The local renormalization group and the anomaly of scale invariance

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Conformal Anomalies: Theory and applications
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from work in collaboration with G. Paci and D. Sauro

Plan

- Discuss the differences between scale and conformal invariance
- Study which properties of conformal invariance may generalize to full scale invariance
- Find in-between "symmetries"

Conformal vs scale anomaly

Classical Weyl symmetry

Local Weyl rescalings by $\sigma = \sigma(x)$

$$g_{\mu\nu} \to g'_{\mu\nu} = \mathrm{e}^{2\sigma} g_{\mu\nu} \qquad \Phi \to \Phi' = \mathrm{e}^{w_\Phi \sigma} \Phi$$

The energy-momentum tensor

$$T^{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

Nöther identities of Diff and Weyl symmetries on-shell

$$\nabla_\mu T^{\mu\nu} = 0 \qquad T^\mu{}_\mu = 0$$

Quantum Weyl symmetry

From the path-integral

$$e^{-\Gamma} = \int [d\Phi]\, e^{-S}$$

The renormalized EMT

$$\langle T^{\mu\nu} \rangle = -\frac{2}{\sqrt{g}} \frac{\delta \Gamma}{\delta g_{\mu\nu}}$$

Conformal anomaly coming from the path-integral on-shell

$$\langle T^{\mu}{}_{\mu} \rangle = \text{beta terms} + \text{anomaly}$$

Scale vs conformal symmetry

For a rigid scale transformation $\sigma = \mathrm{const.}$

$$\int \mathrm{d}^d x \sqrt{g}\, T^{\mu}{}_{\mu} = 0$$

Implies the existence of a "virial" current D_μ

$$T^{\mu}{}_{\mu} = \nabla_\mu D^\mu$$

The current *must not* have anomalous dimension, but I believe

$$\langle T^{\mu}{}_{\mu} - \nabla_\mu D^\mu \rangle = \text{beta terms} + \text{anomaly}$$

Why is this interesting?

Second order phase transitions imply diverging correlation length

$$\xi \sim t^{\frac{1}{\nu}} \qquad \text{e.g.} \quad t = (T - T_c)/T_c$$

$\xi \to \infty$ implies scale invariance not conformal invariance in general.
However, we do study conformally invariant systems most of the times. Our bias?
Old argument: operator $[D^\mu]$ acquires an anomalous dimension, say

$$(d-1) + \gamma_D$$

So $[T] \sim [\partial \cdot D]$ cannot work in general unless $\gamma_D = 0$.

But, we do have *physically interesting* examples of scale-but-not-conformal models:

- **Elastic membranes** (i.e., interacting versions of Cardy-Riva counter-example)
- **Aharony-Fisher dipolar ferromagnets**

A source for D_μ: gauging the Weyl group

Introduce an Abelian gauge potential

$$g_{\mu\nu} \to g'_{\mu\nu} = \mathrm{e}^{2\sigma} g_{\mu\nu} \qquad S_\mu \to S'_\mu = S_\mu - \partial_\mu \sigma \qquad \Phi \to \Phi' = \mathrm{e}^{w_\Phi \sigma} \Phi$$

The unique gauged covariant *compatible* derivative

$$\hat{\nabla}_\mu \Phi = \nabla_\mu \Phi + L_\mu \cdot \Phi + w_\Phi S_\mu \Phi$$

It contains "disformation" because dilatations do not commute with Poincaré

$$(L_\mu)^\alpha{}_\beta = \frac{1}{2}(S_\beta \delta^\alpha_\mu + S_\mu \delta^\alpha_\beta - S^\alpha g_{\beta\mu})$$

By construction it transforms covariantly under Weyl: $\hat{\nabla}_\mu \Phi \to \hat{\nabla}'_\mu \Phi' = \mathrm{e}^{w_\Phi \sigma} \hat{\nabla}_\mu \Phi$

