Gravitational wave memory: displacement or velocity effect ?

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Abstract: The scattering of particles by a burst of Gravitational waves (called **Memory Effect**) has two versions. In the velocity effect (VM) the particles fly apart with constant but nonzero velocity. In the displacement effect (DM), advocated by Zel'dovich and Polnarev, the particle's position has changed but they do not move. The observation of the effect could be a mean to detect gravitational waves. Our study for (i) a Gausssian or (ii) a Pöschl-Teller profile indicates that the generic VM effect can become, for special choices of the wave parameters labeled by an integer, a DM . The claims of Zel'dovich and Polnarev are confirmed.

Joint work with J. Balog, G. W. Gibbons, P-M. Zhang

P. M. Zhang and P. A. Horvathy, "Displacement within velocity effect in gravitational wave memory," [arXiv:2405.12928 [gr-qc]].

Road map:

Memory effect

A. Displacement Zel'dovich, Polnarev

"Radiation of gravitational waves by a cluster of superdense stars," Astron. Zh. **51**, 30 (1974)

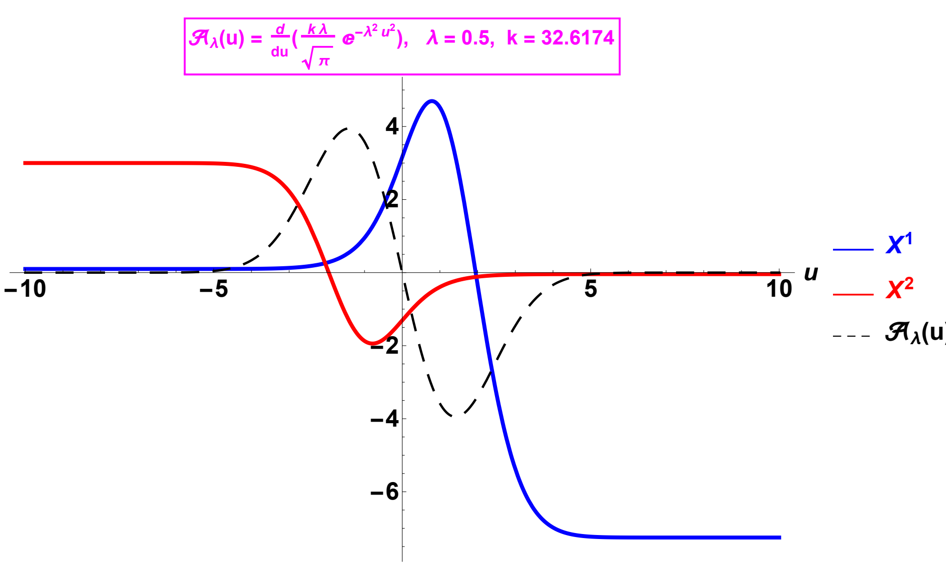
> *. . . [for] two noninteracting bodies (such as satellites). [. . .] the distance should change, and this effect might possibly serve as a nonresonance detector. [. . .] although distance between free bodies will change, their relative velocity will become vanishingly small as* **flyby** *concludes.*

Elaborated by V.B. Braginsky & L. P. Grishchuk

"Kinematic resonance and the memory effect in free mass gravitational antennas," Zh. Eksp. Teor. Fiz. **89** *744-750 (1985)*

Christodoulou : non-linear theory $\rightsquigarrow$ DM

$$\left(|\dot{X}_1 - \dot{X}_2|\right) \to 0 \;\Leftrightarrow\; |X_1 - X_2| \to \text{const} \qquad (1)$$



B. Velocity J. Ehlers and W. Kundt

"Exact solutions of the gravitational field equations," *in Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, London, 1962).

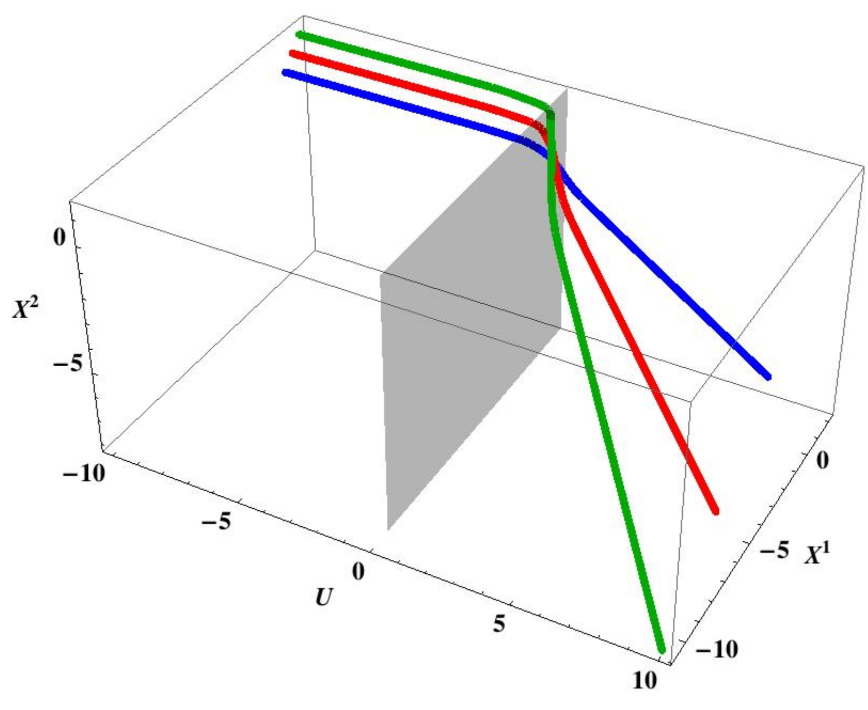
V B Braginsky and K S Thorne

"Gravitational-wave burst with memory and experimental prospects," Nature (London) **327** *123 (1987).*

L. P. Grishchuk and A. G. Polnarev

"Gravitational wave pulses with 'velocity coded memory'," Sov. Phys. JETP **69** *(1989) 653 [Zh. Eksp. Teor. Fiz.* **96** *(1989) 1153].*

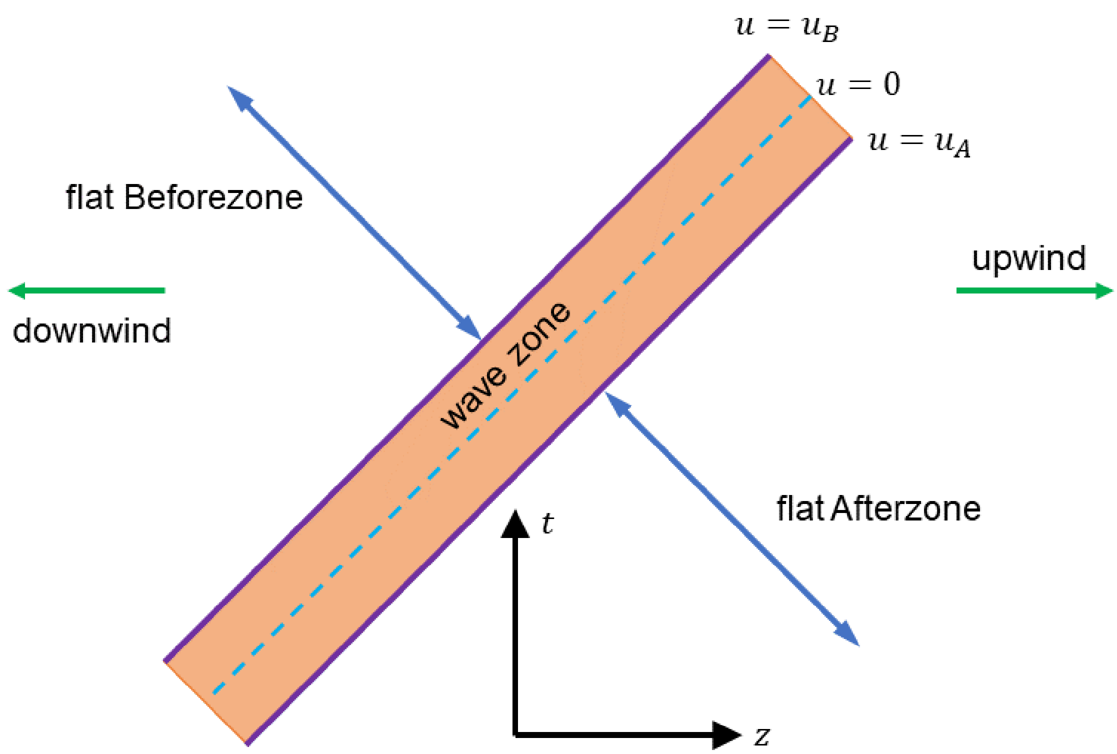
$$\dot{\boldsymbol{X}} \to \text{const} > 0 \qquad |\dot{\boldsymbol{X}}^{(1)} - \dot{\boldsymbol{X}}^{(2)}| > 0 \qquad (2)$$



particles fly apart with constant non-zero velocity

G. W. Gibbons S. W. Hawking *"Theory of the detection of short bursts of gravitational radiation,"* Phys. Rev. D **4** (1971) 2191.

