star), \quad \forall i \in \mathcal{I}. \tag{3}$$

PT game

Variational inequality

Then, (3) is equivalent to the following variational inequality:
A collective strategy $\boldsymbol{x}^\star$ is a CL-NE if and only if for each $i \in \mathcal{I}$ there exists $v_i \in J^c_{\mathcal{O}_i, x_i}(x_i^\star, \boldsymbol{x}^\star_{-i}, \zeta_i)$ such that

$$v^\top(\boldsymbol{x} - \boldsymbol{x}^\star) \leq 0, \quad \forall \boldsymbol{x} \in \mathcal{X}, \tag{4}$$

with $v = \mathrm{col}((v_i)_{i \in \mathcal{I}})$.

Game analysis

We assume the following:

(i) Ξ is compact.

(ii) For all $i \in \mathcal{I}$, $\mathcal{X}_i$ is nonempty, compact, and convex.

(iii) Each $f_i(\cdot, \boldsymbol{x}_{-i}, \xi)$ and $S_i(\cdot, R_i)$ are definable for almost all $\xi \in \Xi$ and $\boldsymbol{x}_{-i} \in \mathcal{X}_{-i}$, and locally Lipschitz.

Proposition (Path-differentiability)

Let $\tilde{f}_i(x_i, \boldsymbol{x}_{-i}, \zeta) = S_i \circ f_i(x_i, \boldsymbol{x}_{-i}, \zeta)_i$. Then, under these assumptions, $\tilde{f}_i(\cdot, \boldsymbol{x}_{-i}, \zeta_i)$ is path-differentiable for almost all $\zeta_i \in \Omega_i$ and $\boldsymbol{x}_{-i} \in \mathcal{X}_{-i}$.

Theorem

Let the previous assumptions hold. Then, the PT-based game (2) admits a CL-NE.

Game analysis

Definition

For a given risk tolerance $\alpha \in (0,1)$, $\boldsymbol{x}^\star \in \mathrm{SOL}(\Gamma_{\mathrm{PT}})$ and $\boldsymbol{y}^\star \in \mathrm{SOL}(\Gamma_{\mathrm{EUT}})$, the α-level Price of Irrationality is defined as

$$\mathrm{PoI}(\boldsymbol{x}^\star, \boldsymbol{y}^\star; \alpha) \overset{\text{def}}{=} \frac{\mathrm{CVaR}_\alpha^{\mathbb{P}}\left[\sum_{i\in\mathcal{I}} f_i(x_i^\star, \boldsymbol{x}_{-i}^\star, \xi)\right]}{\mathrm{CVaR}_\alpha^{\mathbb{P}}\left[\sum_{i\in\mathcal{I}} f_i(y_i^\star, \boldsymbol{y}_{-i}^\star, \xi)\right]}.$$

Local energy community

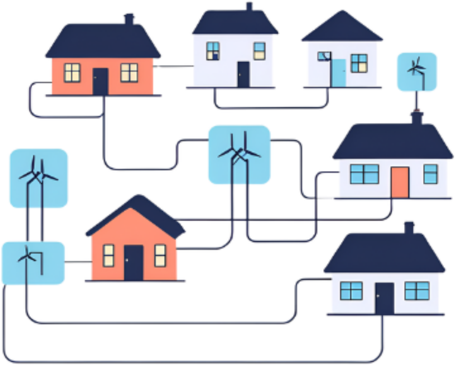


Figure: An energy community.

Settings

- $\mathcal{I} = \{1, \ldots, N\}$ strategic end users
- A supplier which proposes a two-part tariff :
  1. a fixed fee $\bar{P} > 0$ during the first part of the day
  2. a variable charge $P(\xi) = \bar{P} + c + \xi$ during the second part, where $c > 0$ is a constant offset and ξ a random variable
- Let $d^0 \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i=1}^{N} y_i$ be the community collective target, and $\sigma(\boldsymbol{x}) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i=1}^{N} x_i$ be the aggregate purchase
- To encourage social aspects in the community management (e.g., CO_2 emission reduction), the supplier imposes a penalty for deviation from the collective target.