Consequences of gauging Weyl

There is a new dilation current

$$T^{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}} \qquad D^{\mu} = \frac{1}{\sqrt{g}}\frac{\delta S}{\delta S_{\mu}}$$

Classically gauged Weyl and Diff symmetries with $W = \mathrm{d}S$ imply

$$T^{\mu}{}_{\mu} = \nabla^{\mu} D_{\mu} \qquad \hat{\nabla}_{\mu} T^{\mu\nu} + D_{\mu} W^{\mu\nu} = 0$$

In the limit $S_{\mu} \to 0$ we have scale invariance and D_{μ} is virial

$$T^{\mu}{}_{\mu} = \nabla^{\mu} D_{\mu} \qquad \nabla_{\mu} T^{\mu\nu} = 0$$

Local RG analysis of the anomaly

Renormalization with local couplings

Use local couplings to source observables

$$S \supset -\int \mathrm{d}^d x \sqrt{g} \lambda^i(x) \mathcal{O}_i$$

Currents source the expectation values

$$\langle T^{\mu\nu} \rangle = -\frac{2}{\sqrt{g}} \frac{\delta \Gamma}{\delta g_{\mu\nu}} \qquad \langle D^{\mu} \rangle = \frac{1}{\sqrt{g}} \frac{\delta \Gamma}{\delta S_{\mu}} \qquad \langle \mathcal{O}_i \rangle = -\frac{1}{\sqrt{g}} \frac{\delta \Gamma}{\delta \lambda^i}$$

We expect the path-integral to give the anomaly

$$\langle T^{\mu}{}_{\mu} \rangle = \langle \nabla^{\mu} D_{\mu} \rangle + \text{beta terms} + \text{curvatures}$$

Local rg interpretation

Local scale transformation on the geometrical sources

$$\Delta_\sigma^W = \int \left\{ 2\sigma g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} - \partial_\mu \sigma \frac{\delta}{\delta S_\mu} \right\}$$

Local scale transformation caused by the rg beta functions

$$\Delta_\sigma^\beta = -\int \sigma \beta^i \frac{\delta}{\delta \lambda^i}$$

The anomaly of Γ is expressed

$$\Delta_\sigma^W \Gamma = \Delta_\sigma^\beta \Gamma + A_\sigma \qquad A_\sigma \supset \{\partial_\mu \lambda^i, R, S_\mu, W_{\mu\nu} \cdots\}$$

Wess-Zumino consistency

Rewrite

$$\Delta_\sigma \Gamma = (\Delta_\sigma^W - \Delta_\sigma^\beta)\Gamma = A_\sigma$$

For Wess-Zumino's consistency and Abelian transf.

$$[\Delta_\sigma, \Delta_{\sigma'}]\Gamma = 0$$

Consistency condition for the anomaly

$$(\Delta_\sigma^W - \Delta_\sigma^\beta)A_{\sigma'} - (\sigma \leftrightarrow \sigma') = 0$$

The anomaly in two dimensions

Most general parametrization of A_σ using $\hat{R} = R - 2\nabla^\mu S_\mu$

$$A_\sigma = \frac{1}{2\pi}\int \mathrm{d}^2x\sqrt{g}\Big\{\sigma\frac{\beta_\Phi}{2}\hat{R} - \sigma\frac{\chi_{ij}}{2}\partial_\mu\lambda^i\partial^\mu\lambda^j - \partial_\mu\sigma w_i\partial^\mu\lambda^i$$
$$+\sigma\beta_\Psi\nabla_\mu S^\mu + \sigma\frac{\beta_2^S}{2}S_\mu S^\mu - \partial_\mu\sigma\beta_3^S S^\mu + \sigma z_i\partial_\mu\lambda^i S^\mu\Big\}$$

Rich structure of anomaly terms $\Longrightarrow$ cohomological analysis

- $\delta_\sigma\hat{R} = -2\sigma\hat{R}$ is a b-type anomaly
- $\delta_\sigma\nabla_\mu S^\mu = -\nabla^2\sigma$ is a mixed anomaly (a-type + boundary)

Geometry in couplings space

- χ_{ij} is a metric, $\chi_{ij}\sim\langle O_i O_j\rangle$
- Four scalar "charges" β_Φ, β_Ψ, β_2^S and β_3^S

Consistency in two dimensions

The tensors in the anomaly are all observables thorough 2-point functions that we want to constrain using Wess-Zumino's identities.