Sandwich wave: burst of gravitational wave. Spacetime non-flat only in short interval $u_B \leq u \leq u_A$ of retarded time [Wavezone]. Flat both in Beforezone $u < u_B$ that the wave has not reached yet, and in Afterzone $u_A < u$ where has already passed,



(u flows from left to the right, whereas wave advances from right to left.)

Geodesics in Brinkmann* coordinates

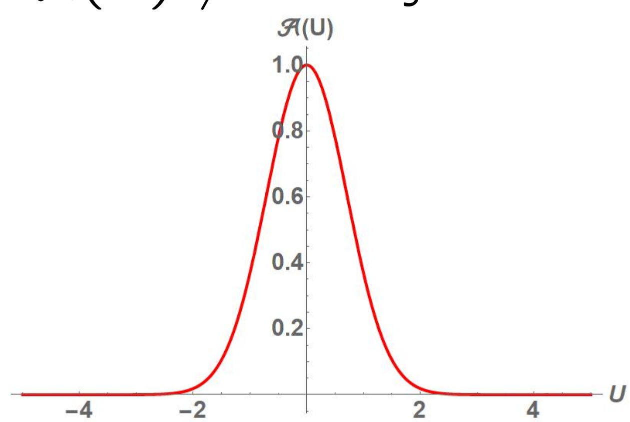
(toy model in 1 space + 2 lighlike dimensions.)
plane GW

$$g_{\mu\nu}X^{\mu}X^{\nu} = dX^2 + 2dUdV - \mathcal{A}(U)X^2dU^2 \qquad (3)$$

$X =$ transverse, U, V light-cone coords.

Sandwich wave: $\mathcal{A}(U) \neq 0$ only in "wave zone"

$U_B < U < U_A$.



For non-tachyonic geodesic: Jacobi invariant

$$\mathfrak{m}^2 = g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu} = \text{const} \leq 0\,. \qquad (4)$$

Massive: $\mathfrak{m}^2 < 0$, Lightlike $\mathfrak{m}^2 = 0$.

* M. W. Brinkmann, "Einstein spaces which are mapped conformally on each other," Math. Ann. **94** (1925) 119–145.

Lightlike geodesics $\mathfrak{m}^2 = 0$:

$$\frac{d^2X}{dU^2} + \frac{1}{2}\mathcal{A}X = 0\,, \tag{5a}$$

$$\frac{d^2V}{dU^2} - \frac{1}{4}\frac{d\mathcal{A}}{dU}(X)^2 - \tfrac{1}{2}\mathcal{A}\,\frac{d(X^2)}{dU} = 0\,. \tag{5b}$$

$V(U)$ horizontal lift of $X(U)$

Coordinate X decoupled from V. Projection into transverse space is V-independent. Conversely, lightlike geo determined by eqn. (5a) with U viewed as Newtonian time *.

*L. P. Eisenhart, *"Dynamical trajectories and geodesics"*, Annals. Math. **30** 591-606 (1928).

C. Duval, G. Burdet, H. P. Kunzle and M. Perrin, *"Bargmann Structures and Newton-cartan Theory,"* Phys. Rev. D **31** (1985), 1841-1853

C. Duval, G. W. Gibbons and P. Horvathy, *"Celestial mechanics, conformal structures and gravitational waves,"* Phys. Rev. D **43** (1991), 3907-3922 [arXiv:hep-th/0512188 [hep-th]].

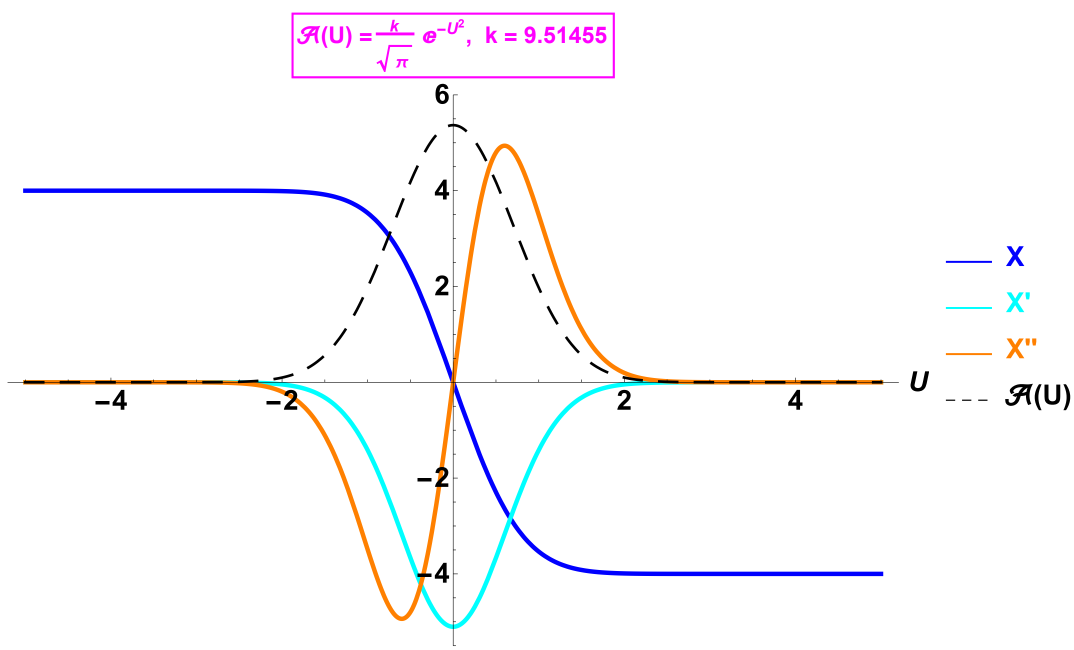
Gaussian profile

$$\mathcal{A}^G(U) = \frac{k}{\sqrt{\pi}} e^{-U^2}, \quad \int \mathcal{A}^G(U)\, dU = k\,. \qquad (6)$$

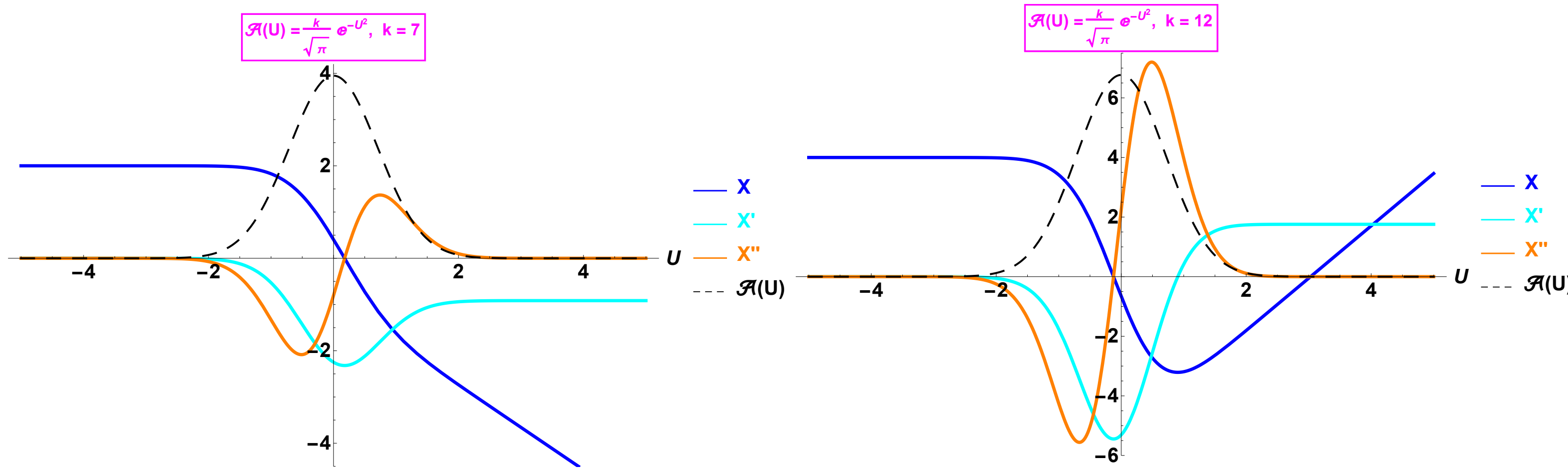
Amplitude $k =$ area below profile.

Earlier work (Duval et al) $\rightsquigarrow$ particles fly apart with non-zero velocity : VM .

