Odd and even trace anomalies

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Summary

Motivation

- Anomalies are a crucial gateway to non-perturbative aspects of Quantum Field Theory (QFT)
  - 't Hooft anomaly matching conditions place constraints in RG flows
  - Gauge anomalies signal quantum inconsistencies which put severe constraints on possible spectra and interactions.
- Conformal symmetry, albeit generically broken either spontaneously or explicitly, characterizes UV and IR fixed points of RG flows and the corresponding anomalies determine protected universal quantities independent of the details of RG flow
- Conformal symmetry is instrumental in addressing such issues in quantum gravity as renormalizability and singularity resolution.

Weyl symmetry

- General coordinate transformations $x^\mu \to x^\mu + \xi^\mu(x)$ induce

$$\delta_\xi g_{\mu\nu} = D_\mu \xi_\nu + D_\nu \xi_\mu, \qquad \xi_\mu = g_{\mu\nu}\xi^\nu \tag{1}$$

- Weyl transformations are

$$\delta_\omega g_{\mu\nu}(x) = 2\omega(x)\, g_{\mu\nu}(x) \tag{2}$$

- The energy momentum tensor $T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{S[g]}{\delta g^{\mu\nu}}$ is both conserved on shell and traceless

$$D^\mu T_{\mu\nu}(x) = 0, \qquad T^\mu_\mu(x) = 0 \tag{3}$$

Cohomological problem

- Due to

$$\delta_\xi \xi^\mu = \xi^\lambda \partial_\lambda \xi^\mu \tag{4}$$

we can define BRST nilpotent transformations

$$\delta_\xi^2 = 0 \quad , \quad \delta_\omega^2 = 0, \qquad \delta_\xi \delta_\omega + \delta_\omega \delta_\xi = 0 \tag{5}$$

- Cocycles

$$(\delta_R + \delta_S)\, W = \Delta_R + \Delta_S \tag{6}$$

satisfy

$$\delta_R^2 = 0 \qquad \delta_S^2 = 0, \qquad \delta_R \delta_S + \delta_S \delta_R = 0 \tag{7}$$

- In known cases, $\Delta_R = \delta_R \mathcal{C}$, and we can redefine $W \to W' = W - \mathcal{C}$, so that

$$\delta_R W' = \Delta'_R = 0 \tag{8}$$

$$\delta_S W' = \Delta_S - \delta_S \mathcal{C} \equiv \Delta'_S \tag{9}$$

Weyl cocycles

Nontrivial cocycles of δ_ω with vanishing diffeomorphism partner in 4d are

$$\Delta[g,\omega] = \int d^4x\,\sqrt{g}\,\omega\,T(x), \qquad \delta_\omega \Delta[g,\omega] = 0 \tag{10}$$

where the density T can be the quadratic Weyl density

$$\mathcal{W}^2 = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2, \tag{11}$$

the Gauss-Bonnet (or Euler) density,

$$E = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \tag{12}$$

and the Pontryagin density,

$$P = \frac{1}{2}\left(\varepsilon^{\mu\nu\mu'\nu'}R_{\mu\nu\lambda\rho}R_{\mu'\nu'}{}^{\lambda\rho}\right). \tag{13}$$

Classically conserved stress-energy tensors for Weyl fermions

- The classically conserved and traceless (on-shell) stress-energy tensor

$$T^{(R)}_{\mu\nu} = \frac{i}{4}\overline{\psi_R}\gamma_\mu \overleftrightarrow{\nabla}_\nu \psi_R + \{\mu \leftrightarrow \nu\}, \qquad \nabla_\mu = D_\mu + \frac{1}{2}\omega_\mu + V_\mu \tag{14}$$

provides the standard coupling to gravity $T_{\mu\nu}(x) = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}$ after redefining $\psi_R \to |g|^{\frac{1}{4}}\psi_R$

- Another possible definition follows from no redefinition

$$\widehat{T}^{(R)}_{\mu\nu} = \frac{i}{4}\overline{\psi_R}\gamma_\mu \overleftrightarrow{\nabla}_\nu \psi_R + (\mu \leftrightarrow \nu) - g_{\mu\nu}\frac{i}{2}\overline{\psi}_R\gamma^\lambda \overleftrightarrow{\nabla}_\lambda \psi_R = T^{(R)}_{\mu\nu} - g_{\mu\nu}T^{(R)\lambda}_{\lambda}, \tag{15}$$

- This possible source of ambiguity affecting the trace at quantum level was checked to be removed by introducing the local counterms needed to preserve diff-WIs, which have a non-Weyl invariant part.

