Boundary terms of the conformal and chiral anomalies

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Motivation:

Sometimes the defects and boundaries are responsible for the failure of extending the symmetries to quantum levels. It is important to investigate the boundary anomaly for various fields, on manifolds with various geometries of boundaries and of course the different legitimate boundary conditions.

This is a review talk based on 1702.00566, 2102.07661 and 2308.10351 with Sergey Solodukhin.

Outlook:

In this talk I will concentrate on the

- Conformal Anomaly
- Chiral Anomaly

In the presence of the boundaries.

Conformal Anomaly

It tells us about the *universal physical features* in quantum theory.

$$Z = e^{-W} = \int \mathcal{D}\Phi \; e^{-S[\Phi,g]} \, ,$$

$$\mathcal{A} = \delta_\sigma W(e^{2\sigma} g_{\mu\nu})\Big|_{\sigma=0} = \int_{\mathcal{M}_d} \langle T^\mu_\mu \rangle \, .$$

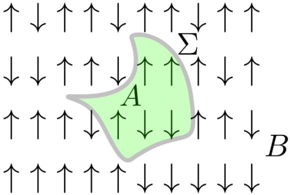
This is the *integrated conformal anomaly.*

Capper and Duff,(1974)

EE in QM

A very good example could be found in quantum information theory

• Consider a quantum mechanical system in a pure ground state which is described by $|\psi\rangle$ ($\rho = |\psi\rangle\langle\psi|$).

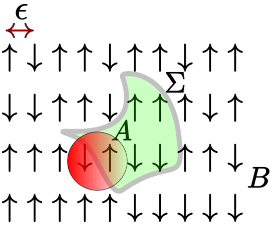


• *Reduced* density operator:

$$\rho_A = \text{Tr}_B\, \rho = \text{Tr}_B |\psi\rangle\langle\psi|\,.$$

Then the EE is

$$S_{EE}(A) = -\,\text{Tr}\, \rho_A \log \rho_A\,.$$



Regularizing the field theory with a cut-off ϵ one gets the following expansion of EE

$$S_{EE}(\Sigma) = \frac{g_1}{\epsilon^{d-2}} + \frac{g_2}{\epsilon^{d-4}} + \cdots + g_0 \log \epsilon \,,$$

• g_is are geometrical quantities and the first one is the area of teh entangling surface

$$g_1 \propto A_\Sigma \,.$$

In particular the logarithmic term is a universal important term.

log term

$$g_0 = -\epsilon \frac{dS_{EE}}{d\epsilon} \sim g_{\mu\nu} \frac{\delta Z}{\delta g_{\mu\nu}} \sim \int \langle T^{\mu}_{\mu} \rangle \,.$$

For instance in 2 dim.

$$g_0 = \frac{c}{3}\,, c \; : \text{central charge.}$$

Heat Kernels

A very useful tool to investigate the conformal anomalies in a given QFT is provided by the *Heat Kernel* method.

A user's manual by Vassilevich (hep-th/0306138)

Lets consider a FT whit D as the Laplacian in Action. Then one can construct the so called *Heat Operator* by introducing an auxiliary time coordinate as e^{-tD}.

the heat kernel

$$K(t; x, y; D) = \langle x|e^{-tD}|y\rangle \,,$$

satisfies the heat equation

$$(\partial_t + D)K(t; x, y; D) = 0 \,,$$

with initial condition

$$K(0; x, y; D) = \delta(x, y) \,.$$

$$(\partial_t + D)K(t; x, y; D) = 0\,,$$

$$K(0; x, y; D) = \delta(x, y)\,.$$

So, it describes the explosion of a hot dot and the propagation of its heat!