Apply Wess-Zumino's

$$[\Delta_\sigma, \Delta_{\sigma'}]\Gamma = \frac{1}{2\pi}\int \mathrm{d}^2x\sqrt{g}\big(\sigma\partial_\mu\sigma' - \sigma'\partial_\mu\sigma\big)\mathcal{Z}^\mu = 0$$

Condition $\mathcal{Z}_\mu = \partial_\mu\lambda^i\mathcal{Y}_i + S_\mu\mathcal{X} = 0$ among tensors becomes (here $\partial_i = \partial/\partial g_i$)

$$\mathcal{Y}_i = -\partial_i\beta_\Psi + \chi_{ij}\beta^j - \beta^j\partial_j w_i - w^j\partial_i\beta_j + z_i$$
$$\mathcal{X} = \beta_2^S - \beta^i\partial_i\beta_3^S - z_i\beta^i$$

A hint of (ir)reversibility and potential gradient-like structure

Define a new charge

$$\tilde{\beta}_\Psi = \beta_\Psi + w_i \beta^i + \beta_3^S$$

Using $\Theta = \beta^i O_i$ and defining $\Theta' = \Theta - \partial_\mu D^\mu$, the observable

$$\langle T(x) T(0) \rangle - \langle \Theta'(x) \Theta'(0) \rangle \sim \tilde{\beta}_\Psi \partial^2 \delta^{(2)}(x)$$

alternatively $\mathcal{T} = T - \partial \cdot D$

$$\langle \mathcal{T}(x) \mathcal{T}(0) \rangle - \langle \Theta(x) \Theta(0) \rangle \sim \tilde{\beta}_\Psi \partial^2 \delta^{(2)}(x)$$

Using both $\mathcal{Y}_i = 0$ and $\mathcal{Z} = 0$

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \tilde{\beta}_\Psi = \beta^i \partial_i \tilde{\beta}_\Psi = \chi_{ij} \beta^i \beta^j + \beta_2^S$$

Conditions for C-like theorems

In the limit in which $D_\mu = 0$, i.e., S_μ is decoupled

$$\beta_\Phi = \beta_\Psi \qquad\qquad \beta_2^S = 0$$

Requiring unitarity we have Zamolodchikov's metric $G_{ij} = \frac{1}{8} |x|^4 \langle \mathcal{O}_i(x) \mathcal{O}_j(0) \rangle > 0$.

Osborn proves that there is a scheme in which $\chi_{ij} \to G_{ij}$ and $\tilde{\beta}_\Psi \sim C$

However unitarity + Poincaré = conformal i.e. no D_μ if unitary
Are there less stringent conditions for $\beta^i \partial_i \tilde{\beta}_\Psi \geq 0$ suitable for scale invariance?

Gradient-like flow and β_2^S obstruction

Assume by contradiction $A(g)$ function such that

$$\beta^i = \gamma^{ij}\partial_j A$$

By construction

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}A = \beta^i\partial_i A = \gamma_{ij}\beta^i\beta^j$$

Suggests through the identification $\tilde{\beta}_\Psi$ that A function exists iff $\beta_2^S = 0$

$$\chi_{ij} \leftrightarrow \gamma_{(ij)} \qquad\qquad A \leftrightarrow \tilde{\beta}_\Psi$$