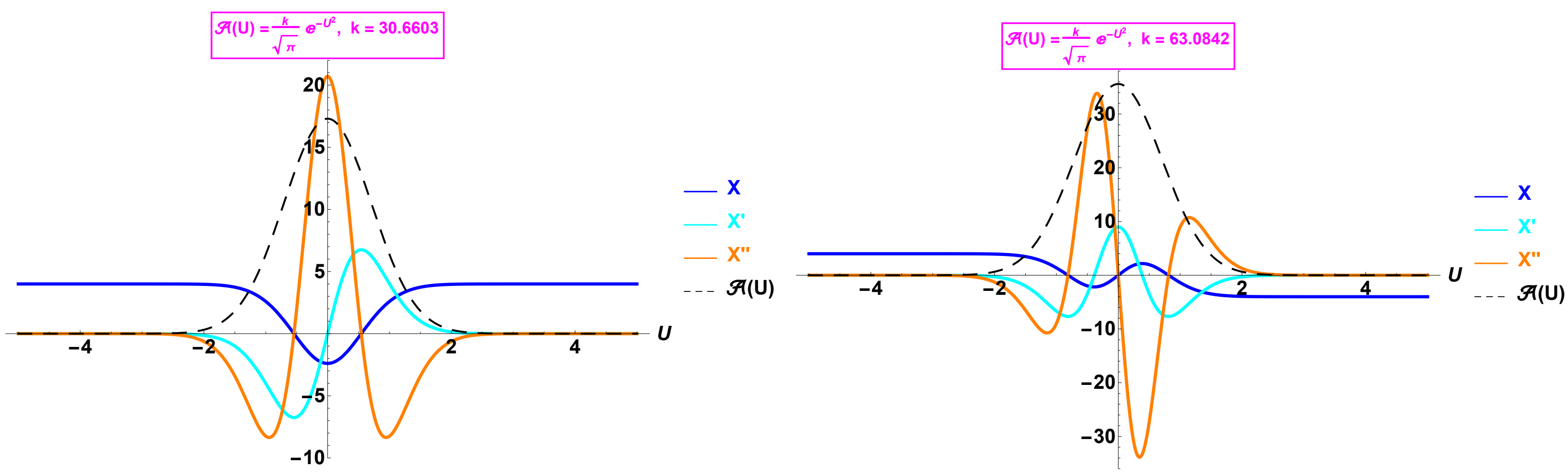
Surprise: numerical investigations $\rightsquigarrow$ *fine-tuning amplitude* k CAN yield (approximate) DM .



Fine-tuning the amplitude to $k = k_{crit}$ *yields DM with* $\mathbf{m = 1}$ *half-wave.* X : **trajectory** , dX/dU : **velocity** , $\mathbf{d^2X}/dU^2$: **force**.



For $k < k_{crit}$ trajectory undershoots and (b) for $k > k_{crit}$ overshoots before being straightened out.



Fine-tuning amplitude yields DM with $\mathbf{m=2}$ *and* $\mathbf{m=3}$ **half-waves** *as trajectories.*

"Miracle" explained by : at (approximate) boundaries of the Wavezone $U_B < U < U_A$ both

velocity and force vanish

— outside Wavezone motion governed by Newton's laws.

Further fine-tuning yields DM for "magic amplitudes",

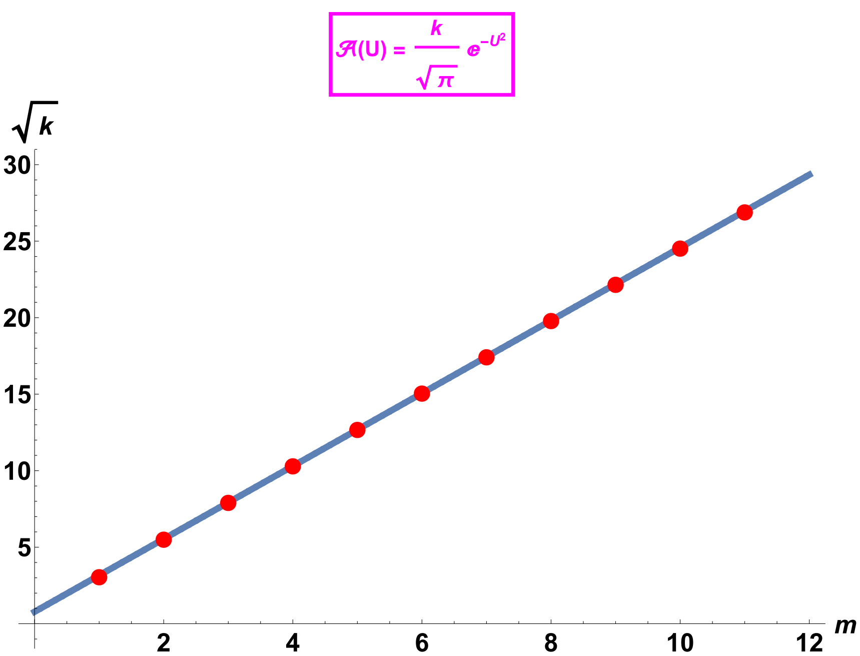
$$\begin{cases} k_1 \approx 9.5 & m = 1, \\ k_2 \approx 30.7 & m = 2, \\ k_3 \approx 63.1 & m = 3, \\ k_4 \approx 106.7 & m = 4, \; \ldots \end{cases} \tag{7}$$

Trajectories consist of m half-waves. Outgoing position depends on parity:

$$X_{out} = (-1)^m X_{in} \, . \tag{8}$$

Higher wave number requires higher amplitude.

$$\sqrt{k_m} \approx 0.78 + 2.38m \, . \tag{9}$$

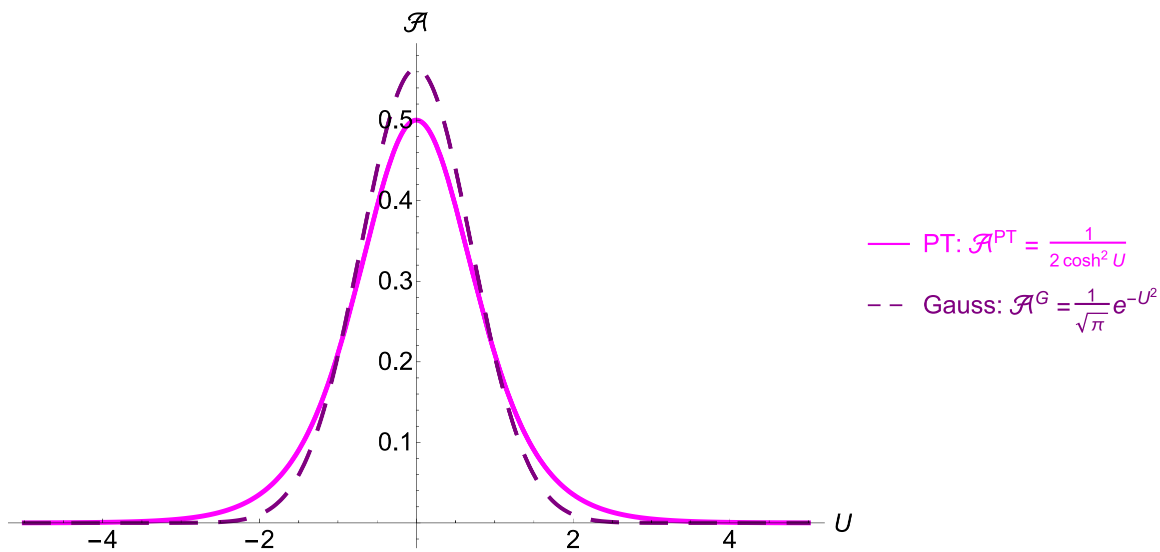


Relation between # of half-wave trajectories in Wavezone, m, and $\sqrt{k_{crit}}$ for DM is approximately linear.

Pöschl-Teller profile

No analytic solutions for Gauss. Shape of $\mathcal{A}^G$ reminiscent of Pöschl-Teller (PT) frequency,

$$\mathcal{A}^{PT}(U) = \frac{k}{2\cosh^2 U}\,, \qquad (10)$$



Gaussian bell (dashed) approximated by Pöschl-Teller potential (10) (solid line), which <u>does</u> admit analytic solutions. Parameters chosen so that area below both profiles be identical, equal to k.

