Approximation of large bike/car sharing systems with product-form

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Outline

- Motivation
- Model and analysis
- On-going work

based on two papers:
F-Tibi (AAP'18) and F-Mohamed (Arxiv'+24).

Motivation: large bike/car sharing systems

Autolib'

Velib'

Flex Communauto Montreal

Bike/car-sharing networks

- M bikes/cars
- N stations
- users take a bike/car at any station, make a trip to another station then return it there.

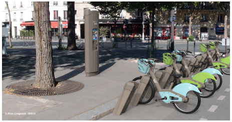
Two types of failures

no bike/car= rejection, no parking space

There is an imbalance

empty

saturated



well-balanced

Inhomogeneity Inevitable: popular stations $\neq$ popular destinations

Aim of the work: probability of failures

- no bike/car
- no parking space

First models: with product-form

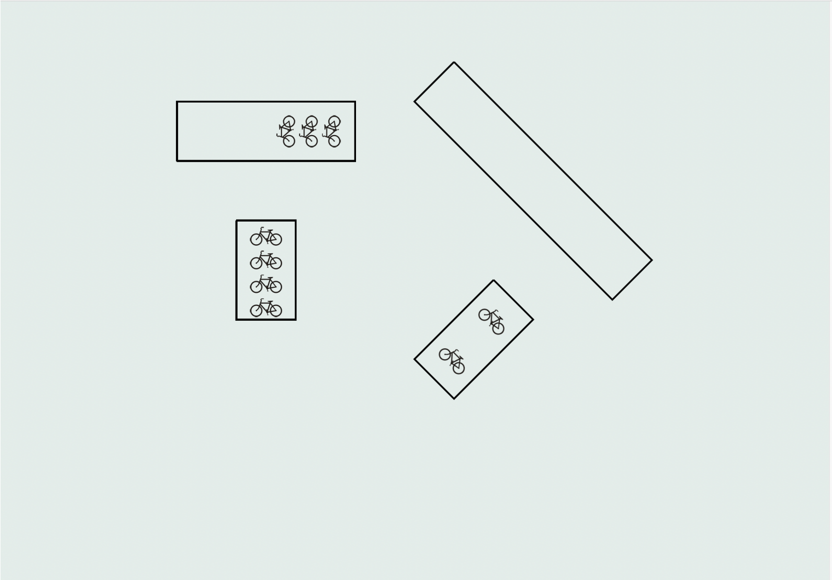
- unlimited capacity
  (Fayolle-Lasgouttes'96; George-Xia'10)

First models: with product-form

- unlimited capacity
  (Fayolle-Lasgouttes'96; George-Xia'10)
  - No blocking of bikes/cars (parking is always possible)
  - but rejection of users (empty stations)

  Assume:
  - Poisson p.p (μ_i) of user arivals at station i
  - trip from station i to destination j with probability q_{ij}
    with Q Markovian irreducible with invariant vector ν
  - trip duration from i to j exp($\mu_{[ij]}$)

A drawing.



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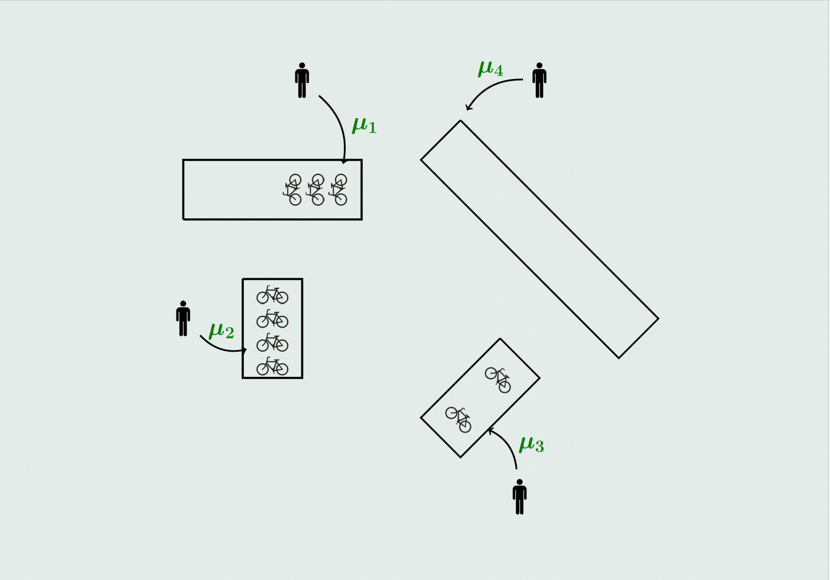