Simple applications

Higher derivative scalar

Higher derivative free scalar is a (log)CFT in flat space in $d = 2$

$$\mathcal{L} = \frac{1}{2}(\partial^2 \varphi)^2$$

Notice that $\langle \varphi(x)\varphi(0)\rangle \sim |x|^2$ for φ primary, in contrast with $(\partial_x^2)^2 \langle \varphi(x)\varphi(0)\rangle \sim \delta(x)$

Does not admit a conformal action in $d = 2$ because of the obstruction

$$S_{\mathrm{conf}}[\varphi, g] = -\frac{1}{2}\int \mathrm{d}^2 x \sqrt{g}\,\varphi \Delta_4 \varphi$$

$$\Delta_4 \varphi = (\nabla^2)^2 \varphi + 2\nabla^\mu \Big(P_{\mu\nu}\nabla^\nu \varphi + \cdots\Big) - (d-4)\Big(P^{\mu\nu}P_{\mu\nu} + \cdots\Big)\varphi$$

$$P_{\mu\nu} = \frac{1}{d-2}\Big\{R_{\mu\nu} - \frac{1}{2(d-1)} R g_{\mu\nu}\Big\}$$

Gauging the higher derivative scalar

Assign the weight $w(\varphi) = \frac{4-d}{2} \to 1$

$$S[\varphi, g_{\mu\nu}, S_\mu] = -\frac{1}{2}\int \mathrm{d}^2x\sqrt{g}\varphi(\hat{\nabla}^2)^2\varphi$$

Does admit a gauged action in $d = 2$

$$(\hat{\nabla}^2)^2\varphi = (\nabla^2)^2\varphi + B^{\mu\nu}\nabla_\mu\partial_\nu\varphi + C^\mu\partial_\nu\varphi + D\varphi$$
$$B_{\mu\nu} = 2g_{\mu\nu}S^\rho S_\rho - 4S_\mu S_\nu + 4\nabla_{(\mu}S_{\nu)}$$

Using heat kernel methods (Barvinsky-Wachowski) $\beta_2^S = 0$, $\beta_\Phi = \frac{1}{3}$ and $\beta_\Psi = \frac{4}{3}$

$$A_\sigma = \frac{1}{2\pi}\int \mathrm{d}^2x\sqrt{g}\sigma\left\{\frac{R}{6} + \nabla^\mu S_\mu\right\}$$

Another application: theory of elasticity

Elastic $2d$ membrane with strain $u_{\mu\nu} = \partial_{(\mu} u_{\nu)}$ considered by Cardy-Riva

$$S[u] = \frac{1}{2} \int \mathrm{d}^2 x \Big\{ 2\, g\, u_{\mu\nu} u^{\mu\nu} + \, k\, u_\mu{}^\mu u_\nu{}^\nu \Big\}$$

Gauging $u_{\mu\nu} \to \hat{\nabla}_{(\mu} u_{\nu)}$ we find

$$A_\sigma = \frac{1}{2\pi} \int \mathrm{d}^2 x \sqrt{g} \Big\{ \frac{13g + 5k}{6(2g+k)} R - \frac{3g+k}{2g+k} \nabla^\mu S_\mu - \frac{(3g+k)^2}{4g(2g+k)} S_\mu S^\mu \Big\} + \cdots$$

Charges $\beta_\Phi = \beta_\Psi = \frac{2}{3}$ and $\beta_2^S = 0$ in the global conformal limit $3g + k = 0$, in general:

$$\beta_\Phi = \frac{5}{3} + \frac{g}{(2g+k)}\,, \qquad \beta_\Psi = \frac{2}{3}\,, \qquad \beta_2^S = -\frac{(3g+k)^2}{4g(2g+k)}$$

Further and future developments

Analysis with the fundamental field

Gimenez-Grau + Nakayama + Rychkov: in actual models with a field φ_μ, the virial current must be protected by some hidden symmetry.

