Phase Transitions in Unimodular Trees

François Baccelli

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Survey of joint research with M.O. Haji Mirsadeghi, O. Gascuel, S. Khaniha, A. Khezeli, B. Roy-Choudhury, & A. Sodre

Outline

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Unimodular Discrete Spaces

- A **discrete (metric) space**: D
- A **rooted** discrete space: $[D, o] \in \mathcal{D}_*$
  - o: the **origin** or the **root**
  - Every ball $N_r(o)$ contains finitely many points (boundedly finite)
- A **random** rooted discrete space: $[\boldsymbol{D}, \boldsymbol{o}]$
- **Unimodular** if (heuristically) "*$\boldsymbol{o}$ is uniformly distributed in $\boldsymbol{D}$*"

$$\forall g : \mathbb{E}\left[\sum_{v \in \boldsymbol{D}} g[\boldsymbol{D}, \boldsymbol{o}, v]\right] = \mathbb{E}\left[\sum_{v \in \boldsymbol{D}} g[\boldsymbol{D}, v, \boldsymbol{o}]\right] \quad \text{(mtp)}$$

Outline

1. Unimodular Random Graphs (with M.O. Haji-Mirsadeghi & A. Khezeli)
   - Examples
   - Covariant Processes
   - Poincaré recurrence lemma
2. Vertex and point shifts (with M.O. Haji Mirsadeghi & A. Khezeli)
3. EFTs Everywhere
4. Conclusions

Example 1: Palm probabilities

Palm version of stationary point processes

- A random discrete subsets of $\mathbb{R}^k$
- Distribution invariant under translations
- Conditioned on containing the origin

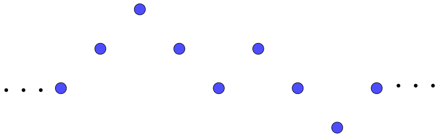
All covariant graphs on stationary point processes under their Palm version are unimodular

Example 2: Graph of a process with stationary increments

$\{X_n\}_{n\in\mathbb{Z}}$ a **stationary stochastic process** with values on $\mathbb{R}^d$

$$
\begin{aligned}
S_0 &= 0, \quad S_i - S_{i-1} = X_{i-1},\ i \in \mathbb{Z} \\
S_i &= \sum_{n=0}^{i-1} X_n,\ i > 0, \quad S_i = -\sum_{n=-i}^{-1} X_n,\ i < 0
\end{aligned}
$$

The **graph** $[\mathrm{G}, (0,0)]$ with $\mathrm{G} = \{i, S_i\}_i$ **is unimodular**



Example 3: Finite graph limits

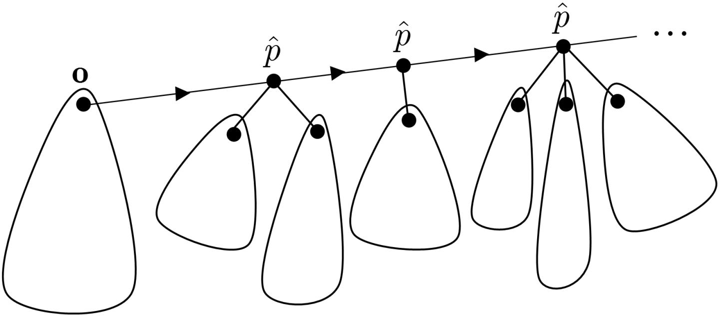
An **a.s. finite random graph with a root picked at random** in the set of vertices is a unimodular rooted discrete space for graph distance

A **local weak limit** of such a random rooted graph is unimodular **[Aldous Lyons 07]**

Canopy Tree Example

- Binary tree with say N generations
- Choose a root o_N at random and let N tend to infinity
- The local weak limit is the **Canopy EFT** which has infinitely many generations, numbered like $\mathbb{N}$
- The index (w.r.t. the generation of the root) of the last generation in this limit is geometrically distributed with parameter $1/2$

Example 4 Eternal Galton-Watson Tree



- π distribution on $\{0, 1, 2, 3, \ldots\}$ with mean $m(\pi) = 1$ and $\pi(1) < 1$
- Size-biased distribution of π, $\hat{\pi}(k) = k\pi(k)$ for all $k \geq 0$

Outline

Covariant process

D: a deterministic discrete space which is boundedly-finite

Marking of D: a function from $D \times D$ to Ξ a measure space

Covariant process $\boldsymbol{Z}$ with values in Ξ
map that assigns to every D a random marking $\boldsymbol{Z}_D$ s.t.