Heat Kernel

The one-loop effective action would be then

$$W = -\frac{1}{2}\int_{\epsilon}^{\infty} \frac{dt}{t} K(t, D)\,,$$

where K is the trace of the heat kernel, defined as

$$K(t, D) = \int d^d x\, K(t; x, x; D)\,.$$

Asymptotic expansion of heat kernel

$$K(t, D) = t^{-d/2} \sum_k a_{2k} t^k \to \text{expansion of } W \text{ in } \epsilon$$

The universal term is the log term since

$$\log \lambda\epsilon = \log \epsilon + \cdots$$

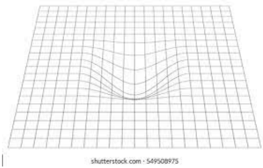
and it can be shown that the corresponding coefficient in the heat kernel expansion is related to the conformal anomaly. In d dimensions it is

$$\mathcal{A} = \int a_d \,.$$

The question is that how heat would propagate in front of a wall?

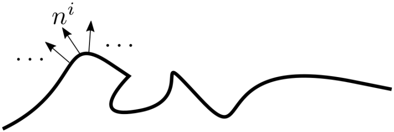
In front of a wall, a_d would depend on scalars made of the intrinsic and extrinsic curvature tensors

- Intrinsic geometry of the base manifold:



$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\partial_\nu \partial_\rho g_{\mu\sigma} + \cdots .$$

- Extrinsig geometry of the boundary:



$$K_{ij} = \nabla_i n_j .$$

Boundary conformal invariants

But which combinations appear?

Topological invariants + Contractions of good tensors

What do we mean by good tensors?

The (projected) Weyl tensor. If

$$\delta g_{ij} = -2\sigma g_{ij},$$

then

$$\delta_\sigma W_{ikj\ell} = 2\sigma W_{ikj\ell}\,,$$

and the (traceless) extrinsic curvature tensor would be transformed as

$$\hat{K}_{ij} = K_{ij} - \frac{1}{d} h_{ij} K\,,$$

then

$$\delta_\sigma \hat{K}_{ab} = \sigma \hat{K}_{ab}\,.$$

But the heat kernel coefficients are not available with this resolution. Our task is to rewrite them in terms of the conformal invariant blocks and extract their coefficients as the conformal charges.

a_ds are known up to $d = 5$ in the presence of the boundaries and for general mixed boundary conditions. So lets talk about the boundary conditions, briefly.

Series of papers by Branson, Gilkey, Kirsten, Vassilevich, ...

Conformal boundary conditions, scalar theory

We need to chose the appropriate boundary conditions, the so called, conformal boundary conditions in a given representation.
The simplest set is governed for scalar theory,

$$D = -(\nabla^2 + E)\ ,\ E = -\frac{d-2}{4(d-1)} R\,.$$

- conformal scalar field in d dimensions:

$$\begin{aligned} &\text{Dirichlet b. c. :} && \phi|_{\partial\mathcal{M}_d} = 0\,, \\ &\text{Robin b. c. :} && (\nabla_n - S)\phi|_{\partial\mathcal{M}_d} = 0\,, \end{aligned}$$

$$S = -\frac{(d-2)}{2(d-1)} K\,.$$

What about the Fermions?

Dirac operator in curved spacetime

$$S = \int d^dx\sqrt{-g}\,\bar{\psi}\,i\gamma^k\hat{\nabla}_k\,\psi\ ,$$

where $\gamma^k = e^k_p\gamma^p$ and γ^p are the Dirac matrices satisfying the Clifford algebra,

$$\begin{aligned}&\gamma^p\gamma^q + \gamma^q\gamma^p = 2\eta^{qp}\,,\ q,p = 0,\,1,\,\cdots d-1\,,\\&\eta = diag(-1,+1,\cdots,+1)\,.\end{aligned}$$

The covariant derivative is defined as

$$\hat{\nabla}_i = \nabla_i + A_i\,.$$

and consequently the field strength tensor will be found as

$$\Omega_{ij} = [\hat{\nabla}_i,\hat{\nabla}_j] = F_{ij} + \frac{1}{4}R_{ij}{}^{pq}\gamma_p\gamma_q\,.$$

The Laplace type operator for Dirac theory is

$$\Delta^{(\frac{1}{2})}\psi \equiv (i\gamma^k \hat{\nabla}_k)^2 = -(\hat{\nabla}^2 + E)\psi \,.$$

For E one finds

$$E = -\frac{1}{4}R + \frac{1}{4}[\gamma^i, \gamma^j]F_{ij} \,.$$

What are the conformal boundary conditions for fermions?