They find in all examples of scale-but-not conformal models a shift-symmetry with respect to some field U (Abelian + commuting with Poincaré)
$\Longrightarrow$ new quantum number to multiplets, including $[\varphi_\mu]$

The vector $U\varphi_\mu$ has scaling dimension $d-1$, there is a candidate virial current

$$D_\mu \sim U\varphi_\mu + \cdots$$

Can we see this as emerging from the local rg analysis? Need fundamental fields

Four dimensions

The analysis of the $d = 4$ anomaly with gauged-Weyl transformations is ongoing.
It may have implications for Weyl-invariant higher derivative quantum gravity.

Gregorio Paci (in the audience) has just completed the cohomological analysis of the anomaly with constant couplings.

The potential S_μ can be interpreted as the torsion vector in metric-affine geometries as discussed with **Dario Sauro** (also in the audience).

Somewhere between scale and conformal invariance: the restricted Weyl group

Grupoids

It is possible to define group-like substructures of Weyl

$$g_{\mu\nu} \to g'_{\mu\nu} = \Omega^2 g_{\mu\nu} \qquad \text{for} \qquad H_g(\Omega) = \Box_g \Omega + \frac{d-4}{2\Omega} g^{\mu\nu} \partial_\mu \Omega \partial_\nu \Omega = 0$$

They are partial but associative, i.e.

$$H_g(\Omega_1) = 0 \qquad \text{and} \qquad H_{(\Omega_1)^2 g}(\Omega_2) = 0 \qquad \Longrightarrow \qquad H_g(\Omega_2 \Omega_1) = 0$$

The above is the harmonic/restricted Weyl "subgroup" unique such that

$$\sqrt{g} R \to \sqrt{g'} R' = \sqrt{g} R$$

We have shown the uniqueness of some substructures under certain assumptions.

Nöther identities

Classically, they imply that there is a scalar Φ such that

$$T^{\mu}{}_{\mu} = \Box_g \Phi$$

Quantum mechanically

$$\langle T^{\mu}{}_{\mu} \rangle = \Box_g \langle \Phi \rangle + \mathcal{A} + (\beta \text{ terms})$$

For example, a nonminimally coupled scalar field in $d = 4$

$$(4\pi)^2 \mathcal{A} = \frac{1}{120} W^2_{\alpha\beta\mu\nu} - \frac{1}{360} E_4 + \frac{1}{2}\left(\xi - \frac{1}{6}\right)^2 R^2$$
$$(4\pi)^2 \langle \Phi \rangle = \left(\xi - \frac{1}{6}\right)\left(3\varphi^2 - \frac{R}{6}\right)$$

Application: higher derivative gravity

Take conformal gravity: $S_{\rm wg} = \frac{1}{2\lambda}\int W^2$ and partially gauge fix Weyl symmetry with

$$\frac{1}{2\alpha}\int \sqrt{g}R^2$$

There is a BRST

$$\delta_B g_{\mu\nu} = 2cg_{\mu\nu} \qquad \delta_B \overline{c} = -2c\overline{c} + b \qquad \delta_B c = 0 \qquad \delta_B b = -2cb$$

The gauge fixed action has residual harmonic invariance

$$S = S_{\rm wg} + S_{\rm gf} + S_{\rm gh} = \int {\rm d}^4x\sqrt{g}\Big\{\frac{1}{2\lambda}W^2 + \frac{1}{2\alpha}R^2\Big\} + 6\int {\rm d}^4x\sqrt{g}\,\overline{c}\,\Box_g c$$

Asymptotic freedom in conformal vs higher derivative gravity

Conformal gravity is asymptotically free but anomalous at 2-loops

$$\beta_\lambda = -\frac{1}{(4\pi)^2}\frac{199}{15}\lambda^2$$

Higher derivative gravity is also free but requires a tachyon

$$\beta_\lambda = -\frac{1}{(4\pi)^2}\frac{133}{10}\lambda^2$$

They differ precisely by the contribution of two scalars (i.e., the ghosts).

