From the trace anomaly to non-conformal theories

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Is there an universal quantity $\mathcal{A} = g^{\mu\nu}\langle T_{\mu\nu}\rangle - \langle g^{\mu\nu} T_{\mu\nu}\rangle$?

(Is there an imaginary term in the trace anomaly of Weyl fermions in $D = 4$?)

Based on:

R. Ferrero, SFV, M. Fröb and W. Lima (2024)

S. Abdallah, SFV and M. Fröb (2023, 2021).

Outline

Consider a classical field theory. If the system under study possesses a continuous symmetry, then from Noether's theorem (in a classical, i.e. non-quantum theory)

$$\exists j^\mu, \partial_\mu j^\mu \overset{*}{=} 0. \tag{1}$$

If the system is quantized, $\partial_\mu \langle \hat{j}^\mu \rangle \overset{?}{=} 0$?

If $\neq 0$, the symmetry is said to be anomalous.

Why anomalies?

- Pion decay (Bell and Jackiw): $\pi \to \gamma\gamma$ (anomaly in J_A^μ).
- Applications in condensed matter/hydrodynamics (review Chernodub et al., 2021, experiment Gooth et al., 2017): $\overrightarrow{J}_\epsilon = \left(a_{F\tilde{F}} + a_{R\tilde{R}} T^2\right) \overrightarrow{B}$ (anomaly in J_A^μ).
- Kuzmin, Rubakov and Shaposhnikov (1985); still new proposals as QCD baryogenesis [1911.01432v2 [hep-ph]].
- Bouncing universes (instead of Big Bang) from anomalies, Fabris, Pelinson, Shapiro [gr-qc/9810032], Asorey et al. [2202.00154], Camargo and Shapiro [2206.02839] (trace anomaly).
- Axionic dark matter (Basilakos, Mavromatos, Solà [gr-qc/2001.03465]) (anomaly for a Kalb–Ramond field).
- BRST formalism and cohomological aspects.
- Anomaly cancellation (t' Hooft, but also recently Navarro Salas or Nicolai).

Why nonconformal theories?

- (small?) deviations from conformality generally expected;
- in renormalization group flow, conformality achieved only at fixed points;
- conserved quantities along such flow? c-theorem (Komargodski et al. 2023, Schwimmer et al. 2023)
- trace cancellation in supertheories (Meissner and Nicolai, 2018);
- Weyl fermions and the anomaly? (Bonora, 2022);

The trace anomaly

Consider a field theory in curved space, and analyze the (local) symmetry under changes of scales

$$g_{\mu\nu} \to e^{-2\sigma} g_{\mu\nu}, \; \Phi_i \to e^{k_i \sigma} \Phi,$$

where the ($n = 4$) Weyl weights for matter fields are $k_i = (-1, -3/2, 0)$ for scalar, fermionic and vector fields.

$$\begin{aligned} \frac{\delta S}{\delta \sigma(x)} &= \int \mathrm{d}^4 y \frac{\delta g_{\mu\nu}(y)}{\delta \sigma(x)} \frac{\delta S}{\delta g_{\mu\nu}(y)} + \#\text{EOM} \\ &= -2 g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}}(x) + \#\text{EOM} \\ &= -\sqrt{-g} g_{\mu\nu} T^{\mu\nu} + \#\text{EOM} \end{aligned}$$

$\implies$ if scale invariant, $T^{\mu}{}_{\mu} \overset{*}{=} 0$.

Does it survive the quantization, i.e. $g_{\mu\nu} \langle T^{\mu\nu} \rangle = 0$?

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Trace Anomalies in Dimensional Regularization.

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M. J. Duff
Department of Astrophysics, University of Oxford - Oxford

(ricevuto il 22 Gennaio 1974)

Summary. — An unusual perturbation theory anomaly is pointed out. If there exists a trace identity valid in an arbitrary number of dimensions, then employing dimensional regularization can result in an amplitude satisfying the identity in an arbitrary number of dimensions, but the finite part of the amplitude violating it in four dimensions. An example given here is the one-loop neutrino contribution to the graviton propagator. Anomalous behaviour, of a different origin, also occurs in the one-loop photon contribution. Both kinds of anomaly can be removed at the expense of introducing *n*-dimensional, rather than 4-dimensional, counterterms.