1. $\boldsymbol{Z}$ is compatible with isometries: $\forall$ isometries $\rho : D_1 \to D_2$, $\boldsymbol{Z}_{D_1} \circ \rho^{-1}$ of D_2 has the same distribution as $\boldsymbol{Z}_{D_2}$
2. For every measurable subset $A \subseteq D'_*$, the function

$$[D, o] \mapsto \mathbb{P}\left[[D, o; \boldsymbol{Z}_D] \in A\right]$$

is measurable

Marked Spaces

Lemma
Let $[\boldsymbol{D}, \boldsymbol{o}]$ be a unimodular discrete space.
If $\boldsymbol{Z}$ is a covariant process on $\boldsymbol{D}$, then $[\boldsymbol{D}, \boldsymbol{o}; \boldsymbol{Z}_{\boldsymbol{D}}]$ is also unimodular

Examples:

- **Deterministic**: in a one ended tree, mark each edge incident to a node with its direction to the end
- **Random**: in a graph, declare the directed edge from a node to one of its neighbors independently for all neighbors but with a probability that depends on the degree of the node

Marked unimodular graphs are referred to as **networks**

Covariant subset, Intensity

Covariant subset:
Set $\boldsymbol{S}$ of points with mark 1 in some $\{0,1\}$-valued covariant process

Intensity:
If $[\boldsymbol{D}, \boldsymbol{o}]$ is a unimodular discrete space, then the **intensity** of $\boldsymbol{S}$ in $\boldsymbol{D}$ is defined by $\rho_{\boldsymbol{D}}(\boldsymbol{S}) := \mathbb{P}[\boldsymbol{o} \in \boldsymbol{S}_{\boldsymbol{D}}]$

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Poincaré Recurrence Lemma

Unimodular Poincaré recurrence lemma
[F.B. Haji-Mirsadeghi, Khezeli 18], [Lovász 20]
Let $[\boldsymbol{G}, \boldsymbol{o}]$ be a unimodular network s.t. $V(\boldsymbol{G})$ is a.s. infinite.
Then any covariant subset S of $V(G)$ is a.s. **either empty or infinite**:

$$\mathbb{P}\left[\#S_{\boldsymbol{G}} \in \{0, \infty\}\right] = 1$$

Several other **unimodular extensions of the theory of measure preserving transformations** have been discussed

Outline

Dynamics on a Discrete Random Structure

Dynamics: point-shifts, vertex-shifts

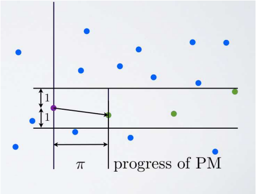
1. **Select one node** (point/vertex)
   - **in** the discrete rooted structure
   - **as a function of** the discrete rooted structure
2. **Move the origin/root there**

Point-shifts in the literature

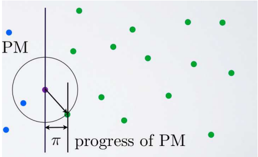
- **[Mecke 75]**: mass stationarity
- **[Thorisson 00]**: this terminology
- **[Holroyd & Peres 05]**: allocation rule

Examples of Point-Shifts on Poisson Point Processes

Strip Routing PS on $\mathbb{R}^2$
[Ferrari, Landim & Thorisson 05]



Directional PS on $\mathbb{R}^2$
[F.B. & Bordenave 07]
(radial spanning tree)