Writing $k_m = 4m(m+1)$, geo eqn. (5a) becomes,

$$\frac{d^2X}{dU^2} + \frac{m\,(m+1)}{\cosh^2 U} X = 0\,. \qquad (11)$$

Particle at rest before burst arrives:

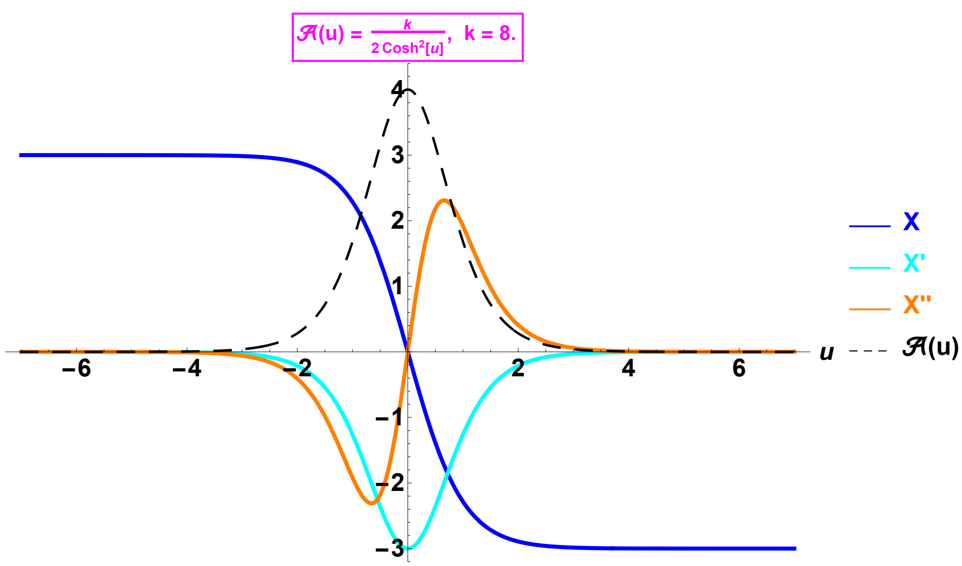
$$X(U = -\infty) = X_0, \; \dot{X}(U = -\infty) = 0\,. \qquad (12)$$

$t = \tanh(U) \rightsquigarrow$ Legendre eqn,

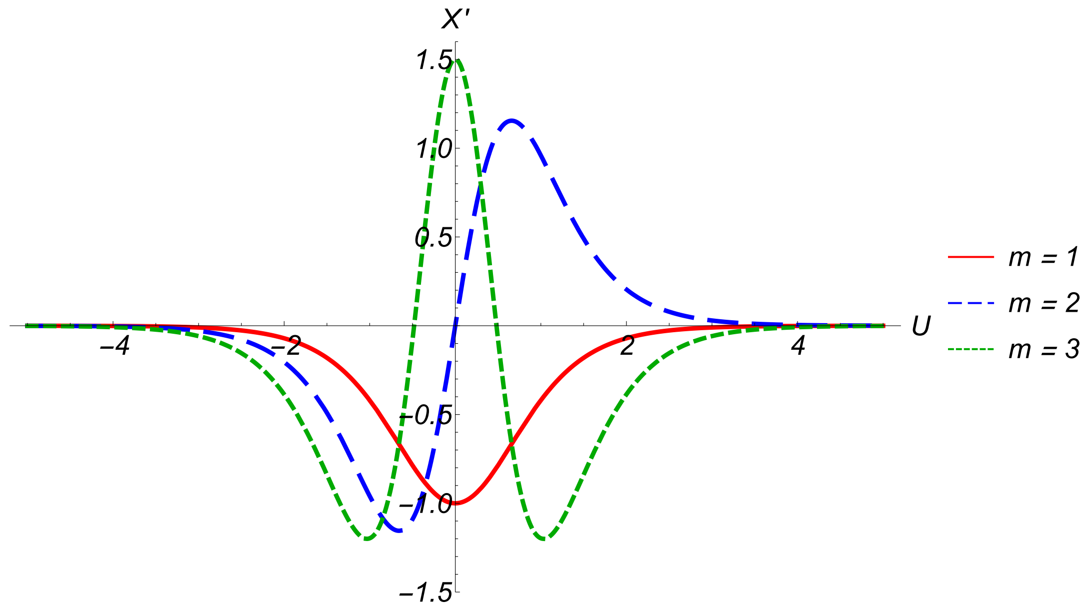
$$\left(1 - t^2\right) \frac{d^2X}{dt^2} - 2t\frac{dX}{dt} + m\,(m+1)\,X = 0\,. \quad (13)$$

DM requires $X(U) \to$ const for $U \to \infty \;\Rightarrow$ solution of (13) extends to $t = \pm 1 \rightsquigarrow m$ positive integer $\Rightarrow$ solution propto Legendre polynomial,

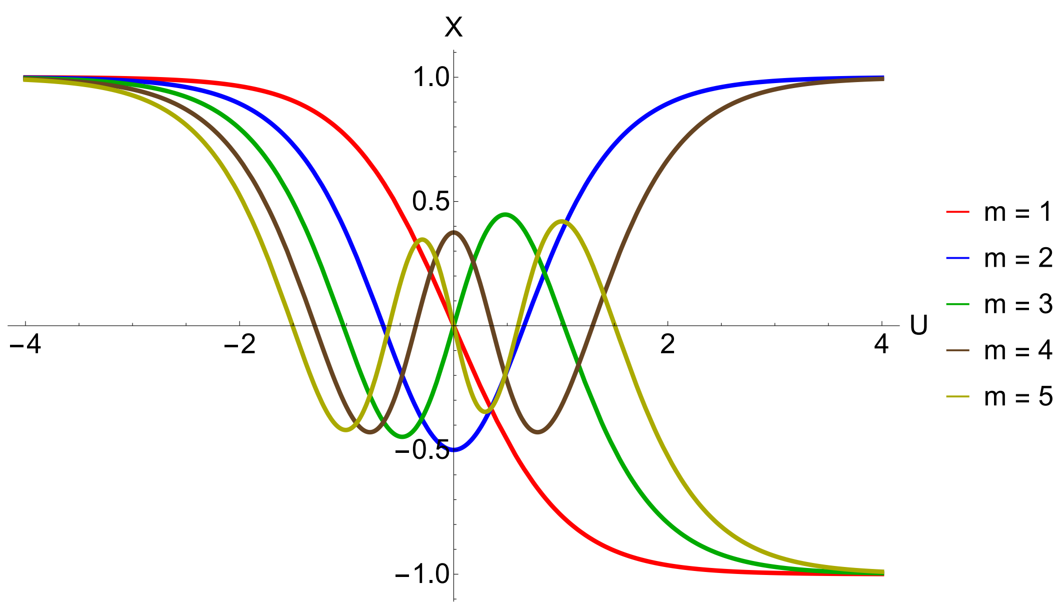
$$X_m(U) = P_m(\tanh U), \;\; m = 1,\, 2,\, \ldots\,, \qquad (14)$$



For Pöschl-Teller with $k_1 = 8$ i.e. $\mathbf{m = 1}$, transverse trajectory consistent with DM, (cf. Gaussian).



Transverse velocities for DM amplitudes $k = k_m$ *for Pöschl-Teller profile* (10), *for* $\mathbf{m} = \mathbf{1}, \mathbf{2}, \mathbf{3}$.



DM trajectories for Pöschl-Teller profile with $\mathbf{m} = \mathbf{1}, \ldots, \mathbf{5}$ *half-waves.*

Frequency decreses with U

$$\omega^2(U) = \frac{m\,(m+1)}{\cosh^2 U}\,. \tag{15}$$

DM for vertical component

For lightlike trajectories "vertical" component $V(U)$ (5b)

$$\frac{d^2V}{dU^2}+\frac{1}{4}\frac{d\mathcal{A}}{dU}X^2\big)+\mathcal{A}\Big(X^1\frac{dX^1}{dU}\Big)=0\,. \tag{16}$$

is horizontal lift of the transversal trajectory,

$$V_{null}(U)=V_0-\int\mathcal{L}_{null}\,dU \tag{17}$$

where $\mathcal{L}_{null}$ is the usual Lagrangian

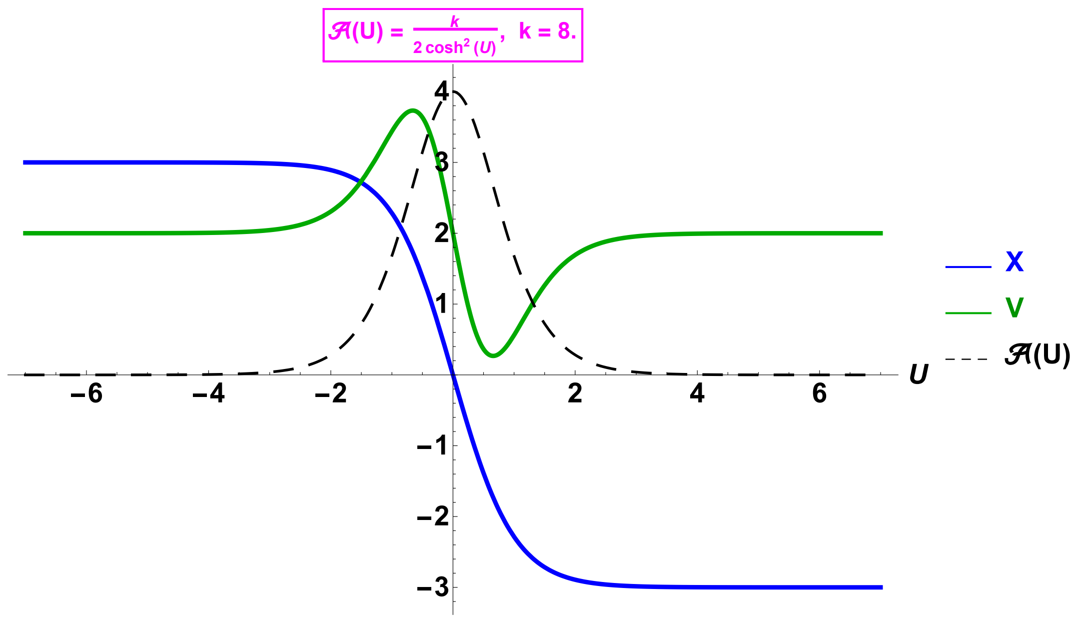
$$\mathcal{L}_{null}=\tfrac{1}{2}(\dot{X})^2-\tfrac{1}{2}\mathcal{A}\,X^2$$