1. – Introduction.

Dimensional regularization (1-3), the scheme whereby the dimension of space-time is used as a regulating parameter in dealing with the divergences of quantum field theory, has gained a good deal of popularity. In gauge theories especially, where it is essential that the regularization scheme respect the Ward identities, this method seems particularly appealing.

In this paper, however, we wish to issue a word of warning. Dimensional regularization, at least in its conventional form, can and does give rise to anomalies. That is to say, symmetries present in the original Lagrangian are

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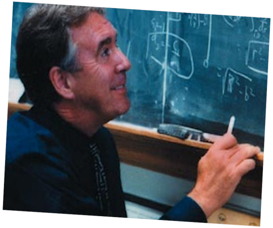

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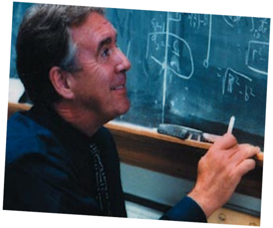

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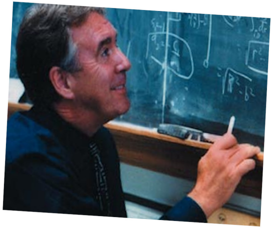


Trace anomalies in one line

[1803:09764 [hep-th] Bruque, Cherchiglia and Pérez-Victoria]: consider a two-dimensional space in which the following integral appears in a perturbative expansion.

$$I_{\mu\nu} = \int \mathrm{d}^2 k \, \frac{k_\mu k_\nu}{(k^2 + m^2)^2} .$$

We need to make sense of it by applying some regularization $R : I_{\mu\nu} \to [I_{\mu\nu}]^R$.

$$\begin{aligned} [I_{\mu\nu}]^R &= \frac{1}{2} \left[\int \mathrm{d}^2 k \, \frac{\delta_{\mu\nu}}{k^2 + m^2} - \int \mathrm{d}^2 k \, \partial_{k^\mu} \left(\frac{k_\nu}{k^2 + m^2} \right) \right]^R \\ &= \frac{1}{2} \left[\int \mathrm{d}^2 k \, \frac{\delta_{\mu\nu}}{k^2 + m^2} \right]^R \\ &= \frac{\delta_{\mu\nu}}{2} \left[\int \mathrm{d}^2 k \, \frac{k^2}{(k^2 + m^2)^2} + \frac{m^2}{(k^2 + m^2)^2} \right]^R \\ &= \frac{\delta_{\mu\nu}}{2} \left[I_{\alpha\alpha} + \pi \right]^R , \end{aligned}$$

$$\Longrightarrow \qquad \delta^{\mu\nu} [I_{\mu\nu}]^R = [\delta^{\mu\nu} I_{\mu\nu}]^R + \pi .$$

Anomaly induced action

In $n = 4$ spacetime dimensions, on dimensional grounds

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle = wC^2 + bE_4 + c\Box R + \alpha R^2 + \beta F^2,$$

E_4: Euler characteristic,

C: Weyl tensor,

w, b, c, α, β: depend on the theory.

Consider semi-classical gravity: integrate out the (free) matter fields.

$$\Downarrow$$

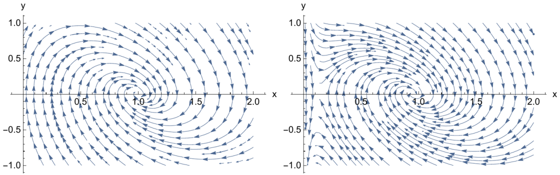
If the matter action is classically conformal invariant, the induced action for the (classical) gravitational field should satisfy the variational equation

$$\frac{\delta\Gamma_{\rm ind}}{\delta\sigma(x)} = -\sqrt{-g}g_{\mu\nu}\langle T^{\mu\nu}\rangle.$$

We can integrate the anomaly to obtain $\Gamma_{\rm ind}$! (Riegert, $\cdots$)

Inflation, bouncing driven by quantum effects:

- Fabris, Pelinson, Shapiro [gr-qc/9810032]
- Hawking, Hertog, Reall [hep-th/0010232]
- ...