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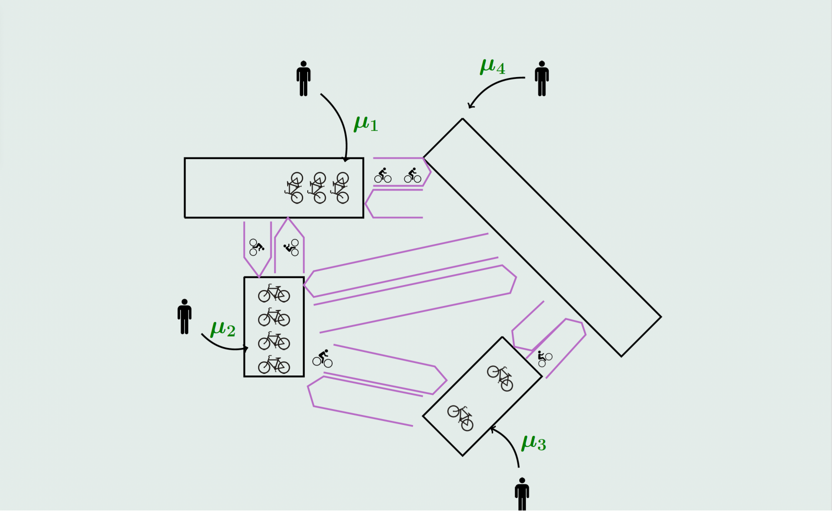

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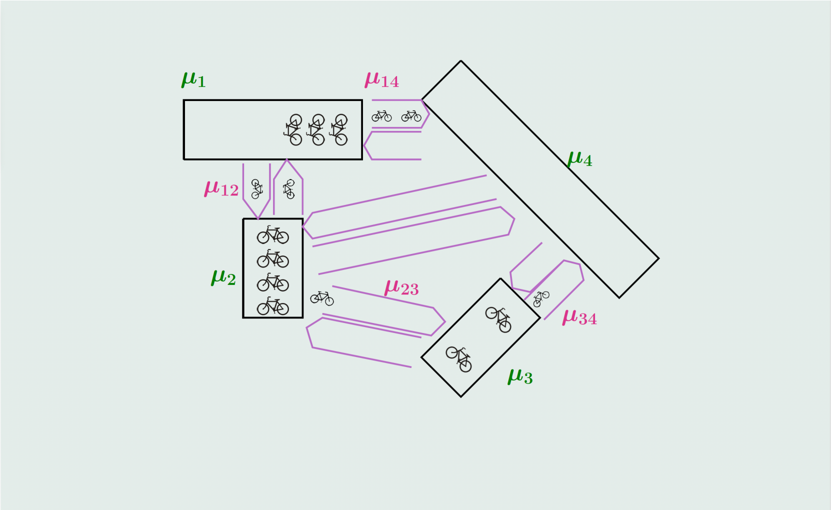

The underlying closed Jackson network

Forget the users. Look only at bikes. Bikes are:

- alternatively at stations and routes
- wait on a line at station: one-server mechanism
- simultaneously handled at routes: infinite-server mechanism

$\Rightarrow$ bikes = customers in a closed Jackson network with two types of nodes

- N one-server nodes : stations i
- N^2 infinite-server nodes : routes $[ij]$

$\Rightarrow$ Product-form invariant measure.