The Dirac operator $\hat{D} = i\gamma^k \hat{\nabla}_k$ is a first order operator so that in this case one has to impose the boundary conditions on a half of fermionic components.

A natural physical condition is to require that the normal component of the fermionic current vanishes on the boundary

$$\psi^\dagger \gamma^n \psi|_{\partial M} = 0\,, \quad \gamma^n = n_k \gamma^k\,.$$

This can be achieved by imposing the Dirichlet boundary condition

$$\Pi_- \psi|_{\partial M} = 0\,,$$

where Π_- is a projector.

Assuming the basic properties of the projectors, such as the unitarity and demanding that $\Pi_-^2 = \Pi_-$ one may conclude some particular commutations relations for Π_- and set

$$\Pi_- = \frac{1}{2}(1-\chi) \quad , \quad \chi = i\gamma^* \gamma^n \, ,$$

where γ^* anti-commutes with all gamma matrices. This is so called the chirality matrix.

A sign of a problem: There is no chirality matrix in odd dimensions!

Starting with $\hat{D}\psi = \lambda\psi$. and applying projector Π_- to both sides of this equation and assuming the condition we arrive at

$$\Pi_- \hat{D}\psi|_{\partial M} = 0\,.$$

After some algebra we arrive at the Robin type boundary condition on the second half of the spinor components,

$$(\hat{\nabla}_n - S)\Pi_+\psi = 0\,, \quad S = -\frac{1}{2}K\Pi_+\,, \quad \Pi_+ = \frac{1}{2}(1+\chi)\,.$$

So in sum the boundary conditions in the case of the fermions (in even dimensions) would be mixed

$$\Pi_{-}\psi|_{\partial M} \oplus (\hat{\nabla}_n - S)\Pi_{+}\psi|_{\partial M} = 0\,.$$

Remind that $\Pi_{\pm} = \frac{1}{2}(1 \pm i\gamma^{*}\gamma^{n})$, where γ^{*} is the chirality matrix and $\gamma^{n} = n_k\gamma^{k}$.

Fermions in even and odd dimensions

In even dimensions

$$\gamma^* \sim \gamma^0\gamma^1 \cdots \gamma^{d-1}\,, \quad (\gamma^*)^2 = 1\,.$$

is the chirality matrix. In odd dimensions, this matrix belongs to the Clifford algebra by itself.

For example in 2 dimensions we have σ^1 and σ^2 which satisfy the Clifford algebra and σ^3 stands for the chirality matrix. But in 3 dimensions σ^3 belongs to the Clifford algebra itself.
Thus, in odd dimensions there does not exist a matrix, γ^*

How do we overcome this problem?
A good solution is to use the doubling trick. See for instance Ivanov and Vassilevich (2022)

Doubling trick

Following the trick we define

$$\Gamma^k = \gamma^k \otimes \sigma^2 \ , \ k = 0, 1, \cdots, d-1 \, ,$$

These new gamma matrices satisfy the Clifford algebra relations,

$$\Gamma^k \Gamma^\ell + \Gamma^\ell \Gamma^k = 2\eta^{k\ell} \, .$$

The respective Dirac fermions have $2^{\frac{d+1}{2}}$ components, twice as the standard Dirac fermions.

Interestingly, now there exist two candidates for the chiral matrix,

$$\Gamma_1^* = \mathbb{I} \otimes \sigma_1 \, , \quad \Gamma_2^* = \mathbb{I} \otimes \sigma_3 \, , \quad (\Gamma_1^*)^2 = 1 \, , \quad (\Gamma_2^*)^2 = 1 \, .$$

We use one to define the appropriate b.c. and save the other one to define the chiral transformation.

Now let us move back to the heat kernel expansion and in particular the coefficient a_d which takes contributions both from the bulk and the boundary

$$a_d = \int_M \mathrm{tr} A_d(x) + \int_{\partial M} \mathrm{tr} B_d(x) \,,$$

where the trace is taken over spinor and group indexes, $A_d(x)$ and $B_d(x)$ are local invarinats constructed from the curvature, gauge fields and the extrinsic curvature.