$\Longrightarrow$ relation with the a-gravity proposal for the UV-completion of GR

Conclusions

- Largely unexplored field discussing the anomaly and the boundary between scale and conformal invariance
- Potentially interesting applications in both statistical mechanical models and quantum gravity

Thank you for listening

Extras on the integration of the anomaly and ambiguities

In two dimensions

For zero beta functions $\beta = 0$ the anomaly is

$$\langle T^\mu{}_\mu \rangle = aR$$

We want to integrate the anomaly, take $g_{\mu\nu} = \mathrm{e}^{2\sigma}\bar{g}_{\mu\nu}$

$$\sqrt{g}R = \sqrt{\bar{g}}(\bar{R} - 2\bar{\nabla}^2\sigma)$$

Using $\frac{\delta}{\delta\sigma}\Gamma \sim \langle T \rangle$, find $\Gamma_{\mathrm{ind}} \subset \Gamma$

$$\Gamma_{\mathrm{ind}} = a \int \mathrm{d}^2x\sqrt{g}\left(\sigma R + \sigma\nabla^2\sigma\right)$$

On-shell in σ we get Polyakov's

$$\Gamma_{\mathrm{ind}} = \frac{a}{4}\int \mathrm{d}^2x\sqrt{g}R\frac{1}{-\nabla^2}R$$

In four dimensions

The anomaly is

$$\langle T^{\mu}{}_{\mu} \rangle = bW^2 + a\tilde{E}_4 + a' \Box R$$

Having defined

$$\tilde{E}_4 = E_4 - \frac{2}{3}\Box R = E_4 + \nabla^{\alpha}\left(-\frac{2}{3}\nabla_{\alpha} R\right)$$

The transformations

$$\sqrt{g}\tilde{E}_4 = \sqrt{\bar{g}}\left(\bar{\tilde{E}}_4 + 4\bar{\Delta}_4\sigma\right) \qquad \sqrt{g}W^2 = \sqrt{\bar{g}}\bar{W}^2$$

$$\sqrt{g}\Box R = -\frac{1}{4}\frac{\delta}{\delta\sigma}\int \mathrm{d}^4x\sqrt{g}R^{\mu\nu}R_{\mu\nu}$$

Four dimensional anomaly

We can integrate each term separately

$$\Gamma = \Gamma_{\rm conf}[g] + \frac{a_1'}{12} \int d^4x \sqrt{g}\, R^2 + \int d^4x \sqrt{g} \left(b_1 W^2 + a_1 \tilde{E}_4 \right) \frac{1}{\Delta_4} \tilde{E}_4$$

Applications

- Quantum field theory $\longrightarrow$ C- and A-theorems
- Black holes $\longrightarrow$ corrections to BH entropy
- Cosmology $\longrightarrow$ expanding universe

In general even d

The anomaly is conjectured (Cardy)

$$\langle T^{\mu}{}_{\mu} \rangle = \sum_i b_i \mathcal{W}_i + a\tilde{E}_d + \nabla_\mu \mathcal{J}^\mu$$

Such that

$$\tilde{E}_d = E_d + \nabla_\mu \mathcal{V}^\mu$$

The transformations

$$\sqrt{g}\tilde{E}_d = \sqrt{\bar{g}}\left(\bar{\tilde{E}}_d + d\bar{\Delta}_d \sigma\right) \qquad \sqrt{g}\mathcal{W}_i = \sqrt{\bar{g}}\bar{\mathcal{W}}_i$$

$$\sqrt{g}\nabla_\mu \mathcal{J}^\mu = \frac{\delta}{\delta\sigma}\int \mathrm{d}^4 x \sqrt{g}\mathcal{L}_{\mathrm{local}}(g, \partial g, \cdots)$$

d-dimensional anomaly

We can integrate each term separately

$$\Gamma = \Gamma_c[g] + \int \mathrm{d}^d x \sqrt{g}\mathcal{L}_{\rm local} + \int d^d x \sqrt{g} \left(b_i \mathcal{W}_i + a_1 \tilde{E}_d \right) \frac{1}{\Delta_d} \tilde{E}_d$$