Ward Identities

- In terms of the generating functional

$$W[h,V] = W[0] + \sum_{n,r=1}^{\infty} \frac{i^{n+r-1}}{2^n n! r!} \int \prod_{i=1}^{n} dx_i \, h^{\mu_i \nu_i}(x_i) \prod_{l=1}^{r} dy_l \, e_{a_l}^{\lambda_l}(y_l) V_{\lambda_l}(y_l)$$
$$\cdot \langle 0 | \mathcal{T} T^{(R)}_{\mu_1 \nu_1}(x_1) \dots T^{(R)}_{\mu_n \nu_n}(x_n) j_R^{a_1}(y_1) \dots j_R^{a_r}(y_r) | 0 \rangle \,, \quad (16)$$

  where $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$ and $J_{R\mu} = i\bar{\psi}_R \gamma_\mu \psi_R$, WIs can be written in compact form in terms of $\langle\langle T_{\mu\nu}(x) \rangle\rangle = 2 \frac{\delta W[h,V]}{\delta h^{\mu\nu}(x)}$.
- If we define trace anomaly *in the perturbative case* as

$$g^{\mu\nu} \langle\langle T_{\mu\nu}(x) \rangle\rangle - \langle\langle g^{\mu\nu} T_{\mu\nu}(x) \rangle\rangle = T[g](x) \quad (17)$$

  any ambiguity in the relevant amplitudes is removed.

Gauge-induced odd parity trace anomaly

- For $T^{(R)}_{\lambda\rho} = \langle 0|\mathcal{T} T^{\mu}_{R\mu}(x) j_{R\lambda}(y) j_{R\rho}(z)|0\rangle$,

$$\widetilde{T}^{(R)}_{\lambda\rho}(k_1,k_2)\Big|_{odd} + \widetilde{T}^{(R)}_{\rho\lambda}(k_2,k_1)\Big|_{odd} = \frac{1}{24\pi^2}\varepsilon_{\mu\nu\lambda\rho}k_1^{\mu}k_2^{\nu}, \tag{18}$$

- For $T^{(R)\mu}_{\mu\lambda\rho} = \eta^{\mu\nu}\langle 0|\mathcal{T} T_{R\mu\nu}(x) j_{R\lambda}(y) j_{R\rho}(z)|0\rangle$,

$$\widetilde{T}^{(R)\mu}_{\mu\lambda\rho}(k_1,k_2) + \widetilde{T}^{(R)\mu}_{\mu\rho\lambda}(k_2,k_1)\Big|_{odd} = \frac{1}{48\pi^2}\varepsilon_{\mu\nu\lambda\rho}k_1^{\mu}k_2^{\nu}\,. \tag{19}$$

- Formula (17) implies

$$\mathcal{A}^{(odd,R)}_{\omega} = g^{\mu\nu}\langle\!\langle T_{R\mu\nu}(x)\rangle\!\rangle\Big|_{odd} - \langle\!\langle g^{\mu\nu}T_{R\mu\nu}(x)\rangle\!\rangle\Big|_{odd} = -\frac{i}{384\pi^2}\varepsilon_{\mu\nu\lambda\rho}\mathrm{tr}(F^{\mu\nu}F^{\lambda\rho}) \tag{20}$$

Gravity-induced odd parity trace anomaly (I)

- Since for a CFT $\langle 0|\mathcal{T}T^{(R)}_{\mu\nu}(x)T^{(R)}_{\mu'\nu'}(y)T^{(R)}_{\alpha\beta}(z)|0\rangle^{(odd)}$ vanishes identically for algebraic reasons,

$$\eta^{\mu\nu}\langle\!\langle T^{(R)}_{\mu\nu}(x)\rangle\!\rangle^{(odd)} = 0, \tag{21}$$