As I told before, a_d is generally known up to a_5. Let us tak a look at them.

Conformal Invariants in various dimensions

Conformal Invariants, $d = 2$

Let us look at the simplest (and somehow) trivial case, $d = 2$

$$\mathcal{A} = \chi[\mathcal{M}_2|_{\text{bdy.}=\partial\mathcal{M}_2}] = \int_{\mathcal{M}_2} R + 2\int_{\partial\mathcal{M}_2} K\,.$$

Question: So are the boundary invariants the same as the Gibbons-Hawking terms for the bulk anomaly terms? (S.Solodukhin, 2015)

Conformal Invariants, $d = 3$

in $d = 3$, the whole contribution comes from the boundary.

$$\mathcal{A} = a\chi[\partial\mathcal{M}_3] + c_1 I_1 \,,$$

where

$$\chi[\partial\mathcal{M}_3] = \frac{1}{4\pi}\int_{\partial\mathcal{M}_3} \bar{R}$$

is the Euler term. and

$$I_1 \equiv \frac{1}{4\pi}\int_{\partial\mathcal{M}_3} \mathrm{Tr}\,\hat{K}^2$$

Using the Gauss-Codazzi equation we arrive at

$$\chi = \frac{1}{4\pi}\int_{\partial M_3} R - 2R_{nn} - \mathrm{tr}K^2 + K^2 \,.$$

We also have

$$I_1 = \frac{1}{4\pi}\int_{\partial\mathcal{M}_3} \mathrm{tr}K^2 - \frac{1}{2}K^2 \,.$$

Conformal Invariants, $d = 4$

$$\mathcal{A} = -\frac{a}{180}\chi[\mathcal{M}_4|_{\text{bdy.}=\partial\mathcal{M}_4}] + \frac{b}{1920\pi^2}\int_{\mathcal{M}_4} W^2_{ijkl}$$
$$+ \frac{c_1}{240\pi^2}\int_{\partial\mathcal{M}_4} \hat{K}^{ab}W_{anbn} + \frac{c_2}{280\pi^2}\int_{\partial\mathcal{M}_4} \operatorname{Tr}\hat{K}^3$$

Conformal Invariants in $d = 5$ A.F.A, S. Solodukhin 2021

Euler term:

$$E_4 = \int_{\partial \mathcal{M}_5} (\bar{R}^2_{ikj\ell} - 4\bar{R}^2_{ij} + \bar{R}^2)\,.$$

K^4 terms

$$I_1 = \int_{\partial\mathcal{M}_5} (\,\mathrm{Tr}\,\hat{K}^2)^2 = \int_{\partial\mathcal{M}_5} [(\,\mathrm{Tr}\,K^2)^2 - \frac{1}{2}K^2\,\mathrm{Tr}\,K^2 + \frac{1}{16}K^4]\,,$$

and

$$I_2 = \int_{\partial\mathcal{M}_5} \mathrm{Tr}\,\hat{K}^4 = \int_{\partial\mathcal{M}_5} (\,\mathrm{Tr}\,K^4 - K\,\mathrm{Tr}\,K^3 + \frac{3}{8}K^2\,\mathrm{Tr}\,K^2 - \frac{3}{64}K^4)\,.$$

W^2 terms

$$I_3 = \int_{\partial\mathcal{M}_5} W^2_{ikj\ell}$$
$$= \int_{\partial\mathcal{M}_5} \left(R^2_{ikj\ell} - \frac{16}{9} R^2_{ij} + \frac{5}{18} R^2 + \frac{8}{3} R^{ij} R_{injn} + \frac{4}{9} R^2_{nn} - \frac{8}{9} R R_{nn} \right) ,$$

and

$$I_4 = \int_{\partial\mathcal{M}_5} W^2_{injn}$$
$$= \int_{\partial\mathcal{M}_5} \left(\frac{1}{9} R^2_{ij} - \frac{1}{36} R^2 + R^2_{injn} - \frac{2}{3} R^{ij} R_{injn} - \frac{4}{9} R^2_{nn} + \frac{2}{9} R R_{nn} \right) .$$