Main points

- Existence of $\tilde{E}_d$
- Existence of Δ_d
- Ambiguities in $\mathcal{L}_{\rm local}$
- Enumeration of $\mathcal{W}_i$

Conformal geometry
and the Fefferman-Graham ambient space

Lightcone embedding in flat space

Move from $\mathbb{R}^d$ to $\mathbb{R}^{d+2}$ on the lightcone

$$Y^A = (Y^\mu, Y^+, Y^-) \qquad \eta_{AB} Y^A Y^B = 0 \qquad Y^A \sim \lambda Y^A$$

Spacetime embedding in the lightcone

$$x^\mu \to Y^A = (Y^\mu, Y^+, Y^-) = Y^+(x^\mu, 1, -x^2)$$
$$Y^A \to x^\mu = \frac{Y^\mu}{Y^+}$$

Embedding Lorentz generates conformal on spacetime

$$(Y'^+)^2 \eta_{\mu\nu} dx'^\mu dx'^\nu = (Y^+)^2 \eta_{\mu\nu} dx^\mu dx^\nu$$

Fefferman-Graham ambient space

Use Cartesian coordinates, $X^2 = 2t^2\rho$, $t = X^+$

$$Y^A \to X^A = (X^\mu, X^{d+1}, X^{d+2}) \stackrel{*}{=} t\Big(x^\mu, \frac{1+2\rho-x^2}{2}, \frac{1-2\rho+x^2}{2}\Big)$$

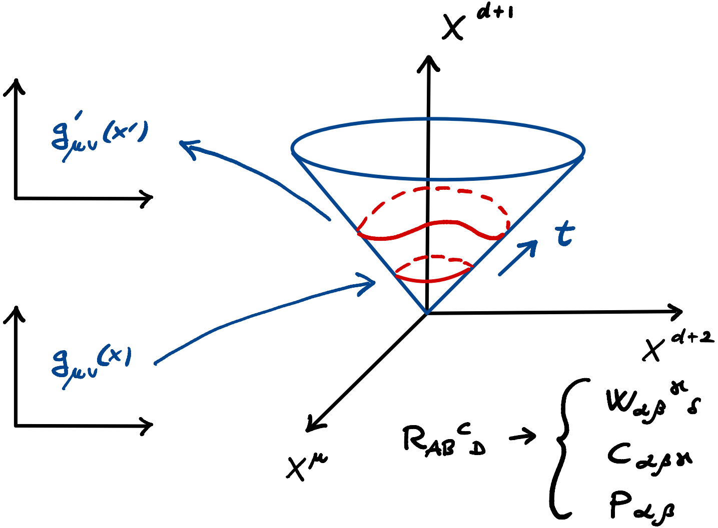
The flat embedding metric

$$\tilde{\eta} = \eta_{AB} dx^A dx^B \stackrel{*}{=} 2\rho dt^2 + 2t dt d\rho + t^2 \eta_{\mu\nu} dx^\mu dx^\nu$$

In curved space: FG metric with $R_{AB} = 0$, $\mathcal{L}_{t\partial_t}\tilde{g} = 2\tilde{g}$ and $h_{\mu\nu}(x, \rho = 0) = g_{\mu\nu}$

$$\tilde{g} = \tilde{g}_{AB} dx^A dx^B \stackrel{*}{=} 2\rho dt^2 + 2t dt d\rho + t^2 h_{\mu\nu}(x,\rho) dx^\mu dx^\nu$$

Ambient Space in a nutshell



PBH diffeomorphisms

A diffeomorphism of the ambient

$$\delta_\zeta \tilde{g}_{AB} = \mathcal{L}_\zeta \tilde{g}_{AB} = \zeta^C \partial_C \tilde{g}_{AB} + \tilde{g}_{AC} \partial_B \zeta^C + \tilde{g}_{BC} \partial_A \zeta^C$$