Locally defined "navigation rule" on the support of the point process

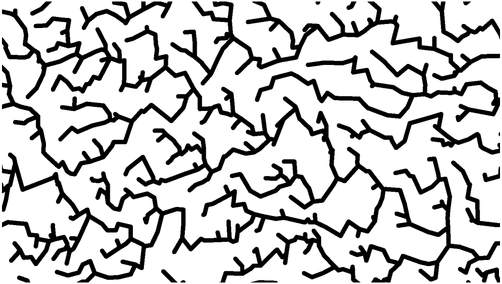
Mecke's Invariance Theorem

Theorem [J. Mecke 75]
Let f be a point-shift on a stationary point process Φ. Then θ_f preserves the Palm distribution of Φ if and only if f is almost surely bijective on the support of Φ

Unimodular Mecke Theorem [B-H-K 18]
Let f be a vertex-shift and $[\boldsymbol{G}, \boldsymbol{o}]$ be a unimodular network. Then θ_f preserves the distribution of $[\boldsymbol{G}, \boldsymbol{o}]$ if and only if $f_{\boldsymbol{G}}$ is almost surely bijective on V_{G}

Vertex/Point-Shift Graph

f-**Graph** of (point/vertex)-shift f:
directed graph with vertices $V(\mathrm{D})$ and edges $\{(v, f(v))\}_{v \in V(\mathrm{D}}$



Euclidean instance: **union of all orbits**, starting from all v

Foliation of a Point/Vertex-Shift

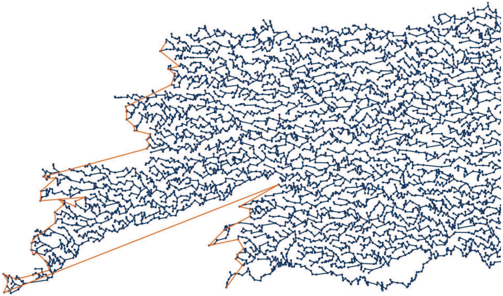
Discrete analogue of the **stable manifold of a smooth dynamics**
Foil partition of the set of points equivalence relation

$$x \sim_f y \Leftrightarrow \exists n \in \mathbb{N}; f^n(x) = f^n(y)$$

f-**foliation**: $\mathcal{L}^f$, equivalence classes of the set of nodes w.r.t. $\sim_f$
The partition $\mathcal{L}^f$ is a **refinement** of the partition $\mathcal{C}^f$
The foil of the root is a **unimodular discrete space**

Illustration: f-graph and foliation of strip PS on a P.P.P.

Φ Poisson P.P. in $\mathbb{R}^2$ Strip Point-Shift
The f-Graph has a.s. one component



Foil of origin

Outline

1 Unimodular Random Graphs (with M.O. Haji-Mirsadeghi & A. Khezeli)

2 Vertex and point shifts (with M.O. Haji Mirsadeghi & A. Khezeli)
- Phase Classification theorem
- Family Trees and EFTs

3 EFTs Everywhere

4 Conclusions

Phase Classification

Theorem [F.B., Haji-Mirsadeghi, Khezeli 18]
Let f be a covariant point-shift on a unimodular random discrete space $[\mathrm{D}, \mathrm{o}]$.
Almost surely the component C of the origin is a unimodular discrete space that belongs to **one of the following three phases**:

1. $\mathcal{F}/\mathcal{F}$-**Phase**: C is finite, each of its f-foils is finite
2. $\mathcal{I}/\mathcal{F}$-**Phase**: C is a two-end directed tree with all its f-foils finite
3. $\mathcal{I}/\mathcal{I}$-**Phase**: C is a one-end directed and all its f-foils are infinite

Proof based on the Unimodular Poincaré Recurrence Lemma

F/F Phase

Class $\mathcal{F}/\mathcal{F}$:

- C is finite (no infinite end)
- each of its f-foils is finite

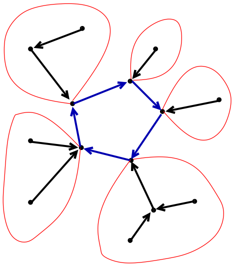
$\#$ **foils**: $1 \leq n < \infty$

C has a **unique cycle** of length n

Vertices of this cycle: $f^{\infty}(C)$

Example

nearest neighbor point-shift on the P.P.P.