- For $\widetilde{\mathsf{T}}_{\mu\nu\mu'\nu'} = \langle 0|\mathcal{T}T^{(R)\mu}_{\nu}(x)T^{(R)}_{\mu'\nu'}(y)T^{(R)}_{\alpha\beta}(z)|0\rangle^{(odd)}$,

$$\widehat{\mathsf{T}}^{(\mathrm{tot})}_{\mu\nu\mu'\nu'}(k_1,k_2) = -\frac{i}{3072\pi^2}k_1^{\alpha}k_2^{\beta}\left(\left(k_1^2+k_2^2+k_1\cdot k_2\right)t_{\mu\nu\mu'\nu'\alpha\beta} - t^{(21)}_{\mu\nu\mu'\nu'\alpha\beta}\right). \tag{22}$$

where

$$t^{(21)}_{\mu\nu\mu'\nu'\kappa\lambda} = k_{2\mu}k_{1\mu'}\varepsilon_{\nu\nu'\kappa\lambda} + k_{2\nu}k_{1\nu'}\varepsilon_{\mu\mu'\kappa\lambda} + k_{2\mu}k_{1\nu'}\varepsilon_{\nu\mu'\kappa\lambda} + k_{2\nu}k_{1\mu'}\varepsilon_{\mu\nu'\kappa\lambda} \tag{23}$$

A few comments

- The term $k_1^\alpha k_2^\beta\,(k_1^2+k_2^2)t_{\mu\nu\mu'\nu'\alpha\beta}$ vanishes on the shell $k_1^2=k_2^2=0$. It produces a cocycle removed by a diff-breaking counterterm

$$\sim \int d^4x\, h\, \varepsilon^{\mu\nu\lambda\rho}\partial_\mu \Box h_\nu^\alpha \partial_\lambda h_{\rho\alpha} \tag{24}$$

  which however can be removed by field redefinition.
- We can define a restricted cohomology defined up to terms $\Box h_{\mu\nu}$ and $\Box\xi^\mu$.
- To lowest order we get

$$\langle\!\langle T_\mu^\mu(x)\rangle\!\rangle^{(odd)} \approx -\frac{i}{768\pi^2}\varepsilon^{\mu\nu\lambda\rho}\left(\partial_\mu\partial_\sigma h_\nu^\tau\,\partial_\lambda\partial_\tau h_\rho^\sigma - \partial_\mu\partial_\sigma h_\nu^\tau\,\partial_\lambda\partial^\sigma h_{\tau\rho}\right) \tag{25}$$

  covariantized to

$$\langle\!\langle T_\mu^\mu(x)\rangle\!\rangle^{(odd)} = -\frac{i}{768\pi^2}\,\frac{1}{2}\,\varepsilon^{\mu\nu\lambda\rho}R_{\mu\nu}{}^{\sigma\tau}R_{\lambda\rho\sigma\tau} \tag{26}$$

Gravity-induced odd parity trace anomaly (II)

- Formula (17) implies

$$T[g](x) = \frac{i}{768\pi^2}\,\frac{1}{2}\,\varepsilon^{\mu\nu\lambda\rho} R_{\mu\nu}{}^{\sigma\tau} R_{\lambda\rho\sigma\tau}. \tag{27}$$

- The anomalies (20) and (27) violate P, but, due to the imaginary coefficient in front, do not violate T. CPT invariance implies they do not violate CP either. However the quantum effective Hamiltonian becomes complex, which may signal violation of unitarity.
- The Kimura-Delbourgo-Salam anomaly

$$\partial^\mu j_{5\mu} = \frac{1}{768\pi^2} \varepsilon^{\mu\nu\lambda\rho} R_{\mu\nu}{}^{\sigma\tau} R_{\lambda\rho\sigma\tau}. \tag{28}$$

is an ABJ-type anomaly of a Dirac fermion theory rigidly linked to odd parity trace anomalies.