KKW terms

$$
\begin{aligned}
I_5 =& \int_{\partial\mathcal{M}_5} \hat{K}^{ij}\hat{K}^{k\ell}W_{ikj\ell} \\
=& \int_{\partial\mathcal{M}_5} \Big(K^{ij}K^{k\ell}R_{ikj\ell} + \frac{2}{3}(K^2)^{ij}R_{ij} - \frac{5}{6}KK^{ij}R_{ij} - \frac{1}{12}\operatorname{Tr}K^2R + \frac{1}{8}K^2R \\
&+ \frac{1}{2}KK^{ij}R_{injn} - \frac{1}{6}K^2R_{nn}\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
I_6 =& \int_{\partial\mathcal{M}_5} \hat{K}^i_k\hat{K}^{kj}W_{injn} \\
=& \int_{\partial\mathcal{M}_5} \Big(-\frac{1}{3}K^i_kK^{kj}R_{ij} + \frac{1}{6}KK^{ij}R_{ij} + \frac{1}{12}\operatorname{Tr}K^2R - \frac{1}{24}K^2R \\
&+ K^i_kK^{kj}R_{injn} - \frac{1}{2}KK^{ij}R_{injn} - \frac{1}{3}\operatorname{Tr}K^2R_{nn} + \frac{1}{6}K^2R_{nn}\Big).
\end{aligned}
$$

Invariants with derivatives

$$I_7 = \int_{\partial\mathcal{M}_5} W^2_{nijk}\,.$$

Using the *Gauss-Codazzi* equation

$$W_{nijk} = 2\bar{\nabla}_{[k}K_{j]i} + \frac{2}{3}\gamma_{i[j}\left(\bar{\nabla}_\ell K^\ell_{k]} - \bar{\nabla}_{k]}K\right)\,.$$

we get

$$I_7 = \int_{\partial\mathcal{M}_5}\Big(2\bar{\nabla}_k K_{ij}\bar{\nabla}^k K^{ij} - 2\bar{\nabla}_k K^{ij}\bar{\nabla}_j K^k_i + \frac{4}{3}\bar{\nabla}_i K\bar{\nabla}_j K^{ij} - \frac{2}{3}\bar{\nabla}_i K^{ij}\bar{\nabla}_k K^k_j - \frac{2}{3}(\bar{\nabla}K)^2\Big)\,,$$

And the missing invariant

$$I_8 = \int_{\mathcal{M}_5} \Big(\hat{K}^{ij} \nabla_n W_{injn} + \frac{1}{2} K \hat{K}^{ij} W_{injn} + \frac{2}{9} \bar{\nabla}_i \hat{K}^i_j \bar{\nabla}_k \hat{K}^{kj} - 2(\hat{K}^2)^{ij} \bar{S}_{ij} + \operatorname{Tr} \hat{K}^2 \bar{S}^i_i \Big) \, ,$$

where

$$\bar{S}_{ij} = \frac{1}{2} (\bar{R}_{ij} - \frac{1}{6} \bar{R} \gamma_{ij}) \, .$$

is the 4d Schouten tensor computed using the intrinsic metric of the boundary.

Now the question is that how can we rewrite a_d in terms of these building blocks and what are the corresponding charges?

We can do this for scalars with Dirichlet and Robin boundary conditions as well as for fermions with mixed boundary condition. The 5 dimensional case would be the most non-trivial example, so let us focus on it.

In 5 dimensions we arrive at 27 algebraic equations for 9 charges, let us take a look at them.