$\mathcal{I}/\mathcal{F}$ Phase

Class $\mathcal{I}/\mathcal{F}$:

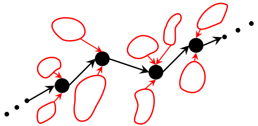
- C is infinite
- Each of its f-foils is finite

C is a **unimodular directed tree**

Each foil has a **junior foil**

$f^{\infty}(C)$: unique **2 end path**

Example: later in the talk



Infinite number of descendants
Finite foil

$\mathcal{I}/\mathcal{I}$ Phase

Class $\mathcal{I}/\mathcal{I}$:

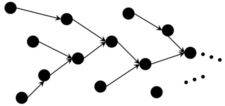
- C is infinite
- All its f-foils are infinite
- Foils order like $\mathbb{N}$ or like $\mathbb{Z}$

C is a **one-ended unimodular tree**

$f^{\infty}(C) = \emptyset$

Examples:

- **Strip PS** on 2 dim. P.P.P.



Finite number of descendants
Infinite foil

Outline

Family Trees

Family Tree (FT):
Directed tree T in which the out-degree of each vertex is at most 1

Eternal Family Tree (EFT)
When the out-degrees of all vertices are exactly 1

Rooted FT or EFT:
as above

Parent:
For a vertex v with one outgoing edge vw, $F(v) := w$

Descendants:

- **of generation n** of x: $D_n(x) := \{y : F^{(n)}(y) = x\}$, $d_n(x) := \# D_n(x)$
- Tree of **descendants** $D(x)$ of x, the subtree with vertices $\cup_{n=0}^{\infty} D_n(x)$

Random Family Tree

Random Family Tree:
a random network with values in $\mathcal{T}_*$ almost surely

Unimodular FT:
defined as above via mtp

Proper random FT:
a random $\mathtt{FT}$ in which $0 < \mathbb{E}[d_n(\boldsymbol{o})] < \infty$ for all $n \geq 0$

Properties of Unimodular EFTs

Proposition

Let $[\boldsymbol{T}, \boldsymbol{o}]$ be a unimodular $\texttt{FT}$

(i) If $\boldsymbol{T}$ has infinitely many vertices a.s., then it is eternal a.s. Moreover, $[\boldsymbol{T}, \boldsymbol{o}]$ is a proper random $\texttt{EFT}$, with
   - $\mathbb{E}[d_n(\boldsymbol{o})] = 1$ for all $n \geq 0$
   - $\mathbb{E}[d(\boldsymbol{o})] = \infty$

(ii) If $\boldsymbol{T}$ is finite with positive probability, then $\mathbb{E}[d_n(\boldsymbol{o})] < 1$ for all $n > 0$

The subtree of descendants of the root of an EFT can be seen as some **generalized branching process**

- No independence assumption

A unimodular EFT is always **critical** in the sense that the mean number of children of the root is 1

Unimodular EFTs and Dynamics on Unimodular Discrete Spaces

From vertex-shift on unimodular network to unimodular EFT

- $[\boldsymbol{G}, \boldsymbol{o}]$: a unimodular network and f a vertex-shift
- $C_{(\boldsymbol{G},\boldsymbol{o})}$: the connected component of the f-graph $\boldsymbol{G}^f$ containing $\boldsymbol{o}$
- Then $[C_{(\boldsymbol{G},\boldsymbol{o})}, \boldsymbol{o}]$, conditioned on being infinite, is a **unimodular EFT**

Conversely

- $[\boldsymbol{T}, \boldsymbol{o}]$: a unimodular EFT
- p: the parent vertex-shift is covariant
- The p-graph is $[\boldsymbol{T}, \boldsymbol{o}]$ itself