- Generalizing Bardeen's approach we consider the metric $\widehat{g}_{\mu\nu}(\hat{x}) = g_{\mu\nu} + \gamma_5 f_{\mu\nu}$ with $\widehat{x} = x_1 + \gamma_5 x_2$ with a natural conjugate coordinate $\bar{\hat{x}} = x_1 - \gamma_5 x_2$.
- One must introduce axially-extended (AE) diffeomorphisms

$$\widehat{x}^\mu \to \widehat{x}^\mu + \widehat{\xi}^\mu(\widehat{x}^\mu), \qquad \widehat{\xi}^\mu = \xi^\mu + \gamma_5 \zeta^\mu \tag{29}$$

and AE Weyl transformations

$$\widehat{g}_{\mu\nu} \longrightarrow e^{2\widehat{\omega}} \widehat{g}_{\mu\nu}, \qquad \widehat{\omega} = \omega + \gamma_5 \eta \tag{30}$$

- The geodesic

$$\ddot{\widehat{x}}^\mu + \widehat{\Gamma}^\mu_{\nu\lambda} \dot{\widehat{x}}^\nu \dot{\widehat{x}}^\lambda = 0 \tag{31}$$

is parametrized by $\widehat{s}$

$$\frac{d\widehat{s}}{d\hat{t}} = \sqrt{\widehat{g}_{\mu\nu} \dot{\widehat{x}}^\mu \dot{\widehat{x}}^\nu}. \tag{32}$$

Fermions in a MAT background

- The action is given by

$$\widehat{S} = \int d^4\widehat{x} \left(i\overline{\psi}\sqrt{\widehat{\overline{g}}}\gamma^a \widehat{e}_a^\mu \left(\partial_\mu + \frac{1}{2}\widehat{\Omega}_\mu \right) \psi \right)(\widehat{x}) \tag{33}$$

- The MAT e.m. tensor

$$\mathbf{T}_{\lambda\rho} = -\frac{i}{2}\overline{\psi}\widehat{\gamma}_\lambda \left(\partial_\rho + \frac{1}{2}\widehat{\Omega}_\rho \right) \psi + (\lambda \leftrightarrow \rho) = -\frac{i}{2}\overline{\psi}\widehat{\gamma}_\lambda \widehat{\nabla}_\rho \psi + (\lambda \leftrightarrow \rho) \tag{34}$$

where $\widehat{\gamma}_\lambda = \gamma_a \widehat{e}_\lambda^a$ and $\mathbf{T}_{\lambda\rho} = T_{\lambda\rho} + \gamma_5 T_{5\lambda\rho}$ satisfies the classical WIs

$$\mathcal{T}(x) \equiv T_{\mu\nu} g^{\mu\nu} + T_{5\mu\nu} f^{\mu\nu} = 0, \tag{35}$$

$$\mathcal{T}_5(x) \equiv T_{\mu\nu} f^{\mu\nu} + T_{5\mu\nu} g^{\mu\nu} = 0, \tag{36}$$

Schwinger-DeWitt method for the trace anomaly

- The effective action is

$$\frac{\widehat{L}(x)}{\mu^{\mathsf{d}}} = -\frac{i}{2}(4\pi\mu^2)\,\mathrm{tr}\int_0^\infty d\widehat{s}\,(4\pi i\mu^2\widehat{s})^{-\frac{\mathsf{d}}{2}-1}\sqrt{\widehat{g}}e^{-im^2\widehat{s}}[\widehat{\Phi}(\widehat{x},\widehat{x},\widehat{s})] \tag{37}$$

where $\widehat{\Phi}(\widehat{x},\widehat{x}',\widehat{s}) = \sum_{n=0}^{\infty}\widehat{a}_n(\widehat{x},\widehat{x}')(i\widehat{s})^n$ is determined by a heat kernel equation.

- By dimensional regularization the trace anomaly is determined as

$$\int d^4\widehat{x}\,\mathrm{tr}\left(\widehat{\omega}\sqrt{\widehat{g}}\,\widehat{g}^{\mu\nu}\widehat{\Theta}_{\mu\nu}\right) \quad = \quad \frac{1}{16\pi^2}\int d^4\widehat{x}\,\mathrm{tr}\left(\sqrt{\widehat{g}}\,\widehat{\omega}\,[\widehat{a}_2]|_{m=0}\right) \tag{38}$$

where $\frac{2}{\sqrt{\widehat{g}}}\frac{\delta}{\delta\widehat{g}^{\mu\nu}}\widehat{L} = \widehat{\Theta}_{\mu\nu}$ and

$$\begin{aligned}[\widehat{a}_2] \quad = \quad & \frac{1}{2}m^4 - \frac{1}{12}m^2\widehat{R} + \frac{1}{288}\widehat{R}^2 - \frac{1}{120}\widehat{R}_{;\mu}{}^{\mu} - \frac{1}{180}\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu} + \frac{1}{180}\widehat{R}_{\mu\nu\lambda\rho}\widehat{R}^{\mu\nu\lambda\rho} \\ & + \frac{1}{48}\widehat{\mathcal{R}}_{\mu\nu}\widehat{\mathcal{R}}^{\mu\nu}\end{aligned} \tag{39}$$