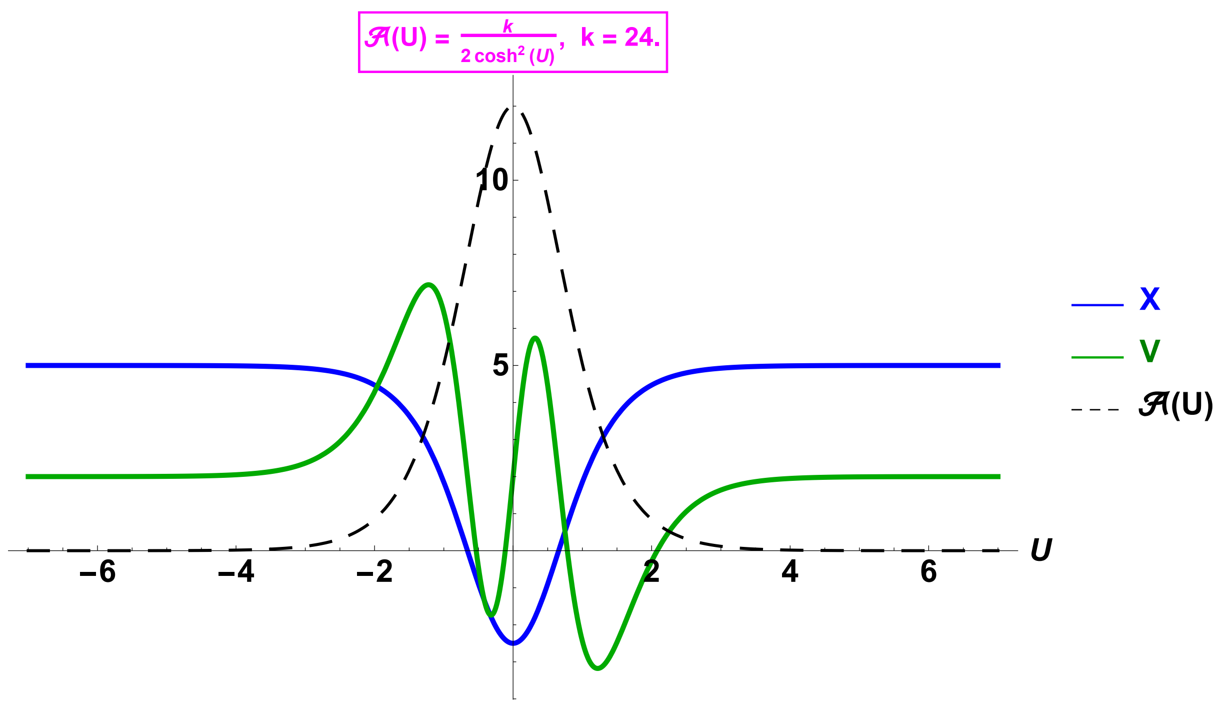
of non-relativistic particle in $D=1$ of unit mass in $1+1$ dim, moving in time-dependent oscillator potential.

In Afterzone both velocity and potential vanish $\Rightarrow$ $\mathcal{L}_{null}$ $\Rightarrow$ action zero along geodesic $\Rightarrow$ DM for V coordinate,

$$V_{null}(out)=V_0=V_{null}(in) \tag{18}$$

More precisely: NO-DM for V_{null} ! Motion purely transverse !





For Pöschl-Teller profile with k_m (shown for $\mathbf{m = 1, 2}$*) both* **transverse,** $\mathbf{X}$*, and* **vertical,** $\mathbf{V}$*, trajectories behave consistently with DM .*

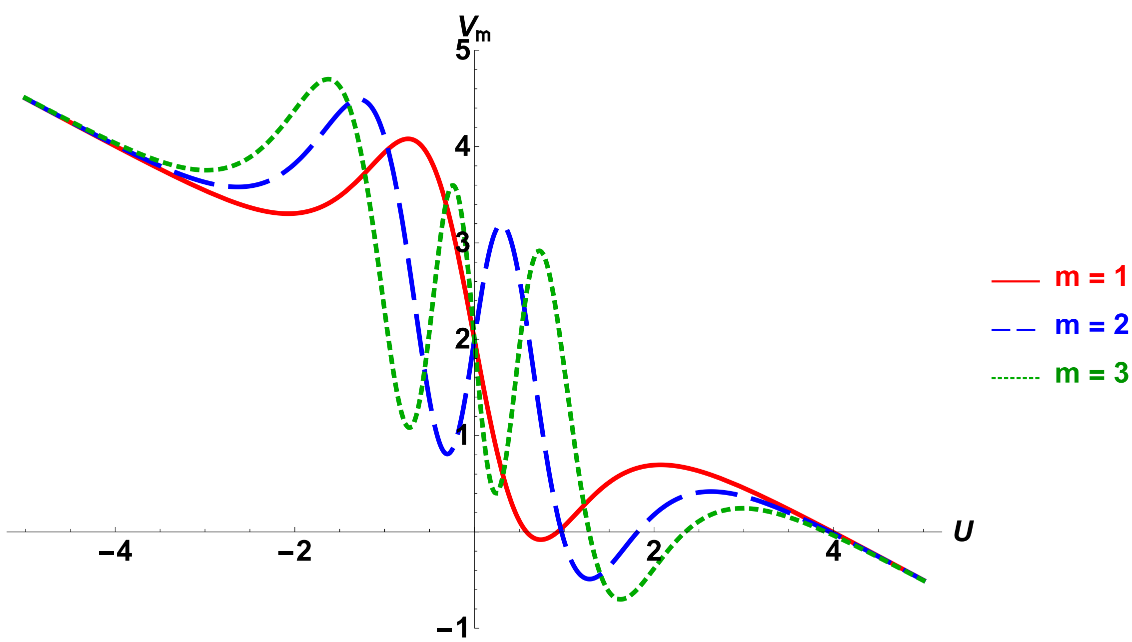
Massive geodesics

Results extend to particles with nonzero relativistic mass, $\mathfrak{m} \neq 0$.

Then* (17) picks up linear-in-U term,

$$V_{\mathfrak{m}}(U) = V_{null}(U) - \left(\frac{\mathfrak{m}}{2m}\right)^2 U\,, \tag{19}$$

where $m = p_v$ is conserved quantity generated by Killing vector ∂_V (non-relativistic mass in E-D framework). In units where $\mathfrak{m} = -1$ and $m = 1$, vertical coordinate (19) gets extra term $-\frac{1}{2}U$.



*M. Elbistan et al. Annals Phys. **418** (2020), 168180 [arXiv:2003.07649 [gr-qc]], eqn. # (VI.2).