Our work: finite capacity stations

1. Modeling
   - propose finite capacity Jackson networks for bike systems
   - having product-form invariant measure
2. Analysis
   - at stationarity , in the large network limit ($M, N \to \infty$)
   - prove *asymptotic independence*, with explicit marginals
   - by proving the equivalence of ensembles

Equivalence of ensembles

Consider a product-form probability measure

$$\pi(x) = \mathbb{P}(\xi_1 = x_1, \ldots, \xi_N = x_N) = \frac{1}{Z} \prod_{l=1}^{N} \phi_l(x_l)$$

on non-product state-space

$$\mathcal{S} = \{x \in \mathbb{N}^N, \sum_{l=1}^{N} x_l = M, x_l \leq c_l \text{ if } c_l < +\infty\}$$

where $1 \leq c_l \leq +\infty$ and Z the normalizing constant.

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Then for any $\gamma > 0$,

$$\pi(x) = \frac{1}{Z(\gamma)} \prod_{l=1}^{N} \gamma^{x_l} \phi_l(x_l) = \mathbb{P}(\eta_1^\gamma = x_1, \ldots \eta_N^\gamma = x_N | \sum_{l=1}^{N} \eta_l^\gamma = M)$$

where $\eta_1^\gamma, \ldots, \eta_N^\gamma$ are independent r.v. with distributions

$$\mathbb{P}(\eta_l^\gamma = x_l) = \frac{1}{Z_l(\gamma)} \gamma^{x_l} \phi_l(x_l), \quad x_l \leq c_l \text{ if } c_l < +\infty.$$

Equivalence of ensembles

- Fixed k-dimensional marginal of π is

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- Choose γ (unique) such that $\mathbb{E}\left(\sum_{l=1}^N \eta_l^\gamma\right) = M$.
- Prove Local limit Theorem for independent non i.d. variables.

- If $N, M \to \infty$ with, for $b_N^2 = \mathrm{Var}(\eta_1^\gamma) + \ldots + \mathrm{Var}(\eta_N^\gamma)$, $b_N \to \infty$ and $\sum_{l=1}^N \mathbb{E}\left(|\eta_l^\gamma - \mathbb{E}(\eta_l^\gamma)|^3\right) = o(b_N^3)$ then

$$\mathbb{P}\left(\sum_{l=1}^N \eta_l^\gamma = M\right) \sim \mathbb{P}\left(\sum_{l=k+1}^N \eta_l^\gamma = M - \sum_{l=1}^k x_l\right) \sim (b_N\sqrt{2\pi})^{-1}.$$

Hence $\mathbb{P}(\xi_1 = x_1, \ldots, \xi_k = x_k) \sim \mathbb{P}(\eta_1^\gamma = x_1, \ldots, \eta_k^\gamma = x_k)$.

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References

A. I. Khinchin (1949) (*grand canonical* approximation)
R.L. Dobrushin and B. Tirozzi (1977) (*Gibbs measures*)
C. Kipnis and C. Landim (1999) (*zero range*)

Bike-sharing model with finite capacity

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Theorem (Product-form, F-Tibi'18)

Stationary distribution (for node states):

$$\pi(x) = \frac{1}{Z} \prod_{i=1}^{N} (\nu_i / \mu_i)^{x_i} \prod_{1 \leq i,j \leq N} \frac{(\nu_i q_{ij} / \mu_{[ij]})^{x_{[ij]}}}{x_{[ij]}!}$$

on space $\mathcal{S} = \{x \in \mathbb{N}^{N+N^2}, \sum_i x_i + \sum_{ij} x_{[ij]} = M, \text{ and } x_i \leq c_i\}$.

