Routing Optimization for Asynchronous Federated Learning on Heterogeneous Resources

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Atelier en Évaluation des Performances – December 2, 2024

Outline

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Federated Learning (FL)

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Each client $i \in \{1, 2, \ldots, n\}$:

- Has local objective function $f_i(w) = \mathbb{E}_{(x,y) \in \mathcal{D}_i}[\ell_i(NN(x, w), y)]$.
- Approximates the gradient $\nabla_w f_i(w)$ with a **stochastic gradient estimate** $g_i(w)$.

Synchronous FL

Decentralized SGD: The central server sends (gradient estimation) tasks to $m \leq n$ clients, waits until all m clients answer, and updates the model parameter.

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Central server
Parameter w_0

Client 1

Client 2

Client 3

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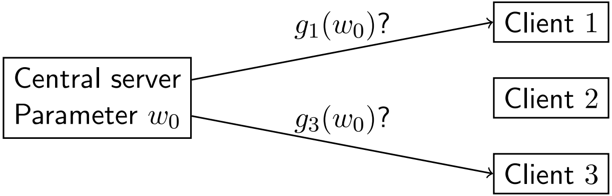

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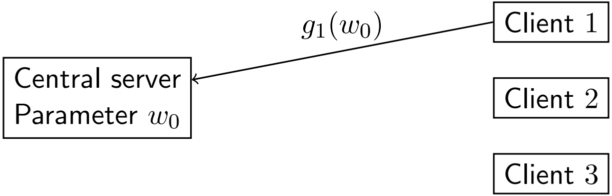

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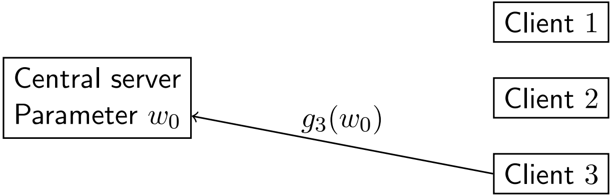

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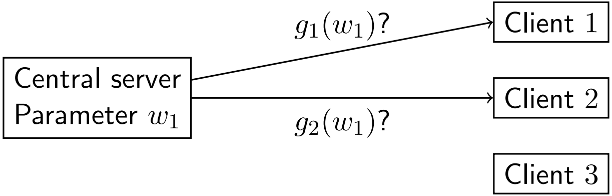

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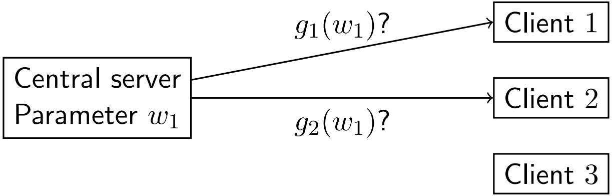

Problems:
1. The straggler effect
2. Coordination

Asynchronous FL

Asynchronicity: The central server updates the model parameter *au fil de l'eau* (that is, as gradient estimates are received)

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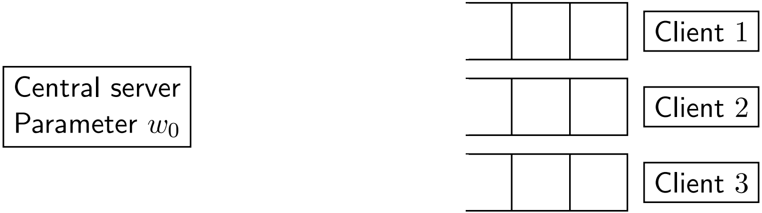

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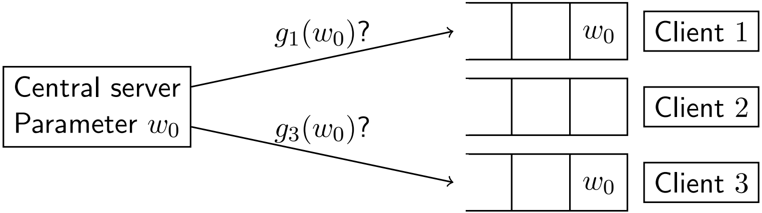