If it preserves the form of the ambient metric

$$\zeta^t = t\sigma(x) \qquad \zeta^\rho = -2\rho\sigma(x) \qquad \zeta^\mu = \xi^\mu(x) + \cdots$$

It generates Diff $\times$ Weyl on spacetime

$$\delta_\zeta h_{\mu\nu}|_{\rho=0} = \delta_\zeta g_{\mu\nu} = \delta_{\sigma,\xi} g_{\mu\nu} = 2\sigma g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\mu \xi_\nu$$

Ricci-flatness determines $h_{\mu\nu}$

Expand in ρ

$$h_{\mu\nu}(x,\rho) = g_{\mu\nu}(x) + \rho {h^{(1)}}_{\mu\nu} + \frac{1}{2}\rho^2 {h^{(2)}}_{\mu\nu} + \cdots$$

The coefficients find obstructions in even d

$$ {h^{(1)}}_{\mu\nu} = 2P_{\mu\nu} = \frac{2}{d-2}\left(R_{\mu\nu} - \frac{R}{2(d-1)} g_{\mu\nu}\right)$$

$$ {h^{(2)}}_{\mu\nu} = -\frac{2}{d-4} B_{\mu\nu} + 2P_{\mu\sigma}{P^{\sigma}}_{\nu}$$

$$ {h^{(3)}}_{\mu\nu} = \frac{2}{(d-6)(d-4)} \nabla^2 B_{\mu\nu} + \cdots$$

Ambient Laplacian

Scalar Laplacian of the embedding

$$-\Box_{\tilde{g}}\Phi = -\frac{1}{t^2}\Box_h\Phi - \frac{2}{t}\partial_t\partial_\rho\Phi - \frac{1}{2t}\partial_t\Phi - \frac{d-2}{t^2}\partial_\rho\Phi + \frac{\rho}{t^2}{h'}_\mu{}^\mu\partial_\rho\Phi$$

Consider an embedding scalar field

$$\Phi = t^{\Delta_\varphi}\varphi(x)$$

The projection of the Laplacian gives Yamabe

$$-\Box_{\tilde{g}}(t^{\Delta_\varphi}\varphi(x))|_{\rho=0} = t^{\Delta_\varphi-2}\left(-\Box_g - \frac{R}{2(d-1)}\right)\varphi$$

We can construct a family of powers of conformal GJMS Laplacians

$$P_{2n}\varphi(x) \equiv t^{-\frac{2n+d}{2}}(-\Box_{\tilde{g}})^n(t^{\frac{2n-d}{2}}\varphi)|_{\rho=0}$$

Conformal Laplacians

There are derivative and constant parts

$$P_{2n}\varphi(x) = \Delta_{2n} + \frac{d-2n}{2} Q_{2n}$$

Constant part transforms nicely: Q-curvatures in $d = 2n$

$$\sqrt{g} Q_d = \sqrt{\bar{g}} (\bar{Q}_d + \bar{\Delta}_d \sigma)$$

In fact we just found in $d = 2n$

$$\tilde{E}_d = dQ_d + \text{conformal invariants}$$

A physicist proof of Cardy's conjecture

The anomaly is best parametrized

$$\langle T^{\mu}{}_{\mu}\rangle = \sum_i b_i \mathcal{W}_i + a Q_d + \nabla_\mu \mathcal{J}^\mu$$

So that the integration is always possible

$$\Gamma = \Gamma_c[g] + \int \mathrm{d}^d x \sqrt{g}\,\mathcal{L}_{\rm local} + \int d^4x\,\sqrt{g}\,(b_i \mathcal{W}_i + a_1 Q_d)\,\frac{1}{\Delta_d} Q_d$$

- Ambient curvatures enumerate conformal invariants
- Scaling analysis dictates local anomaly ($\mathcal{J}^\mu$ is like a "virial" current)
- Ambiguities in defining Δ_d come from embedding Riemann in $d \geq 6$