Scalar charges

Conformal charges	Dirichlet b.c.	Robin b.c.
a	$\frac{17}{8}$	$-\frac{17}{8}$
c_1	$-\frac{681}{32}$	$\frac{39}{32}$
c_2	$\frac{609}{8}$	$\frac{309}{8}$
c_3	$-\frac{81}{8}$	$\frac{81}{8}$
c_4	$-\frac{27}{2}$	$\frac{27}{2}$
c_5	$-\frac{9}{8}$	$\frac{189}{8}$
c_6	$\frac{819}{8}$	$\frac{441}{8}$
c_7	$-\frac{615}{16}$	$\frac{195}{16}$
c_8	$-\frac{45}{2}$	$-\frac{45}{2}$

Fermion charges

Conformal charges	Mixed b.c.
a	0
c_1	$-\frac{429}{4}$
c_2	621
c_3	0
c_4	0
c_5	270
c_6	990
c_7	-210
c_8	-360

A possible flow for the charges

Let us focus on scalar theory on a d-dimensional unit ball, B^d, with Dirichlet, Neumann and Robin boundary conditions defined on the $(d-1)$-dimensional sphere, S^{d-1}.

$$\begin{cases} \text{Dirichlet b.c.} \ : & \phi|_{S^{d-1}} = 0\,, \\ \text{Neumann b.c.} \ : & \nabla_n \phi|_{S^{d-1}} = 0\,, \\ \text{Robin b.c.} \ : & (\nabla_n - S)\phi|_{S^{d-1}} = 0\,. \end{cases}$$

In the following table we have calculated the coefficint $b = b_d$ of the heat kernel expansion on the boundary, for various Dirichlet, Neumann and Robin boundary conditions.

Dimensions	b_d^D	b_d^N	b_d^R
3	$-\frac{1}{48}$	$\frac{7}{48}$	$\frac{1}{48}$
4	$-\frac{2}{45}$	$\frac{58}{45}$	$-\frac{2}{45}$
5	$\frac{17}{1920}$	$\frac{1873}{1920}$	$-\frac{17}{1920}$
6	$\frac{8}{189}$	$\frac{9448}{945}$	$\frac{8}{189}$
7	$-\frac{367}{32256}$	$\frac{291217}{32256}$	$\frac{367}{32256}$
8	$-\frac{368}{4725}$	$\frac{75856}{675}$	$-\frac{368}{4725}$

At least for $3 \leq d \leq 8$, we observe that

$$b_d^R = (-1)^d b_d^D \, ,$$
$$b_d^N > b_d^D \, .$$

Under investigation with S. Solodukhin.

Conformal anomaly, due to the gauge fields

In the case of the fermions, the gauge field may also contributes to the conformal anomaly
in $d = 4$,

$$\int_{\mathcal{M}_4} \langle T_{ij} \rangle g^{ij} = \frac{1}{24\pi^2} \int_{M_4} F_{ij} F^{ij} \, ,$$

and in $d = 5$

$$\int_{\partial M_5} \langle T_{ij} \rangle g^{ij} = -\frac{3}{128\pi^2} \int_{\partial M_5} F_{an} F^{a}{}_{n} \, .$$

Chiral anomaly

Provided there exists a chirality matrix γ^*, the theory classically possesses a conserved axial current $j^i_A = \bar{\psi}\gamma^i\gamma^*\psi$, $\nabla_i j^i = 0$. In quantum theory the conservation is modified by a quantum anomaly:

$$\nabla_i \langle j^i(x)\rangle = -2i\left[\mathrm{tr}(\gamma^* A_d(x)) + \mathrm{tr}(\gamma^* B_d(x))\right] .$$

See a series of papers by Vassilevich, Ivanov, Kurkov and Marachevsky (2004-2022) See also Witten and K. Yonekura (2019)

- We need to use the expressions we had for the heat kernel coefficiants in various dimensions.
- In odd dimensions we need to replace γ^* with Γ^*_2 .

It is very good that we have two chirality matrices, Γ_2^* and Γ_1^*. Since Γ_2^* commutes with Γ_1^* and thus $\chi = i\Gamma_2^*\Gamma^n$, the b.c. would be chiral invariant.

Although the 4 dimensional case is safe, it is a tricky subtle point in even dimensions since there is just one chirality matrix γ^* which anti-commutes with $\chi = i\gamma^*\gamma^n$ in even dimensions.