Joining of a sequence of directed trees

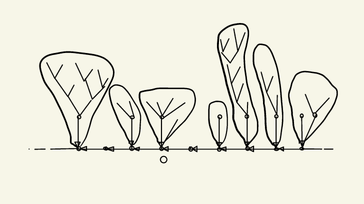
$([\boldsymbol{T}_i, \boldsymbol{o}_i])_{i=-\infty}^{\infty}$ a stationary sequence of random rooted trees

Regard each $[\boldsymbol{T}_i, \boldsymbol{o}_i]$ as a Family Tree by directing edges towards $\boldsymbol{o}_i$

Add a directed edge $\boldsymbol{o}_i\boldsymbol{o}_{i-1}$ for each $i \in \mathbb{Z}$

Let $\boldsymbol{o} := \boldsymbol{o}_0$

The resulting random rooted EFT, denoted by $[\boldsymbol{T}, \boldsymbol{o}]$, is the **joining** of the sequence $([\boldsymbol{T}_i, \boldsymbol{o}_i])_{i=-\infty}^{\infty}$



Decomposition Result on the $\mathcal{I}/\mathcal{F}$ Phase

If $\mathbb{E}\left[\#V(\boldsymbol{T}_0)\right] < \infty$, one can move the root of $\boldsymbol{T}$ to a *typical vertex of* $\boldsymbol{T}_0$:

$$\mathcal{P}'[A] := \frac{1}{\mathbb{E}\left[\#V(\boldsymbol{T}_0)\right]}\mathbb{E}\left[\sum_{v\in V(\boldsymbol{T}_0)} 1_A([\boldsymbol{T}, v])\right] \quad \text{probability measure}$$

Theorem [B-H-K 18]
Let $[\boldsymbol{T}, \boldsymbol{o}]$ be the joining of a stationary sequence of trees $([\boldsymbol{T}_i, \boldsymbol{o}_i])_{i=-\infty}^{\infty}$ such that $\mathbb{E}\left[\#V(\boldsymbol{T}_0)\right] < \infty$.
Let $[\boldsymbol{T}', \boldsymbol{o}']$ be a random rooted EFT with distribution $\mathcal{P}'$

1. $[\boldsymbol{T}', \boldsymbol{o}']$ is a unimodular EFT and of class $\mathcal{I}/\mathcal{F}$ a.s. As a result, all generations of $\boldsymbol{T}$ and $\boldsymbol{T}'$ are finite a.s.
2. Any unimodular non-ordered EFT of class $\mathcal{I}/\mathcal{F}$ can be constructed by joining a stationary sequence of trees

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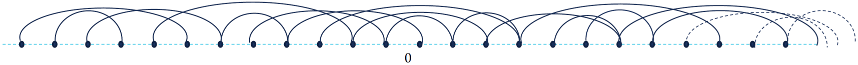
2. Vertex and point shifts (with M.O. Haji Mirsadeghi & A. Khezeli)
3. EFTs Everywhere
   - Renewal EFTs (with O. Sodre & S. Khaniha)
   - Record EFTs (with B. Roy Choudhury)
   - Evolutionary Trees (Ongoing work with O. Gascuel)
4. Conclusions

The Renewal EFT

- Graph: grid on $\mathbb{Z}$
- Marks $m(i)$, $i \in \mathbb{Z}$, i.i.d. with distrib. π on $\mathbb{N}^*$
- Unimodular Network

Vertex Shift

$$F(i) = i + m(i)$$



Finite Mean Interarrival Times

Theorem [F.B., A. Sodre 22]
Assume that π has finite mean and is aperiodic. Then the F-graph is an EFT (the **Renewal EFT**) which

- is unimodular
- is I/F
- has a covariant subset of individuals with infinite progeny

Infinite Mean Interarrival Times

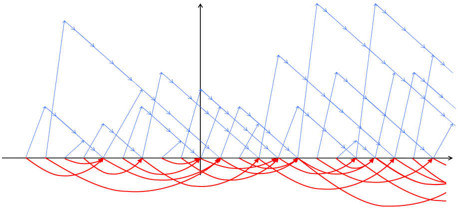
Theorem [F.B., S. Khaniha, M.-O. Mirsadeghi 22]