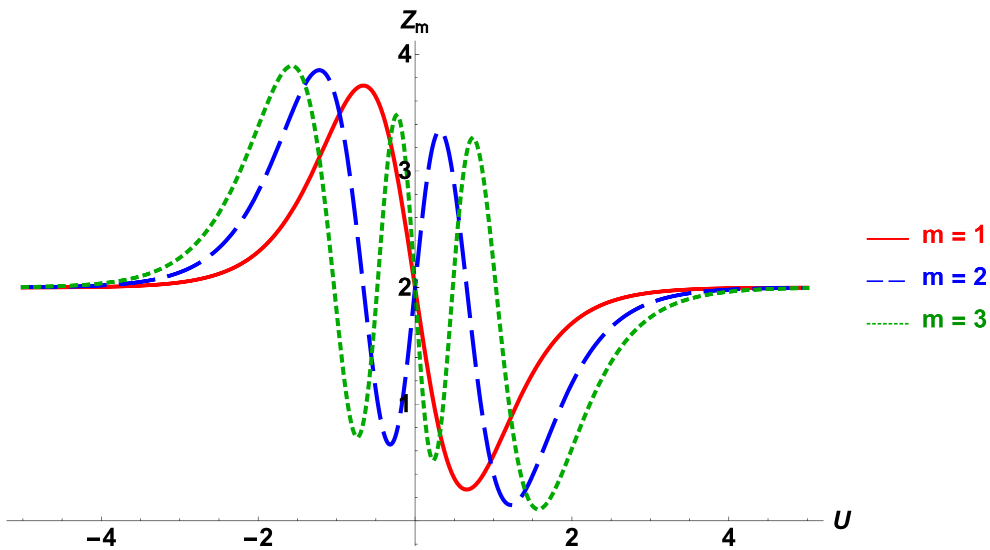
However switching from (lightlike) V to relativistic position coordinate,

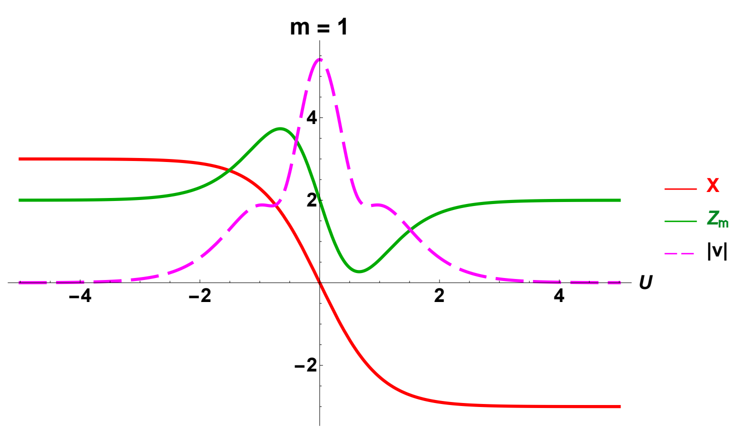
$$Z = V + \tfrac{1}{2}U \tag{20}$$

yields

$$Z_{\mathfrak{m}}(U) = V_0 = \text{const} \tag{21}$$

i.e., NO-DM for Z_m component !!





velocity $|v| = \sqrt{v_x^2 + v_z^2}$

$$|v|^2_{out} = |v|^2_{in}| \,.$$

GW in $D = 3 + 1$ dimensions

$D = 3 + 1$ dim (coords X^1, X^2, U, V, U, V light-like) :

$$\delta_{ij}\, dX^i dX^j + 2dUdV + K_{ij}(U)X^iX^j\, dU^2, \quad (22a)$$

$$K_{ij}(U)X^iX^j = \tfrac{1}{2}\mathcal{A}_+(U)\Big((X^1)^2 - (X^2)^2\Big) + \mathcal{A}_\times(U)\, X^1X^2, \quad (22b)$$

Lightlike Geodesics for linearly polarized ($\mathcal{A}_\times \equiv 0$) GW :

$$\frac{d^2X^1}{dU^2} - \frac{1}{2}\mathcal{A}X^1 = 0, \quad (23a)$$

$$\frac{d^2X^2}{dU^2} + \frac{1}{2}\mathcal{A}X^2 = 0, \quad (23b)$$

$$\frac{d^2V}{dU^2} + \frac{1}{4}\frac{d\mathcal{A}}{dU}\Big((X^1)^2 - (X^2)^2\Big) + \mathcal{A}\Big(X^1\frac{dX^1}{dU} - X^2\frac{dX^2}{dU}\Big) = 0\,. \quad (23c)$$

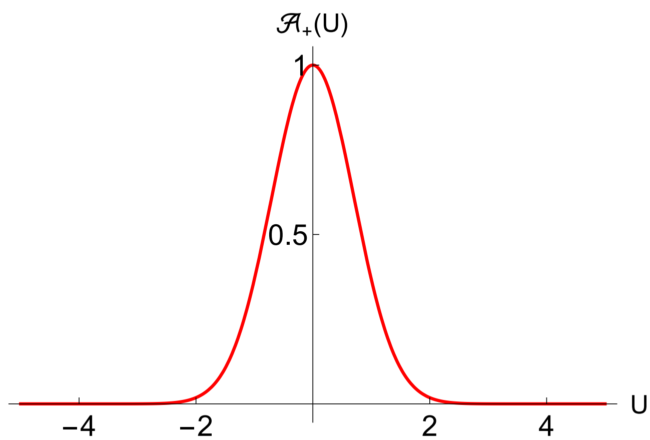
$X^{1,2}$-components decoupled from V. Projection of $4D$ worldline to transverse plane (X^1, X^2) independent of $V(U)$.

NB: eqn (23c) $\sim$ horizontal lift.

In Eisenhart-Duval framework: (23a)-(23b) $\sim$ repulsive/attractive harmonic force with (possibly time-dependent) frequency $\omega^2(U) = \mathcal{A}(U)$.

• Gaussian flyby profile $\sim$ Gibbons-Hawking'71

$$K_{ij}(U)X^iX^j = \frac{e^{-U^2}}{\sqrt{\pi}}\Big((X^1)^2-(X^2)^2\Big). \tag{24}$$

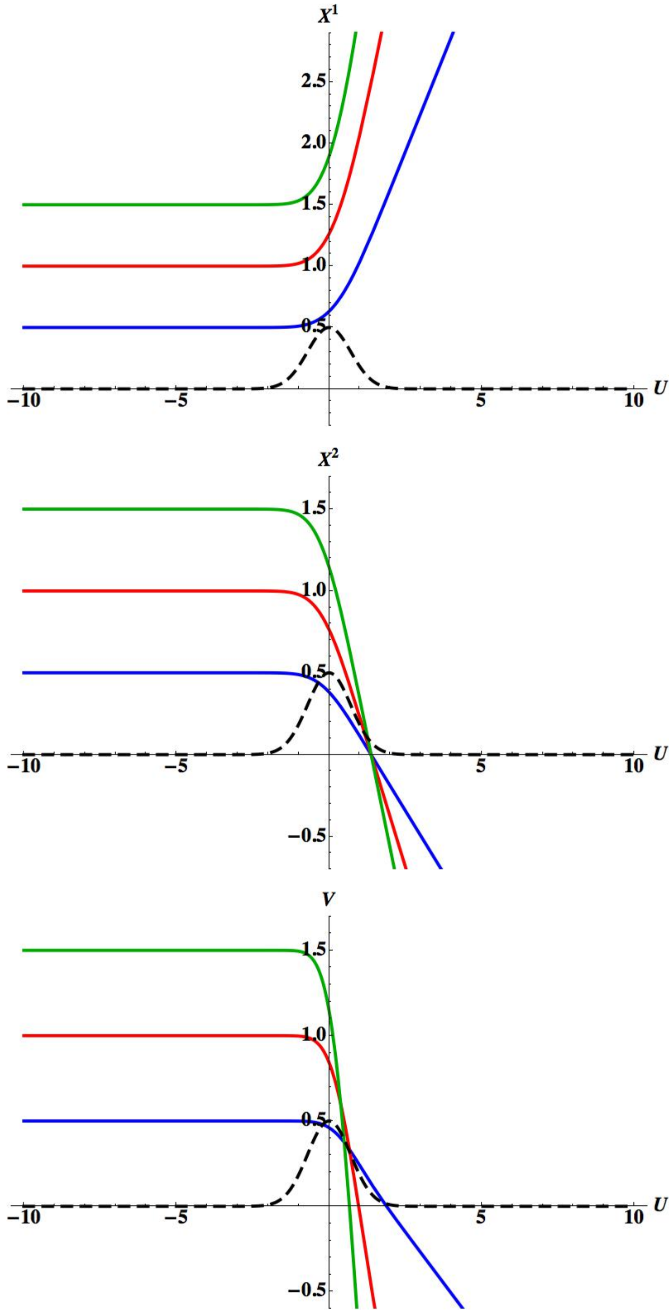


• Pöschl-Teller profile

$$K_{ij}(U)X^iX^j = \frac{k}{2\cosh^2 U}\Big((X^1)^2-(X^2)^2\Big). \tag{25}$$

Bargmann pic $\sim$ anisotropic repulsive/attractive oscillator with "time"-dependent profile. $\Rightarrow$ VM . Can get DM ?