Chiral anomaly takes contributions from both the field strength tensor and the curvature tensor, on the base manifold and the boundary.

Chiral anomaly in $d = 2, 3$

Contribution from the curvature tensor: NO
Contribution from the gauge field: Yes The chiral anomaly in $d = 3$ is entirely due to abelian gauge field A^i. The local chiral anomaly is thus due to a boundary term,

$$\nabla_i \langle j^i \rangle = -\frac{1}{2\pi} \int_{\partial \mathcal{M}_3} \epsilon^{ij} \partial_i A_j \, .$$

In $d = 2$ there would be a similar bulk term.

Chiral anomaly in $d = 4$

Contribution from the curvature tensor: Yes

$$\int_{M_4} \nabla_i \langle j^i \rangle = -\frac{1}{384\pi^2}\left[\int_{M_4} \epsilon^{k\ell pq} R_{ijk\ell} R^{ij}{}_{pq} - \frac{16}{5}\int_{\partial M_4} \epsilon^{abc} K_a^d \bar{\nabla}_c K_{bd}\right],$$

The first term here is $\frac{1}{12}P$, where P is the Pontryagin number defined as

$$P = \frac{1}{32\pi^2}\int_{M_4} \epsilon^{k\ell pq} R_{ijk\ell} R^{ij}{}_{pq}\,.$$

Contribution from the gauge field: Yes

$$\int_{M_4} \nabla_i \langle j^i \rangle = -\frac{1}{16\pi^2}\int_{M_4} \epsilon^{ijk\ell} F_{ij} F_{k\ell}\,.$$

A question for further discussion

Pontryagin anomaly, to be or not to be?

Using the other methods, some people admit that there is no Pontryagin term in chiral anomaly. Is it really there?

Chiral anomaly in $d = 5$

$$\int_{M_4} \nabla_i \langle j^i \rangle = -\frac{1}{96(4\pi)^2}(2J_1 + 3J_2 + 2J_3)\,.$$

<u>Contribution from the curvature tensor</u>: Yes

$$J_1 = \int_{\partial M_5} \epsilon^{abcd} W_{abef} W_{cd}{}^{ef}\,,$$

$$J_2 = \int_{\partial M_5} \epsilon^{abcd} W_{abne} W_{cdn}{}^{e} = 4 \int_{\partial M_4} \epsilon^{abcd} \bar{\nabla}_b \hat{K}_{ae} \bar{\nabla}_d \hat{K}^e_c\,,$$

and

$$J_3 = \int_{\partial M_5} \epsilon^{abcd} \hat{K}^e_a \hat{K}^f_b W_{cdef}\,,$$

Chiral anomaly in $d = 5$

Contribution from the gauge field: Yes

$$\int_{\partial\mathcal{M}_5} \nabla_i \langle j^i \rangle = -\frac{1}{2(4\pi)^2} \int_{\partial M_5} \epsilon^{nabcd} F_{ab} F_{cd} \,.$$

Comment on Holography

This is all that we can do in field theory! Should we stop?

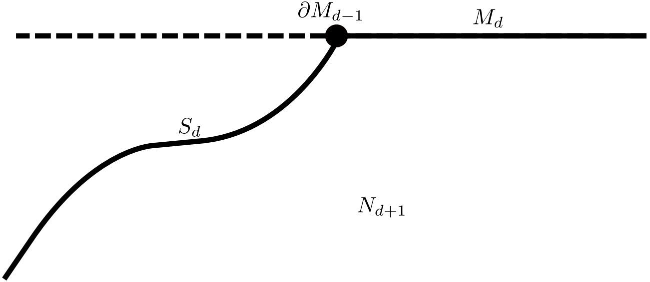
Holography comes to help as usual!

Holographic BCFT

There are many investigations of the holographic BCFTs, done by Takayanagi, Tonni, Herzog, Jensen, Safdi, ... But I want to tell you about an specific prescription, the minimal hypersurface priscription based on my 2017 paper with S. Solodukhin.