MAT trace anomalies

- For a general MAT background

$$\mathcal{T}(x) = \frac{i}{1536\pi^2}\varepsilon^{\mu\nu\lambda\rho}\left(\sqrt{g_+}R^{(+)}_{\mu\nu\alpha\beta}R^{(+)\alpha\beta}_{\lambda\rho} + \sqrt{g_-}R^{(-)}_{\mu\nu\alpha\beta}R^{(-)\alpha\beta}_{\lambda\rho}\right) \tag{40}$$

$$\mathcal{T}_5(x) = \frac{i}{1536\pi^2}\varepsilon^{\mu\nu\lambda\rho}\left(\sqrt{g_+}R^{(+)}_{\mu\nu\alpha\beta}R^{(+)\alpha\beta}_{\lambda\rho} - \sqrt{g_-}R^{(-)}_{\mu\nu\alpha\beta}R^{(-)\alpha\beta}_{\lambda\rho}\right) \tag{41}$$

where $\widehat{\mathcal{R}}_{\mu\nu} = P_+\mathcal{R}^{(+)}_{\mu\nu} + P_-\mathcal{R}^{(-)}_{\mu\nu}$.

- In the chiral limit

$$g_{\mu\nu} \to \eta_{\mu\nu} + \frac{h_{\mu\nu}}{2}, \qquad f_{\mu\nu} \to \frac{h_{\mu\nu}}{2}, \tag{42}$$

$$\mathcal{T}(x) = \frac{i}{1536\pi^2}\varepsilon^{\mu\nu\lambda\rho}R_{\mu\nu\alpha\beta}R_{\lambda\rho}{}^{\alpha\beta} \equiv \mathcal{T}_R(x) \tag{43}$$

Different kinds of anomalies

- Odd-parity trace anomalies trace back their origin to the obstructions to the existence of a Weyl fermion propagator, similarly to chiral gauge anomaly which are encoded by an non-trivial family's index. As such, whenever the gravity background is dynamical, they signal an inconsistency which must be removed through one of the known anomaly cancellation mechanisms.
- Even-parity trace anomalies appear also in a Dirac fermion theory, for which propagators are well defined (their family's index vanishes!) and similarly to odd-parity anomalies of the ABJ type they find large applications in phenomenological applications.

Anomaly cancellations

MSM trace-gravity

The multiplet of the Minimal Stantard Model (MSM), when weakly coupled to gravity, will produce an overall non-vanishing (imaginary) coefficient for the Pontryagin density in the trace anomaly.
$\Longrightarrow$ Sterile neutrinos can naturally cancel it

MSM trace-gauge

- We have six units of the anomaly (20) with curvature $F \equiv F^{\mathfrak{su}(3)}$ and six units with opposite sign. Therefore the MSM multiplet is free of these anomalies.
- We have instead 4 units of the same anomaly with gauge field $F \equiv F^{\mathfrak{su}(2)}$ and positive sign, computed in the doublet representation of $\mathfrak{su}(2)$.
- Finally we have a $U(1)$ gauge-induced trace anomaly with vanishing total coefficient: $6\left(\frac{1}{6}\right)^2 - 3\left(\frac{2}{3}\right)^2 - 3\left(-\frac{1}{3}\right)^2 + 2\left(-\frac{1}{2}\right)^2 - (-1)^2 = 0$

Conclusions and outlook

- Odd-parity trace anomalies have been computed thorugh a number of both perturbative and non-perturbative techniques giving consistent results. Family's index theorem, through the correspondence of quantum numbers (form degree, dimension, parity, Lorentz covariance) for the densities (of the Chern and Pontryagin classes), might provide further evidence.
- They have far-reaching implications on our understanding of the possible extensions of the Standard Model and on Quantum Gravity..
- ...work in progress

Thanks!!!

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