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Proof: Special case of blocking and rerouting policy
(Quadrat-Viot'80, Economou-Fakinos'98)

Bike-sharing model with finite capacity

Theorem (Large system, F-Tibi'18)

Assume $N, M \to \infty$, $M/N \to cst$, c_i **uniformly bounded**, *$\mu_i, \mu_{ij} = O(1)$ and $\nu_i = O(1/N)$.*
Then a finite number of stations and routes are asymptotically independent with limiting distributions

- *geom($\gamma\nu_i/\mu_i, c_i$) for station i*
- *Poisson $(\gamma\nu_i q_{ij}/\mu_{[ij]})$ for route $[ij]$*

where γ solves

$$\sum_i \mathbb{E}(geom(\gamma\nu_i/\mu_i, c_i)) + \sum_{ij} \gamma(\nu_i q_{ij}/\mu_{[ij]}) = M.$$

Proof: proving Local Limit Theorem for mixed truncated geometric and Poisson variables.
Remark: In a one-route version: independence only for stations .

Performance

Assuming known parameters $c_i, \mu_i, \mu_{ij}, q_{ij}$ (thus ν_i),

$$\text{choose } M \iff \text{choose } \gamma.$$

Optimize a performance parameter.
Example: total rate of rejection+blocking

$$\tau = \sum_i \mu_i \mathbb{P}(\xi_i = 0) + \sum_{ij} \mu_{[ij]} \mathbb{E}(\xi_{[ij]} \mathbb{1}_{\xi_j = c_j})$$

is asymptotically

$$\tau \sim \sum_i \mu_i \frac{1 + (\gamma\nu_i/\mu_i)^{c_i+1}}{\sum_{i=0}^{c_i} (\gamma\nu_i/\mu_i)^k} \text{ and can be minimized wrt } \gamma.$$

homogeneous: $\tau \sim 2\mu \frac{N}{c+1}$, $M_{opt} \sim \left(\frac{c}{2} + \frac{\mu_S}{\mu_R}\right) N$ (F-Gast'16).

On going work: other tractable models

Product-form and equivalence of ensembles for

- car reservation (a *first* model for free-floating car-sharing)
  (F-Popescu et al.'21, F-Mohamed+24)
  (truncated geom $\leftarrow$ truncated sum Poisson & geom)

Product-form for

- *our new* model for free-floating car-sharing
  Jackson with two classes of customers:
  - internal (shared cars)
  - *external* (private cars)

  car-sharing in a fast-varying random environment
  (mean-field F-Mohamed-Rigonat'24)
- bike/e-bike system (Velib' now)

No product-form for

- parking space reservation (Autolib'):
  $\Longleftrightarrow$ Jackson Network blocking policy+non-reversible routing.

Conclusion and future work

- a direct approach for the large stationary behavior when product-form
- a probabilistic proof
- to prove equivalence of ensembles in such complex frameworks
    - *our* model with random envirومment (free-floating car-sharing)
      scaling ?
    - bike/e-bike system
      to prove Local Limit Theorem in dimension 2?

Thank you for your attention!

Local Limit Theorem

Let $(\eta_{l,N})_{1\leq l\leq N}$ be independent $\mathbb{Z}$-valued r.v. and

$$S_N=\sum_{l=1}^{N}\eta_{l,N},\quad a_N=\mathbb{E}(S_N) \text{ and } \quad b_N^2=\mathrm{Var}S_N.$$

Assume

1. $\lim_N b_N=+\infty$,
2. $\exists\delta>0,\sum_{l=1}^{N}\mathbb{E}(|\eta_{l,N}-m_{l,N}|^{2+\delta})=o(b_N^{2+\delta})$ as $N\to\infty$,
3.

$\exists\Phi\in L^1(\mathbb{R}),\forall N\geq 1,t\in[-\pi,\pi],\mathbb{E}(|\eta_{l,N}-m_{l,N}|^{2+\delta})\leq\Phi(b_Nt)$.

then

$$\lim_{N\to\infty}\sup_{k\in\mathbb{Z}}\left[b_N\sqrt{2\pi}\mathbb{P}(S_N=k)-exp(-\frac{(k-a_N)^2}{b_N^2})\right]=0.$$