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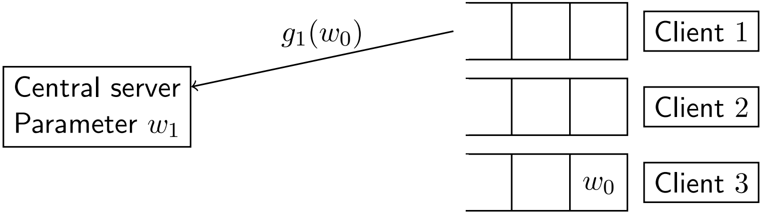

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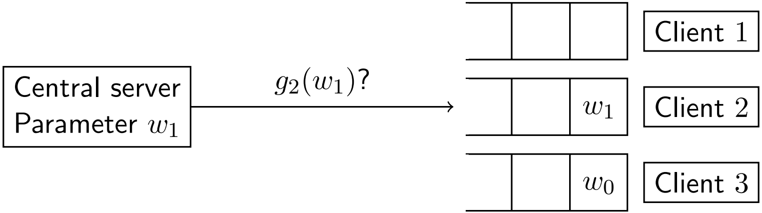

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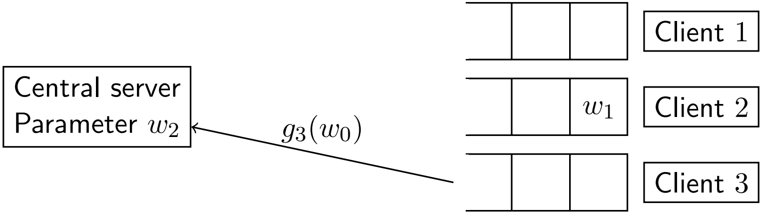

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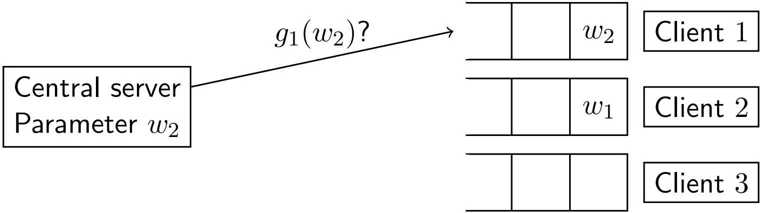

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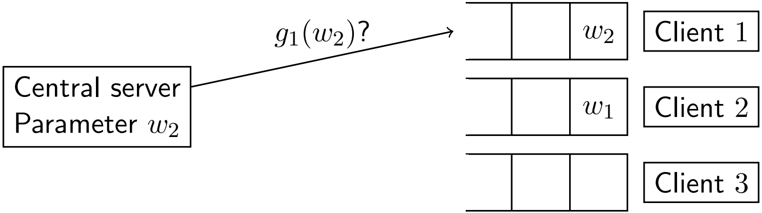

Challenge: Find a tradeoff between two antagonistic performance objectives

- Staleness, measured by the **mean relative delay**
- Speed, measured by the **throughput**, i.e., mean time between successive updates

Related works on FL

- **Introduction of FL** (Konečný, McMahan, and Ramage 2015; McMahan et al. 2017; Wang et al. 2020; Qu, Song, and Tsui 2022; Makarenko et al. 2022; Mao et al. 2022; Tyurin and Richtárik 2022)
- **Limitations of synchronous FL** (Xie, Koyejo, and Gupta 2020; Chen et al. 2021)
- **Algorithms for asynchronous FL**
  - `FedAsync` and its variants (Xie, Koyejo, and Gupta 2020; Chen et al. 2020; Xu et al. 2023)
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  - `Generalized AsyncSGD` (Leconte et al. 2024)
  - `AsGrad` (Islamov, Safaryan, and Alistarh 2024)
- **Performance analysis of asynchronous FL**
  - Impact of dataset heterogeneity (Chen et al. 2020; Chen et al. 2021; Xu et al. 2023; Koloskova, Stich, and Jaggi 2022; Agarwal, Joshi, and Pileggi 2024)
  - Impact of queuing dynamics (Leconte et al. 2024), (Agarwal, Joshi, and Pileggi 2024)