(©Shapiro+ [hep-th/2012.10554], $y \sim \dot{H}$, $x \sim H$)

Outline

Nonconformal theories

On physical grounds, the proposal by Duff 1994, Casarin, Godazgar and Nicolai 2018 in dim reg is:

$$\mathcal{A} := g^{\mu\nu} \langle T_{\mu\nu} \rangle - \langle T_{\mu\nu} g^{\mu\nu} \rangle \ . \tag{2}$$

Consider a **non-conformal-invariant** theory

$$S = -\frac{1}{2} \int \left(\nabla^{\mu}\phi \nabla_{\mu}\phi + m^2\phi^2 + \xi R\phi^2 \right) \sqrt{-g} \mathrm{d}^n x \, .$$

Define in terms of the composite operators $\Phi^{(2)} := \phi^2$ and $\Phi^{(3)}_{\mu\nu} := \nabla_{\mu}\phi\nabla_{\nu}\phi$:

$$T_{\mu\nu} = \left(\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu} - \frac{g_{\mu\nu}}{2} g^{\rho\sigma} \right) \Phi^{(3)}_{\rho\sigma} - \frac{g_{\mu\nu}}{2} m^2 \Phi^{(2)} - \xi \left(\nabla_{\mu}\nabla_{\nu} - g_{\mu\nu}\nabla^2 - R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R \right) \Phi^{(2)} \, ,$$

$$T \, : = g^{\mu\nu} T_{\mu\nu} = -\frac{n-2}{2} g^{\rho\sigma} \Phi^{(3)}_{\rho\sigma} - \frac{1}{2} n m^2 \Phi^{(2)} + \xi(n-1)\nabla^2 \Phi^{(2)} - \xi \frac{n-2}{2} R \Phi^{(2)} \, .$$

Renormalization is needed. . .

Follow Hollands and Wald [arXiv:0103074, 0404074], there is (fortunately) a finite renormalization ambiguity; in $n = 4$

$$\Phi^{(2),\mathrm{ren}} \to \Phi^{(2),\mathrm{ren}} + c_{0,0} m^2 + c_{2,0} R\,,$$
$$\Phi^{(3),\mathrm{ren}}_{\mu\nu} \to \Phi^{(3),\mathrm{ren}}_{\mu\nu} + \sum_{i=1}^{8} c_{4,i} C^{(4,i)}_{\mu\nu} + m^2 \sum_{i=1}^{2} c_{2,i} C^{(2,i)}_{\mu\nu} + c_{0,1} m^4 g_{\mu\nu}\,.$$

Can be used to obtain a conserved EM tensor (diffeo invariance).

What happens in **other regularizations**? It seems that we need more ingredients:

- involve classically vanishing contribution (EOM);
- "analiticity" in the parameters and dimensionality fixes

$$E := \langle \phi\, \mathrm{EOM} \rangle = \nabla^2 \Phi^{(2)} - 2\left(m^2 + \xi R\right)\Phi^{(2)} - 2 g^{\rho\sigma} \Phi^{(3)}_{\rho\sigma}\,.$$

Expressing $\Phi^{(3)}$ in terms of T,

$$\mathcal{A}_{\beta,\beta_{\mathrm{tr}}} := g^{\mu\nu}\left\langle (T_{\mu\nu} + \beta g_{\mu\nu} E)^{\mathrm{ren}} \right\rangle - \left\langle (T + \beta_{\mathrm{tr}} E)^{\mathrm{ren}} \right\rangle$$
$$= \frac{n\beta - \beta_{\mathrm{tr}}}{n - 2 + 4n\beta}\left[4g^{\mu\nu}\left\langle T^{\mathrm{ren}}_{\mu\nu} + \beta g_{\mu\nu} E^{\mathrm{ren}} \right\rangle + 4m^2\left\langle \Phi^{(2),\mathrm{ren}} \right\rangle + [(n-2) - 4(n-1)\xi]\nabla^2\left\langle \Phi^{(2),\mathrm{ren}} \right\rangle\right].$$

It is known that β is not universal, i.e. depends on the renormalization scheme; for instance, $\beta = -1/4$ for locally covariant schemes (Moretti, 1999).