Assume that π has finite mean and is aperiodic. Then the F-graph (**Recurrence Time EFF**)

- Can either be a tree or a forest made of an infinite collection of trees (depending on the tail of the renewal CDF)

In the tree case, the **Renewal EFT**

- is unimodular
- is I/I



Outline

Stochastic processes with stationary increments

- Stationary integer-valued sequence $X = (X_n)_{n\in\mathbb{Z}}$ such that their common mean exists
- Stochastic process $S = (S_n)_{n\in\mathbb{Z}}$ is given by

$$
\begin{aligned}
S_0 &= 0 \\
n > 0,\ \ S_n &= \sum_{i=0}^{n-1} X_i \\
n < 0,\ \ S_n &= \sum_{i=n}^{-1} -X_i
\end{aligned}
$$

- Graph of the process S, $\{(n, S_n) : n \in \mathbb{Z}\}$.

Record graph

- Given a stationary integer-valued sequence $X = (X_n)_{n\in\mathbb{Z}}$, its **record map** $R_X : \mathbb{Z} \to \mathbb{Z}$ is given by

$$i \mapsto R_X(i) = \begin{cases} \inf\{n > i : S_n \geq S_i\} \text{ if inf exists} \\ i \text{ otherwise} \end{cases}$$

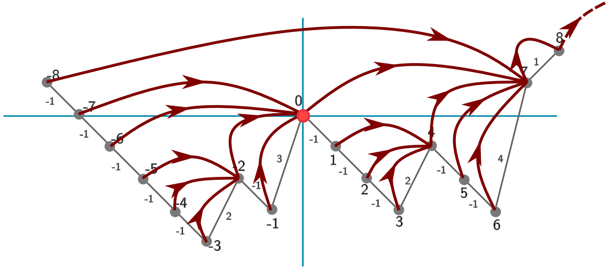
  ($S_n \geq S_i$ is equivalent to $\sum_{k=i}^{n-1} X_k \geq 0$).
- **The Record Graph** $\mathbb{Z}_X^R$ is the random graph given by

$$\text{vertices: } V(\mathbb{Z}_X^R) = \mathbb{Z}$$
$$\text{Directed Edges: } E(\mathbb{Z}_X^R) = \{(i, R_X(i)) : i \in \mathbb{Z} \text{ and } i \neq R_X(i)\}$$

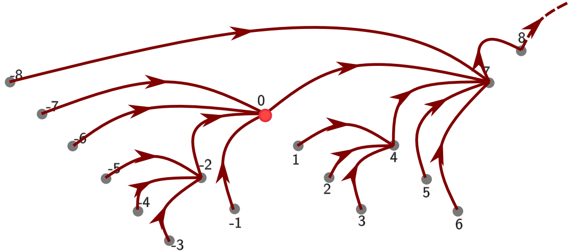
- $\mathbb{Z}_X^R(i)$ denotes the component of integer i in the record graph

Record graph picture

(A)



(B)



$S_0 = 0, S_n = \sum_{k=0}^{n-1} X_k$ for $(n > 0)$ and $S_n = \sum_{k=n}^{-1} -X_k$ for $n < 0$

Phase transition of the record graph

Theorem [F. B., B. Roy Choudhury 24]
Let $X = (X_n)_{n\in\mathbb{Z}}$ be a stationary and ergodic sequence of random variables such that their common mean exists. Let $\mathbb{Z}_X^R$ denote the record graph of the network $(\mathbb{Z}, X)$

- If $\mathbb{E}[X_0] < 0$, then a.s. every component of $\mathbb{Z}_X^R$ is of class $\mathcal{F}/\mathcal{F}$.
- If $\mathbb{E}[X_0] > 0$, then a.s. $\mathbb{Z}_X^R$ is connected, and it is of class $\mathcal{I}/\mathcal{F}$ a.s.
- If $\mathbb{E}[X_0] = 0$, then a.s. $\mathbb{Z}_X^R$ is connected, and it is either of class $\mathcal{I}/\mathcal{F}$ or of class $\mathcal{I}/\mathcal{I}$.