EUT game

We consider agents' profit functions of the form

$$f_i(x_i, \sigma(\boldsymbol{x}), \xi) = U_i(\sigma(\boldsymbol{x})) - C_i(x_i, \xi)$$

where

- $Ui(\sigma(\boldsymbol{x})) = -a_i(\sigma(\boldsymbol{x}) - d^0)^2 + b_i$ captures the social benefit of the end user's behavior
- $C_i(x_i, \xi) = \bar{P}x_i + P(\xi)(y_i - x_i)$ is the trading cost

Let $\Gamma_{\text{EUT}} := (\mathcal{X}, \{f_i\}_{i\in\mathcal{I}}, \mathbb{P})$ be the EUT game, specifying a set of interdependent optimization problems where each player i maximizes the expected profit under the ground truth distribution

$$\forall i \in \mathcal{I}, \ \max_{x_i \in \mathcal{X}_i} \mathbb{E}_{\mathbb{P}}[f_i(x_i, \sigma(\boldsymbol{x}), \xi)]$$

PT game

1. The value function (Logarithmic value function) $S : \mathbb{R} \to \mathbb{R}$ describing the (behavioral) value of gains or losses

$$S_i(y) = \log(1+y)\mathbb{1}_{[y \geq 0]} - \kappa \log(1-y)\mathbb{1}_{[y<0]}$$

2. The weighting function induces a "distorted" distribution $\mathbb{Q}_i \stackrel{\text{def}}{=} \omega_{\#}\mathbb{P}$

$$\omega(p) = e^{-\beta(-\ln\ p)^{\gamma}}$$

We define a PT game as a tuple $\Gamma_{\text{PT}} \stackrel{\text{def}}{=} (\mathcal{X}, \{S_i \circ f_i\}_{i \in \mathcal{I}}, \mathbb{Q}_i)$ specifying a set of interdependent optimization problems of the form

$$\forall i \in \mathcal{I}, \ \max_{x_i \in \mathcal{X}_i} \mathbb{E}_{\mathbb{Q}_i}[\tilde{f}_i(x_i, \sigma(\boldsymbol{x}), \xi)]$$

where $\tilde{f}_i \stackrel{\text{def}}{=} S_i \circ f_i$, $\forall i$.

Main results

Difficulty

The composition $S_i \circ f_i$ yields a nonconvex non-smooth problem

We proved that

- There exists a unique symmetric Nash Equilibrium $\boldsymbol{x}^*$ solution of Γ_{EUT}
- The PT game admits a Clarke-Nash (CL-NE) Equilibrium
- For any CL-Nash Equilibrium $\boldsymbol{x}^*$ solution of Γ_{PT}, the unique Nash Equilibrium $\boldsymbol{y}^*$ solution of Γ_{EUT} when $N \to +\infty$, $\|\sigma(\boldsymbol{y}^*) - \sigma(\boldsymbol{x}^*)\| \to 0$
- Under (mild) assumptions, Stochastic Gradient Descent (SGD) converges to a CL-NE

Numerical results

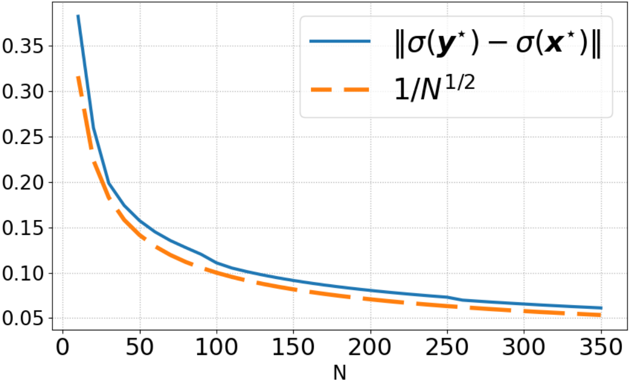


Figure: Distance between the aggregates $\sigma(\boldsymbol{y}^*)$ and $\sigma(\boldsymbol{x}^*)$ at the EUT and PT equilibrium