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Outline

Generalized AsyncSGD

Algorithm 1: `Central server`

Input: $T = \#$ model updates, $n = \#$ clients, $m = \#$ tasks,
$p =$ routing vector, $\eta =$ learning rate

`// Initialization`
Initialize model parameter w_0 randomly;
Send m (gradient estimation) tasks to the clients based on w_0;
`// End Initialization`
for $t = 0, \dots, T$ **do**
  Receive stochastic gradient $g_{C_t}(w_{I_t})$ from a client C_t;
  Update $w_{t+1} \leftarrow w_t - \frac{\eta}{np_{C_t}} g_{C_t}(w_{I_t})$;
  Sample a new client A_{t+1} with $\mathbb{P}(A_{t+1} = i) = p_i$;
  Send new model w_{t+1} to A_{t+1};
end

Algorithm 2: `Client` i

Input: Queue of tasks, $\mathcal{D}_i =$ local dataset

if *queue is not empty* **then**
  Take received parameter w from queue in FIFO order;
  Compute gradient estimate $g_i(w)$;
  Send the gradient estimate to CS;
  Repeat;
end

Assumptions

Queuing assumptions

- Tasks are processed in first-in-first-out order at each client.
- Processing times at client i are i.i.d. exponentially distributed with rate $\mu_i > 0$.
- Neglected: processing times at the central server, communication times.

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Dataset assumptions

- Lower boundedness: The objective function f is bounded from below by some f^*.
- Gradient smoothness: f_i has an L-Lipschitz continuous gradient, for each client i.
- Stochastic gradient properties: $g_i(w)$ is unbiased with variance bounded by $\sigma^2 > 0$, for each i.
- Bounded client heterogeneity: $\mathbb{E}[\|\nabla f(w) - \nabla f_i(w)\|^2] \leq \zeta^2$ for each client i, parameter w.

Definitions and notation

For each $t \in \{1, 2, \dots, T\}$, **step** t proceeds as follows:

- A task is assigned to client A_t, with $\mathbb{P}(A_t = i) = p_i$.
- A task is completed at client C_t.
- The system state at the end of this step is $X_t^{m-1} = (X_{1,t}^{m-1}, X_{2,t}^{m-1}, \dots, X_{n,t}^{m-1})$, with

$$X_{i,t}^{m-1} = X_{i,t-1}^{m-1} + \mathbb{1}[A_t = i] - \mathbb{1}[C_t - i].$$

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The **relative delay** at time t and client i is defined as follows:

$$D_{i,t} = \mathbb{1}[A_t = i] R_{i,t}, \text{ where } R_{i,t} = \min \left\{ r \in \mathbb{N} : \sum_{s=t}^{t+r} \mathbb{1}[C_s = i] = X_{i,t-1}^{m-1} + \mathbb{1}[A_t = i] \right\}.$$

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We assume the Markov chain $(X_t^{m-1}, t \geq 0)$ is **stationary**, hence we drop the t index.

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Results from (Leconte et al. 2024)

- The system dynamics $(X_t^{m-1}, t \geq 0)$ are given by a **Jackson network** (Jackson 1957).

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- The system dynamics $(X_t^{m-1}, t \geq 0)$ are given by a **Jackson network** (Jackson 1957).
- **Upper bound on the empirical mean of the norm-square of the gradient of f:**
  There is $\eta_{\mathsf{max}}(p) > 0$ so that, for any $\eta \in (0, \eta_{\mathsf{max}}(p))$,

$$\frac{1}{T+1} \sum_{t=0}^{T} \mathbb{E}[\|\nabla f(w_t)\|^2] \leq G,$$

where $G = \dfrac{A}{\eta(T+1)} + \dfrac{\eta L B}{n^2} \displaystyle\sum_{i=1}^{n} \dfrac{1}{p_i} + \dfrac{\eta^2 L^2 B m}{n^2} \displaystyle\sum_{i=1}^{n} \dfrac{\mathbb{E}[D_i]}{p_i^2}$.

