On the anomaly inflow

Dmitri Vassilevich

UFABC

Conformal Anomalies: Theory and Applications, Tours, June 2024

Support: CNPq, FAPESP

Basics

Let M be a manifold with boundary ∂M. One speaks about anomaly inflow when an anomaly on M can be expressed through an anomaly in a theory on ∂M. Original motivation [Callan and Harvey, 1985]: a way to obtain anomaly-free theories.

Milestone in Mathematics: the Atiyah–Patodi–Singer theorem [1975] which relates the index of a Dirac operator on M to the η invariant (parity anomaly) for an effective Dirac operator on ∂M. (Non-local boundary conditions.)

Renewed interest: Witten and Yonekura [2019], phase of the Dirac determinant through the η invariant on ∂M. (Non-hermitian and not strongly elliptic boundary conditions).

What can one say about anomaly inflow with local hermitian strongly elliptic boundary conditions?
[Ivanov & D.V., JHEP (2022)]

Example: parity anomaly

Let $\not{D}$ be a Dirac operator with eigenvalues λ. The η function is

$$\eta(s,\not{D}) = \sum_{\lambda>0} \lambda^{-s} - \sum_{\lambda<0} (-\lambda)^{-s}$$

with s being a complex parameter. (We are on a Euclidean manifold.) For $\Re s$ sufficiently large, the sum above is convergent and defines a meromorphic function on $\mathbb{C}$. At $s=0$, $\eta(0,\not{D})$ defines the spectral asymmetry of $\not{D}$ which is the parity anomaly. Note: the parity anomaly is not a smooth functional of background fields.

Th exponentiated η invariant

$$\mathcal{E}_{\not{D}} = \exp\left(-i\pi(\eta(0,\not{D}) + \dim \mathrm{Ker}\,\not{D})\right)$$

is smooth.
The heat kernel expansion at $t \to +0$:

$$\mathrm{Tr}\left(Qe^{-t\not{D}^2}\right) \simeq \sum_{k=0}^{\infty} t^{\frac{k-n}{2}} a_k(Q,\not{D}^2)$$

where $n = \dim M$, Q is a matrix-valued function.
Consider a variation $\not{D} \to \not{D} + \delta\not{D}$. If the eigenvalues λ do not cross 0,

$$\delta\eta(0,\not{D}) = -\frac{2}{\sqrt{\pi}} a_{n-1}(\delta\not{D},\not{D}^2)$$

This equation defines all variations of $\mathcal{E}_{\not{D}}$.

Physics

- $\eta(0, \not{D})$ defines the parity anomaly (a parity odd part of the effective action) and the phase of the partition function in Euclidean theory.
- For fermions in a 3D topologically trivial space in an external electromagnetic field the parity anomaly is a $k = \pm 1/2$ Chern-Simons action (corresponding to anomalous Hall conductivity).
- Parity anomaly is always gauge invariant, also with respect to large gauge

Boundaries

Let $n = \dim M$ be odd and $\gamma_\star$ be the chirality matrix. Let $\partial M = \cup_\alpha \partial M_\alpha$ consist of several connected components. For a Dirac type operator the boundary conditions are of the form

$$\Pi_- \psi|_{\partial M} = 0$$

where Π_- is a rank $1/2$ projector. A good choice is a local projector

$$\Pi_- = \frac{1}{2}(1 - \chi), \qquad \chi = i\varepsilon_\alpha \gamma_* \gamma^n,$$

where $\varepsilon_\alpha = \pm 1$, γ^n is a $\gamma^\mu n_\mu$, n_μ is an inward pointing unit normal to the boundary. These conditions are called the bag boundary conditions. One can check that for these conditions the normal component of the fermion current $\psi^\dagger \gamma^n \psi$ vanishes on the boundary. Thus the Dirac operator is hermitian.

Let us assume that $\not{D}$ is just the usual massless Dirac operator with a gauge connection and M has a product structure near the boundary.
Near the boundary the Dirac operator reads (in a suitable basis)

$$\not{D} = \begin{pmatrix} \mathcal{D} & i\partial_n \\ i\partial_n & -\mathcal{D} \end{pmatrix}$$

One can show that the boundary conditions kill either upper or lower components of the spinor. The Dirac operator restricted to the space of boundary data is $\mathcal{D}_\alpha = -\varepsilon_\alpha \mathcal{D}$.

One can also show that $\mathcal{E}(\not{D})$ is a product of contributions from components of the boundary, and that each contribution is independent of the bulk geometry. Each contribution can be thus computed for any choice of the bulk which make the spectrum simple. [This bulk appears to be a direct product with an interval.]

$$\mathcal{E}(\not{D})^2 = C \prod_\alpha \mathcal{E}(\mathcal{D}_\alpha)$$

where C has vanishing local variations (i.e., it is topological). This results is consistent with explicit calculations for $n = 4$.

Note $\mathcal{E}^2$ in the formula above!

$\underline{n \text{ is odd}}$. To define bag boundary conditions we need a chirality matrix which anticommutes with all gamma matrices. Thus we have to double the number of fermions. The chirality matrix then depends on an angle θ which may be different $(= \theta_\alpha)$ on different components of the boundary.
By proceeding similarly to the case of even n one gets

$$\mathcal{E}(\not{D}) = \exp\left(-i\pi\mathcal{N}_0 + 2\pi i \sum_\alpha \theta_\alpha \mathrm{Ind}(\mathcal{D}, \gamma^n)\right)$$

where $\mathcal{N}_0$ is the number of zero modes of $\not{D}$ when all $\theta_\alpha = 0$.

Both results are surprisingly simple and quite “holographic”.

Conformal anomaly

Let n be odd. Consider a scalar field on M subject to Dirichlet or Neumann boundary conditions on ∂M_α. Again, we assume a product structure near the boundary. Let L be a scalar Laplace operator with conformal coupling. Near the boundary, $L = -\partial_n^2 + \widehat{L}$. One can show that if the function f has vanishing normal derivatives,

$$a_n(f, L) = \frac{1}{4} \sum_\alpha \left(\mp a_{n-1}(f, \widehat{L})\right)$$

where $\mp$ corresponds to Dirichlet/Neumann bc.
L.h.s. is the conformal anomaly, while r.h.s. is not quite since $\widehat{L}$ is not conformal in $n-1$ dimensions.

Boundary anomalies and boundary states

Boundary states are the eigenfunctions of the Dirac operator which decay exponentially fast with the distance from the boundary. Thus we need a mass (which defines the falloff speed) and noncompact manifolds. However, since the formulas for variation of the η function work for localized variations we need compact manifolds.
A way out [Fresneda, Souza and D.V. (2023)]: consider an “infinite radius limit”.
In this limit, there is indeed a relation between boundary anomalies and boundary states if there is a lower bound for the localization distance of boundary modes (bag boundary conditions). There is no suche relation otherwise (Example: chiral bag boundary conditions, Fermi-arc like states).

Comments and Remarks

I. (Technical) The requirement of product structure near the boundary may be

- weakened in low dimensions by considering all possible invariants which may enter the heat kernel coefficients;
- replaced by the requirement of being conformal to a product structure in higher dimensions.

II. (Domain walls) Domain walls may be of many different types. The APS theorems have been derived in two cases: when the gauge field jumps on the wall and when the mass function jumps. (Ask me for the references.)