Numerical results

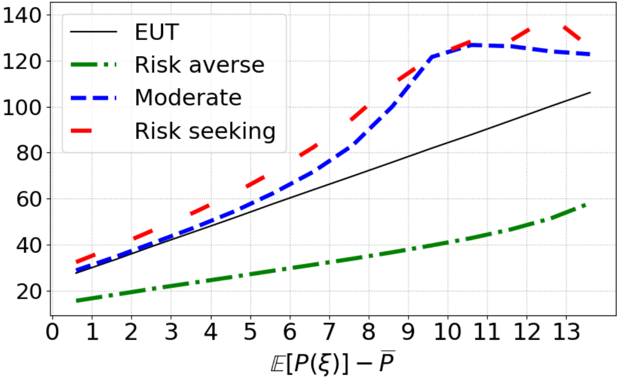


Figure: Optimal consumption behavior versus tariff variable charge favorability

Numerical results

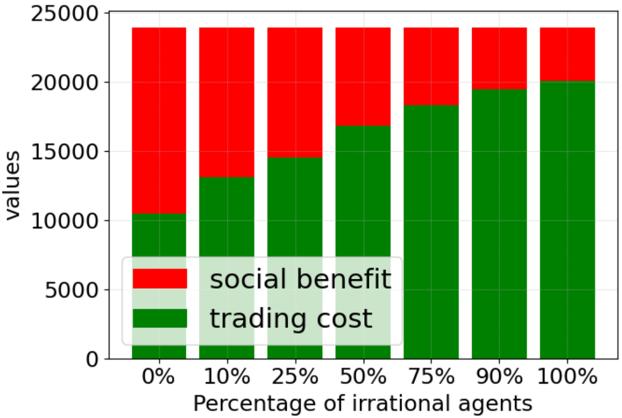


Figure: social benefit vs trading cost for different shares of irrational agents

Numerical results

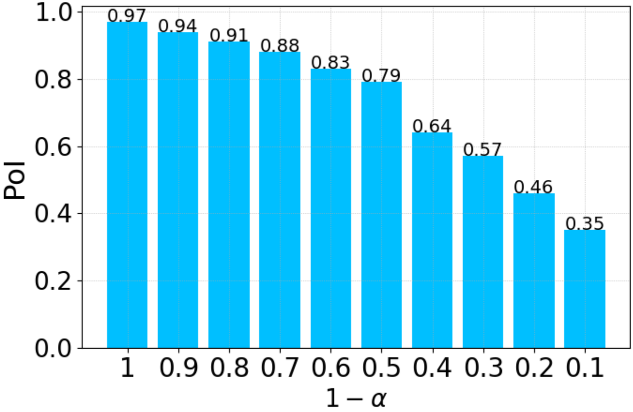


Figure: PoI as a function of the risk tolerance α in the PoI definition

Conclusion and future works

- ▷ We considered N-agents in noncooperative stochastic games, where agents display irrational behaviors due to their risk perception.
- ▷ We showed that these behaviors can be encoded in game theoretical formulations by means of prospect theory while retrieving guarantees on the existence and the algorithmic convergence to PT equilibria.
- ▷ Can we design incentives that steer the system to a desirable outcome and enable the principal to learn the agents' preferences?

Thank you!

♣ M. Fochesato, F. Pokou, H. Le Cadre, J. Lygeros, Noncooperative Games with Prospect Theoretic Preferences, arXiv/HAL Preprint, 2024.

♣ code GitHub `https://github.com/phdPokou/PT-Game`

♣ contact `fredy-vale-manuel.pokou@inria.fr`