The variables A, B, and L depend on the learning problem and are estimated heuristically.

Mean relative delay

Theorem

The mean relative delay $\mathbb{E}[D_i]$ *and its gradient are given by*

$$\mathbb{E}[D_i] = \mathbb{E}[X_i^{m-1}], \qquad \frac{\partial \mathbb{E}[D_i]}{\partial p_j} = \frac{1}{p_j} \operatorname{Cov}[X_i^{m-1}, X_j^{m-1}], \qquad i, j \in \{1, 2, \ldots, n\}$$

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where for each $i, j \in \{1, 2, \ldots, n\}$,

$$\mathbb{E}[X_i^{m-1}] = \sum_{k=1}^{m-1} \left(\frac{p_i}{\mu_i}\right)^k \frac{Z_{n,m-1-k}}{Z_{n,m-1}}, \quad \mathbb{E}[X_i^{m-1} X_j^{m-1}] = \sum_{\substack{k,\ell=1 \\ k+\ell \le m-1}}^{m-1} \left(\frac{p_i}{\mu_i}\right)^k \left(\frac{p_j}{\mu_j}\right)^\ell \frac{Z_{n,m-1-k-\ell}}{Z_{n,m-1}},$$

and the $Z_{n,\mathfrak{m}}$'s are computed using Buzen's algorithm.

Numerical results

$$G = \frac{A}{\eta(T+1)} + \frac{\eta L B}{n^2} \sum_{i=1}^{n} \frac{1}{p_i} + \frac{\eta^2 L^2 B m}{n^2} \sum_{i=1}^{n} \frac{\mathbb{E}[D_i]}{p_i^2}$$

Procedure

- Optimize G using the Adam gradient-descent algorithm, initialized with p^{uniform}.
- Simulate the dynamics of the Jackson network.
- Evaluate `Generalized AsyncSGD` on image classification tasks using the Fashion-MNIST and CIFAR-10 datasets, each containing 10 equally distributed image classes.

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Datasets

- **Homogeneous:** Data is distributed i.i.d. across clients, and all clients have the same number of data points.
- **Heterogeneous:** For each k, we sample a vector $p_k \sim \text{Dir}_n(0.5)$, where $p_{k,j}$ is the proportion of class-k instances allocated to client j and $\text{Dir}_n(\beta)$ the Dirichlet distribution with dimension n and concentration parameter $\beta > 0$.

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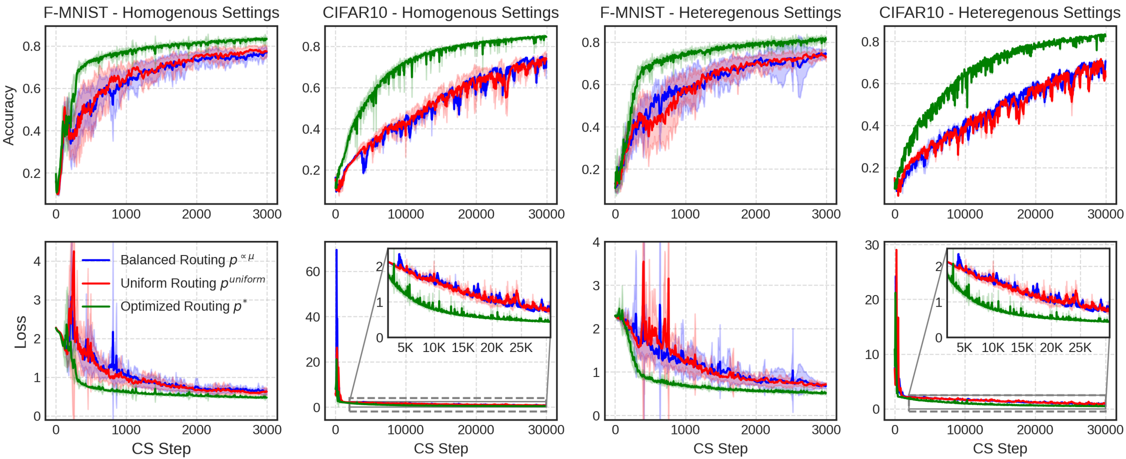

Figure: Performance on the validation set. Parameters: $n = 20$, $m = 100$, $\mu_i = e^{i/100}$, $\eta = 0.01$, $L = 1$.