repulsive in X^1 **attractive** in X^2 $\rightsquigarrow$



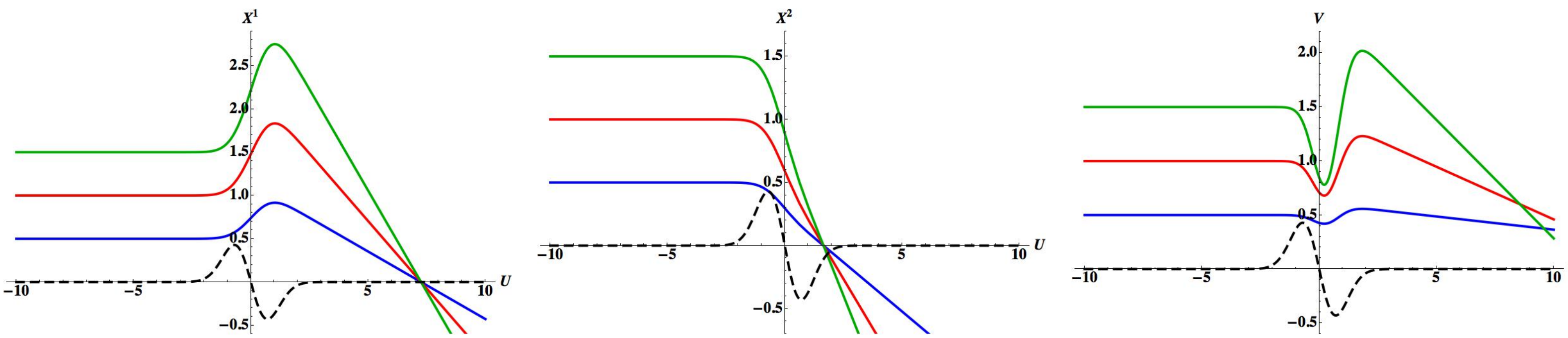
Geodesics for Gaussian burst with for blue/red/green positions in Beforezone.

VM but **no** DM unless reducing to 1 dim by putting $X^1(-\infty) = 0 \Rightarrow X^1(U) \equiv 0$.

DM for flyby ??

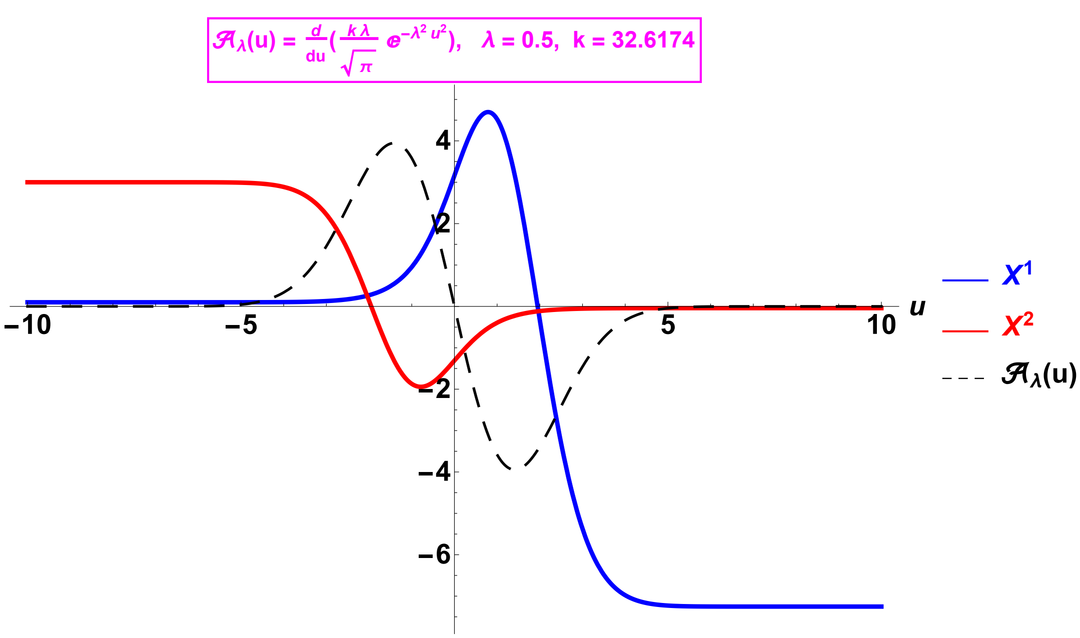
Gibbons-Hawking: flyby profile proportional to *first derivative* of a Gaussian,

$$\mathcal{A}(U) = \frac{d}{dU}\left(k\frac{\lambda}{\sqrt{\pi}}e^{-\lambda^2U^2}\right) \qquad (26)$$

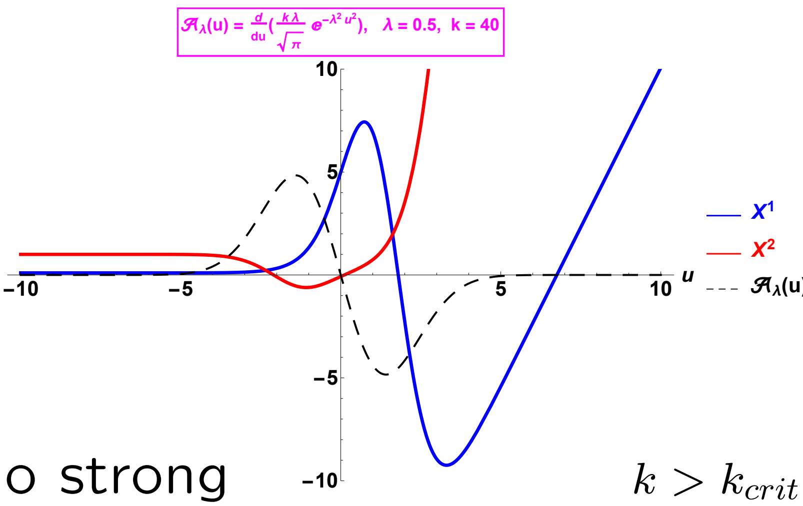


Geodesics for flyby profile (26) with parameters $k = \lambda = 1$.

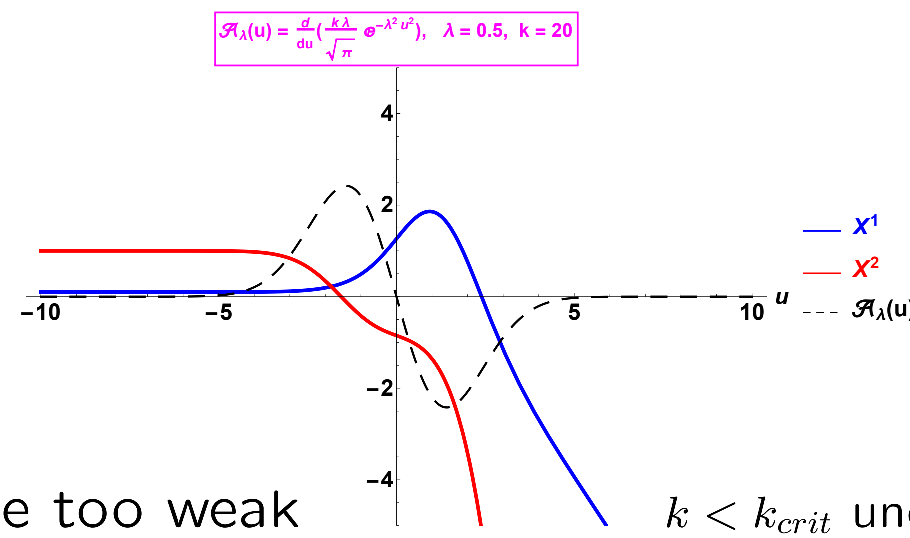
show VM *no* DM . However Miracle ! for (numerically found) specific choices of parameters DM for both components !



For $k \neq k_{crit}$ velocity effect. Transverse velocity approx. constant but non-zero in Afterzone.