Component of 0 in the record graph is a unimodular tree

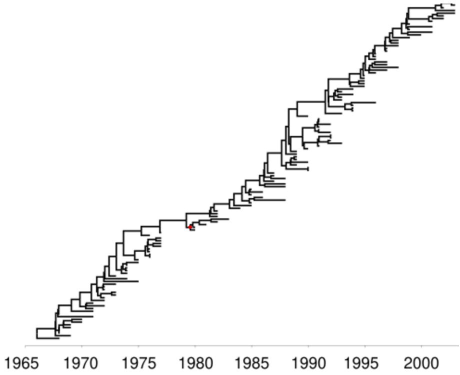
Skip-free random walks, $\mathbb{E}[X_0] = 0$ Example

Theorem [F. B., B. Roy Choudhury 24]
Let $X = (X_n)_{n\in\mathbb{Z}}$ be the increments of skip-free to the left random walk and $\mathbb{Z}_X^R$ be the record graph of the network $(\mathbb{Z}, X)$

If $\mathbb{E}[X_0] = 0$, then $[\mathbb{Z}_X^R(0), 0]$, the component of 0 in the record graph is distributed as the ordered $EGWT(\pi)$, where $\pi \stackrel{\mathcal{D}}{=} X_0 + 1$

Outline

Trigger



The evolution tree of Influenza [Wikipedia]

Reference: Book of M. Steel: Phylogeny, SIAM, 2016

Several classes of models:

- Branching : Bienaymé-Galton-Watson (neutral)
- Coalescent (neutral)
- Yule–Harding (neutral)
- Caterpillar model (non neutral)
- Brunet-Derrida-Mueller-Munier (non neutral)

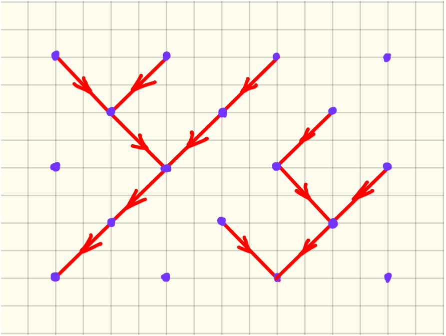
When choosing direction from offspring to parent, and when selecting a node at random as root, each of them admits a local weak limit EFT when letting some size parameter tend to infinity

Ansatz

Prelimit evolution models should belong to one of two phases depending on the phase of their limit

Examples of Limits and Phase Transitions

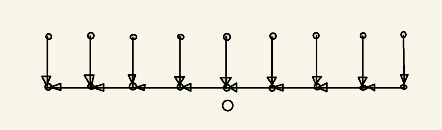
- Critical branching : Unimodular Bienaymé-Galton-Watson EFT
  - $\mathcal{I}/\mathcal{I}$ when variance of offspring distribution is positive
  - $\mathcal{I}/\mathcal{F}$ otherwise
- Coalescent
  - $\mathcal{I}/\mathcal{I}$ when the set of nodes per generation is $\mathbb{Z}$
  - $\mathcal{I}/\mathcal{F}$ otherwise **(Hence some neutral models are $\mathcal{I}/\mathcal{F}$)**



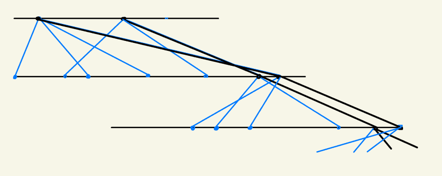
Instance of Coalescent - Local view of case on $\mathbb{Z}$

Examples of Limits and Phase Transitions (Continued)