Outline

Performance bound and mean delay

Conjecture

There exists $\eta_{max}(p) > 0$ *such that, for any* $\eta \in (0, \eta_{max}(p))$,

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left[\tau_t\|\nabla f(w_t)\|^2\right] \le H,$$

where $H = \dfrac{\tilde{A}}{\eta\lambda(p)} + \dfrac{\eta L B}{\lambda(p)n^2}\displaystyle\sum_{i=1}^{n}\frac{1}{p_i} + \dfrac{\eta^2 L^2 B m}{\lambda(p)n^2}\displaystyle\sum_{i=1}^{n}\frac{\mathbb{E}[X_i^m]}{p_i^2}.$

The variables $\tilde{A}$, B, *and* L *depend on the learning problem, and*

$$\lambda(p) = \sum_{i=1}^{n}\mu_i\mathbb{P}(X_i^m > 0) = \frac{Z_{n,m-1}}{Z_{n,m}}.$$

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Procedure

- Optimize H using the Adam gradient-descent algorithm, initialized with p^{uniform}.
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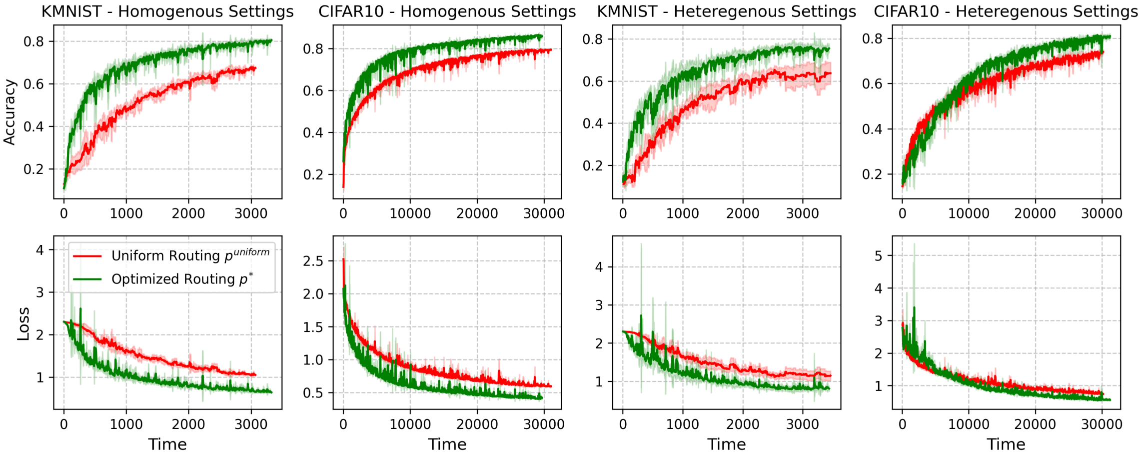

Figure: Performance on the validation set. Parameters: $n = m = 30$, $\mu_{1\ldots10} = \frac{1}{100}$, $\mu_{11\ldots20} = \frac{1}{10}$, $\mu_{21\ldots30} = 1$, $\tilde{A} = 15$, $L = 1$, $\sigma = 3$, $G = 10$, $\eta = 0.01$.

Conclusion

Contributions

- Computed mean relative delay by applying (a variant of) Little's law in Jackson networks.
- Computed its gradient by rewriting it as an expectation.
- Introduced a new clock-time-aware performance metric and derive an upper-bound.
- Designed a gradient-descent algorithm to optimize the upper-bounds.
- Numerically evaluated performance on image-classification tasks (Fashion-MNIST, KMNIST, and CIFAR-10 datasets).