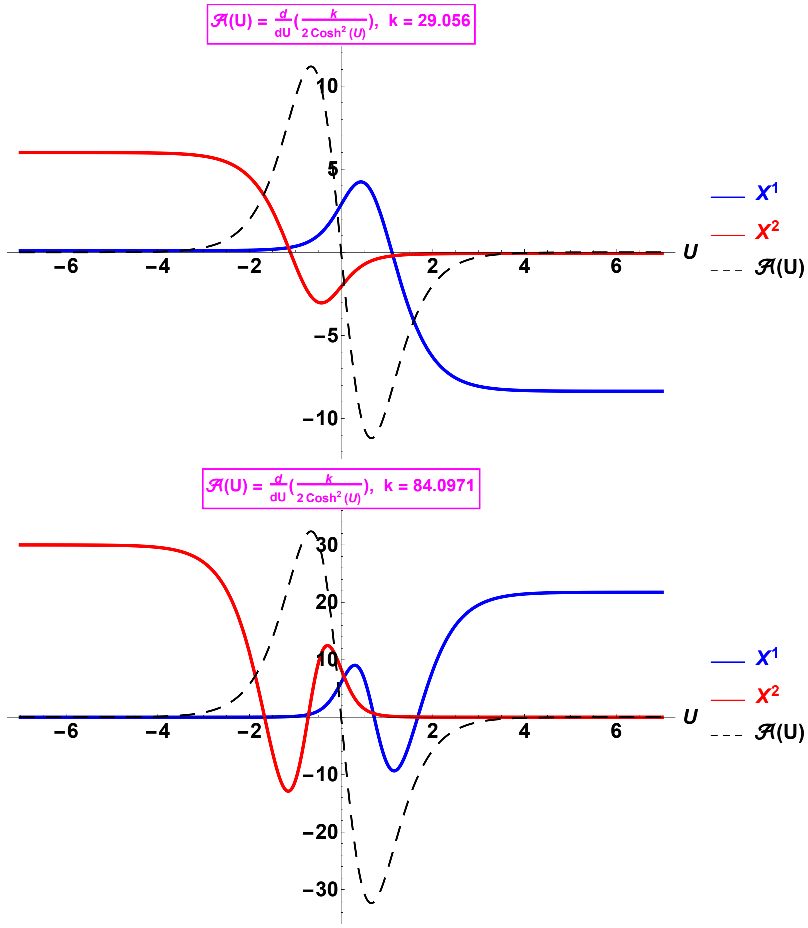


Wave too strong $k > k_{crit}$ overshoot

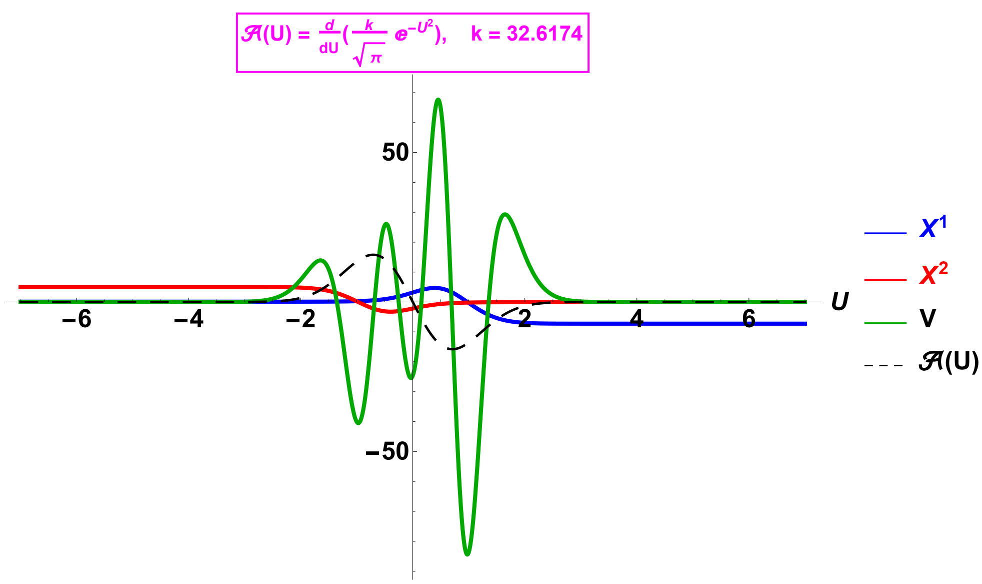


Wave too weak $k < k_{crit}$ undershoot

Idem for flyby - Pöschl-Teller :



For Pöschl-Teller*-flyby profile* (25) *(dashed line), (approximate)* DM *for both components when amplitude is* $k_m = 4m(m+1)$*, shown for* $m = 1$ *and* $m = 2$*.*



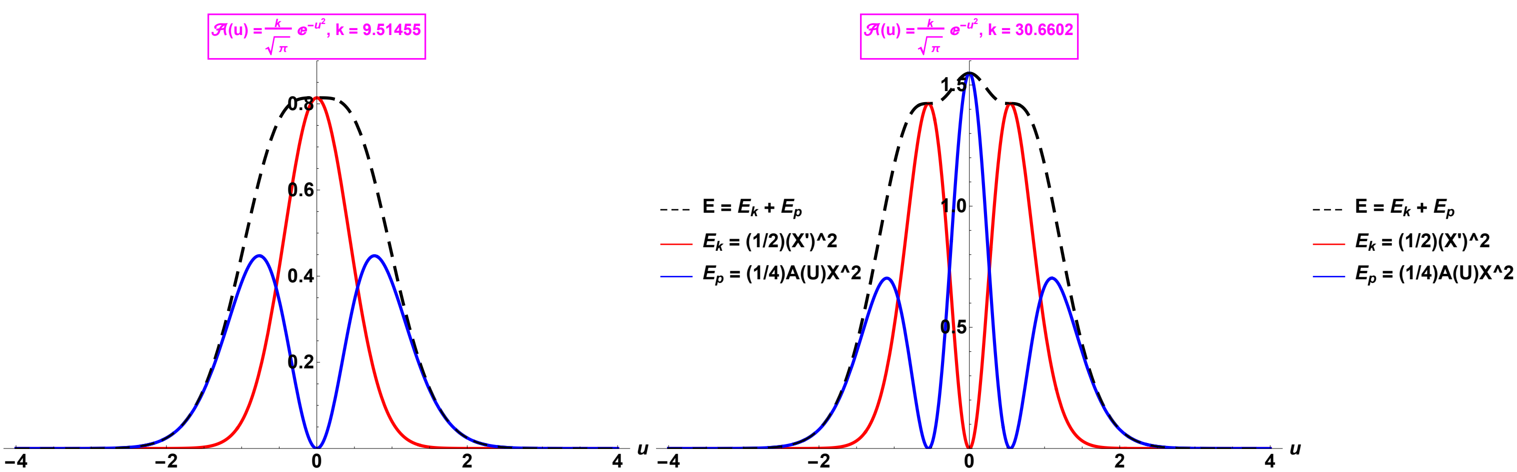
Similar for higher ($d = 1, 2, 3 \ldots$) derivatives of Gaussian : Braginsky - Thorne d=2, "gravitational collapse" d=3 . . .

For order $d = 2n$ even : $\frac{1}{2}$ DM, and for $d = 2n + 1$ odd: DM for both components.

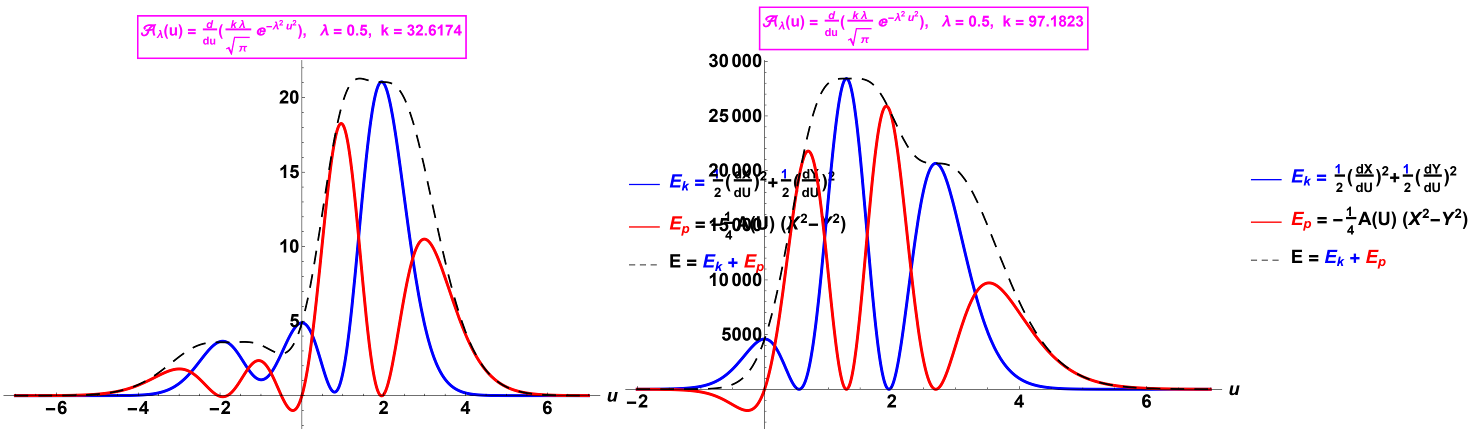
CONCLUSIONS

1. DM for exceptional values of parameters which correspond to to integer # m of standing wave trajectories in wave zone. Full conformation of Zel'dovich-Polnarev prediction

2. For DM parameter k_{crit} the incoming wave first transfers some energy to the particle until reaching a maximal value, but the accumulated energy is then radiated away*, as shown for $\mathbf{m = 1}$ and $\mathbf{m = 2}$.



*Old Testament, Genesis 3:19: *"pulvis es, et in pulverem reverteris"* ["you were made from dust, and to dust you will return."].



Variation of the energy for Gaussian and Pöschl-Teller with DM parameter k_m for $\mathbf{m = 1}$ *and* $\mathbf{m = 2}$*. Almost identical plots obtained for Pöschl-Teller.*