- **Caterpillar model $\to$ Caterpillar EFT: always** $\mathcal{I}/\mathcal{F}$



- **Brunet-Derrida-Mueller-Munier model $\to$ BDMM EFT: always** $\mathcal{I}/\mathcal{F}$
  Individuals reproduce independently like in a branching process
  Each individual has a fitness which is that of its parent plus an increment with positive mean

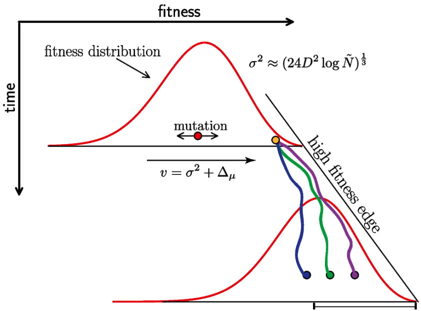


In every generation only the K most fit individuals reproduce

The BDMM Model and the FKPP Universality Class

The Fischer Kolmogorov Petrovskii Piscounov waves

The fitness of the individuals evolves with time as a wave propagating to the right at a constant speed



The BDMM Model belongs to the FKPP Universality Class

The $\mathcal{I}/\mathcal{F}$ phase and the FKPP Universality Class

Let

- $[T, o]$ be an I/F unimodular EFT
- $\{o_l\}_{l\in\mathbb{Z}}$ be the special individual sequence
- $\{[T_l, o_l]\}_{i\in\mathbb{Z}}$ be the trees in the joining decomposition of T
- $g_{l,k}$ be the number of the descendants of order $k \geq 0$ in T_l

Conditionally on $o = o_0$

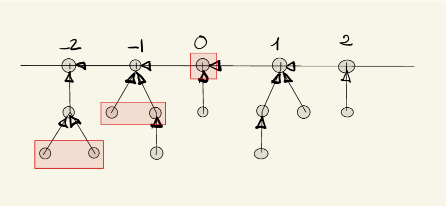
- $\{g_{l,k}\}_{k\geq 0}$ is stationary in l **(joining theorem)**
- $G_0 = \sum_{k\geq 0} g_{0,k}$ has finite mean

$\mathcal{I}/\mathcal{F}$ and FKPP - continued

Define the **fitness** of o_l and of all its descendants in T_l to be l

The fitness of generation l is best represented by the random measure

$$\Phi_l = \delta_l + \sum_{i<0} g_{l+i,l-i}\delta_{l+i}$$



The key observation is that **relative to l,** the random measures Φ_l, have the same probability distributions for all l

Generic extension of the FKPP wave valid for all I/F models

Individual's Success

Call **success** of a species (or an individual) the number of its descendant species of all generations.

- In the $\mathcal{I}/\mathcal{I}$ phase, the success of the typical individual is finite but with infinite mean
  A numbering of generations by $\mathbb{Z}$ implies that when **navigating the foil/generation** of the typical species, one finds a **subsequence of individuals with a success tending to infinity** a.s. This sequence of successes is stationary. If it is ergodic, when exploring the foil, one will find species with a success that dwarfs that of any other node visited earlier
- In the $\mathcal{I}/\mathcal{F}$ phase, **success in a generation is infinite for the individual of the generation belonging to the bi-infinite path** (the special individual of this generation) and finite for the others

What triggers the phase transition?

Renewal EFT

- finite mean jumps: I/F
- infinite mean jumps: I/I

Record EFT when interpreting state as fitness

- negative drift of fitness: I/F
- zero drift of fitness: I/I

Neutral coalescent EFT

- finite state space: I/F
- infinite state space : I/I

UBGW EFT

- zero variance: I/F
- positive variance : I/I

Outline

Conclusions

- Discrete unimodularity **extends Palm calculus** beyond Euclidean
- Several of **ergodic theory like results** despite no measure preserving transformation available
- Unimodular EFTs allow one to describe structural properties of **any dynamics on any discrete unimodular spaces**
- The unimodular Poincaré recurrence lemma leads to a **classification of dynamics** that hold across parametric models
- Such random structures are **ubiquitous**
- In particular, this should have **implications on evolution**

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