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Future works:
1. More tailored scheduling that accounts for staleness?
2. State-dependeng routing mechanism?

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To appear soon on HAL and arXiv

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$$G = \frac{A}{\eta(T+1)} + \frac{\eta L B}{n^2} \sum_{i=1}^{n} \frac{1}{p_i} + \frac{\eta^2 L^2 B m}{n^2} \sum_{i=1}^{n} \frac{\mathbb{E}[D_i]}{p_i^2}$$

Dependency on the number m of tasks?

- $\mathbb{E}[D_i]$ and G are increasing with m, and

$$\sum_{i=1}^{n} \mathbb{E}[D_i] = \sum_{i=1}^{n} \mathbb{E}[X_i^{m-1}] = m - 1.$$

- Performance is best with $m = 1$ task!

Dependency on the routing policy p?

- Second term minimized by p^{uniform}
- Third term is non-monotonic

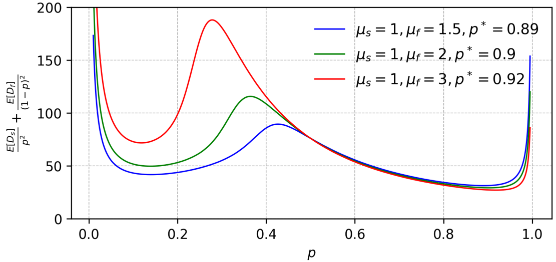

Figure: Third term of the bound G vs. the routing probability to the slowest client, in a toy example with $n = 2$ clients and $m = 20$ tasks, for various speed vectors $\mu = (\mu_s, \mu_f)$.

Uniform routing p^{uniform}

- The default in many applications: $p_i^{\text{uniform}} = \frac{1}{n}$.
- $G(p^{\text{uniform}}) = \dfrac{A}{\eta(T+1)} + \eta L B + \eta^2 L^2 B m(m-1)$.

Proportional routing $p^{\propto \mu}$

- A load-balancing heuristic: $p_i^{\propto \mu} = \frac{\mu_i}{\sum_j \mu_j}$.
- $G(p^{\propto \mu}) = \dfrac{A}{\eta(T+1)} + \dfrac{\eta L B |\mu|}{n^2} \sum_i \dfrac{1}{\mu_i} + \dfrac{\eta^2 L^2 B m(m-1)|\mu|^2}{n^3} \sum_i \dfrac{1}{\mu_i^2}$.

$\mathbb{E}[S_i] = \mathbb{E}[X_i^m]$: Consequences

$$H = \frac{\tilde{A}}{\eta\lambda(p)} + \frac{\eta L B}{\lambda(p) n^2} \sum_{i=1}^{n} \frac{1}{p_i} + \frac{\eta^2 L^2 B m}{\lambda(p) n^2} \sum_{i=1}^{n} \frac{\mathbb{E}[X_i^m]}{p_i^2}$$

Non-monotonicity with respect to the number m of tasks

- Contrary to G, H is not increasing in m, as it accounts for the duration of a step.
- H is minimized by some $m^* > 1$.
- m^* decreases with the learning rate η.

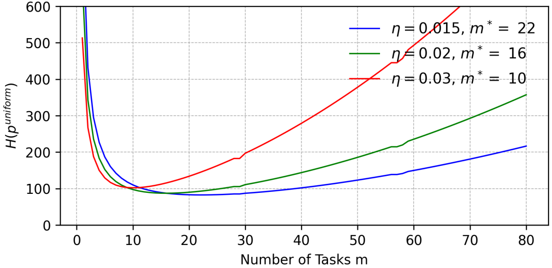

Figure: Bound $H(p^{\text{uniform}})$ as a function of the number m of tasks for different values of the step size η. The system consists of $n = 50$ clients, and the service speeds are $\mu_i = e^{i/100}$.