Holographic set-up

Extending the boundary into the bulk:



Holographic Calculation, Minimal hypersurface prescription

We modify the gravitational action by adding a boundary term on S_d

$$W_{gr} = -\frac{1}{16\pi G}\int_{N_{d+1}}(R-2\Lambda) - \frac{1}{8\pi G}\left(\int_{M_d} K_M + \int_{S_d} K_S\right).$$

where N_{d+1} is a portion of AdS_{d+1} which is enclosed by S_d.
We impose the minimality condition on the holographic hypersurface

$$K_{\mathcal{S}} = 0\,.$$

and replace the boundary term with a volume term. Referring to AdS/CFT correspondence, the holographic integral anomaly will be defined as

$$W_{gr} = -\int_{\mathcal{M}_d}\langle T\rangle \ln\epsilon\,.$$

But what does this minimality condition mean?
To answer to this question we investigate a theory with the known spectrum in the context of AdS/CFT, *e.g.*, $\mathcal{N} = 4$ SYM on the boundary.

Charges in $4d$

$$\text{real scalar}: a = 1\,,\quad b_1 = b = 1\,,\quad c = 1\ (\text{Dirichlet b. c.})\,,$$
$$\text{real scalar}: a = 1\,,\ b_1 = b = 1\,,\ c = \frac{7}{9}\ (\text{Robin b. c.})\,,$$
$$\text{Dirac fermion}: a = 11\,,\ b_1 = b = 6\,,\ c = 5,\ (\text{mixed b. c.})\,,$$
$$\text{gauge boson}: a = 62, b_1 = b = 12, c = 8\ (\text{absolute or relative b. c.}).$$

Conformal anomaly in $d = 4$: $\mathcal{N} = 4$ $SU(N)$ SYM

The free field multiplet consists of $n_s = 6$ scalars, $n_f = 2$ Dirac fermions and $n_v = 1$ gauge bosons, each field in the adjoint representation of $SU(N)$. Only scalars are sensitive to the choice of boundary conditions. The total number $n_s^D + n_s^R = 6$ is fixed. Introducing $\Delta n = n_s^D - n_s^R$ we find,

$$a = 90(N^2 - 1)\,, \quad b = b_1 = 30(N^2 - 1)\,, \quad c = (\frac{70}{3} + \frac{1}{2}\Delta n)(N^2 - 1)\,.$$

and hence the integral anomaly is (we focus only on the boundary terms)

$$\int_{\mathcal{M}_4} \langle T \rangle_{SYM} = \frac{(N^2 - 1)}{24\pi^2} \int_{\partial \mathcal{M}_4} [\frac{3}{2}(K^{\mu\nu} + K n^\mu n^\nu - \frac{2}{3} K g^{\mu\nu}) R_{\mu\nu} + (K \mathrm{tr} K^2 - \frac{5}{9} K^3) + \frac{3\Delta n}{70} \mathrm{tr} \hat{K}^3]\,.$$

The last term which is sensitive to the choice of the boundary conditions, disappears if $n_s^D = n_s^R = 3$. This is exactly what we obtain from our minimal-surface prescription.

Note that in this case one imposes the Dirichlet boundary condition on a half of scalars and the Robin boundary condition on the other half of scalars. This is exactly the condition for maximal preservation of SUSY in $\mathcal{N} = 4$ superconformal theory. In this case $1/2$ of SUSY will be preserved Gaiotto-Witten.

We can also find the holographic conformal charges in 5d, following the minimal hypersurface construction

Charges	Holographic value
a	$-\frac{1}{64}$
c_1	$\frac{5}{192}$
c_2	$-\frac{1}{32}$
c_3	$\frac{1}{64}$
c_4	$-\frac{1}{16}$
c_5	$\frac{1}{8}$
c_6	0
c_7	$-\frac{1}{32}$
c_8	$-\frac{1}{16}$

Unpublished work with S. Solodukhin.

Summary

- We have investigated the boundary terms of the conformal anomaly and the chiral anomaly.
- In odd dimensions it needs a special traetment.
- We have discussed a possible holographic picture of BCFTs demanding the maximal preservation of SUSY.

Thank You!