This creates a preferred value $\hat{\beta}_{\mathrm{tr}} = \frac{n-2}{4}$ $\rightarrow$ independence on β, suggesting the definition

$$\mathcal{A} := g^{\mu\nu}\left\langle T^{\mathrm{ren}}_{\mu\nu} \right\rangle^{\star} - \left\langle T^{\mathrm{ren}} \right\rangle^{\star} - \hat{\beta}_{\mathrm{tr}}\left\langle E^{\mathrm{ren}} \right\rangle^{\star} ,$$

($*$ means conserved EM tensor).

NB:

- Conformal case: the second and third contribution cancel.
- Nonconformal case (not simply going on-shell):

$$T + \hat{\beta}_{\mathrm{tr}} E = (n-1)(\xi - \xi_{\mathrm{cc}})\nabla^2\phi^2 - m^2\phi^2 .$$

Computation with the HK

Spectral dim reg (other options include dim reg, point splitting; Feynman diagrams, Fujikawa, $\cdots$)

$$G^{\mathrm{F}}(x,x') = -\mathrm{i} \lim_{\epsilon\to 0^+} \int_0^\infty \tau^\alpha K(x,x';\tau)\, \mathrm{e}^{-\mathrm{i}(m^2-\mathrm{i}\epsilon)\tau} \mathrm{d}\tau \,.$$

K is the heat kernel, satisfying

$$\partial_\tau K(x,x';\tau) = \mathrm{i}\left(\nabla^2 - \xi R\right) K(x,x';\tau)\,.$$

With the SDW ansatz

$$K(x,x';\tau) \sim \frac{\sqrt{\Delta(x,x')}}{(4\pi\tau)^{\frac{n}{2}}} \mathrm{e}^{\frac{\mathrm{i}\sigma(x,x')}{2\tau} - \mathrm{i}\frac{n-2}{4}\pi} \sum_{k=0}^{\infty} (\mathrm{i}\tau)^k A_k(x,x')\,.$$

we get a recursion relation

$$\left(\nabla^\nu \sigma \nabla_\nu + k\right) A_k(x,x') = \left[\nabla^2 - \xi R + \left(\nabla^\nu \ln\Delta\right)\nabla_\nu + \frac{1}{4}\left(\nabla^\nu \ln\Delta\right)^2 + \frac{1}{2}\nabla^2 \ln\Delta\right] A_{k-1}(x,x')\,.$$

and the A_i coefficients can be computed

$$A_0(x,x) = 1\,, \qquad A_1(x,x) = \left(\frac{1}{6} - \xi\right) R\,,$$

$$A_2(x,x) = \frac{1}{180}\left[R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - R^{\alpha\beta}R_{\alpha\beta} + \frac{5}{2}(1-6\xi)^2 R^2 + 6(1-5\xi)\nabla^2 R\right].$$

Using our definition

$$\mathcal{A}_{s=0}^{n=4} = \frac{3C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} - \mathcal{E}_4 + 5(1-6\xi)^2 R^2}{360(4\pi)^2}\,,$$

which is written in terms of the square of the four-dimensional Weyl tensor C and the Euler density:

$$C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 2R^{\mu\nu}R_{\mu\nu} + \frac{1}{3}R^2\,, \quad \mathcal{E}_4 := R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2\,.$$

- Agrees with Christensen 1976, $\cdots$.
- The R^2 not Wess-Zumino consistent!

$$(\delta_X\delta_Y - \delta_Y\delta_X)W = \delta_{[X,Y]}W.$$

For a non-minimally coupled vector

$$S_{s=1} = -\frac{1}{2}\int \left[\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + (m^2 + \xi R)A^\mu A_\mu + (\zeta - 1)A_\mu A_\nu R^{\mu\nu} + \alpha^{-1}\left(\nabla_\mu A^\mu\right)^2\right]\sqrt{-g}\mathrm{d}^n x\,,$$

two possible "EOMs"

$$T^{\alpha_i}_{\mu\nu} := T_{\mu\nu} + \alpha_1\left(A_\mu M_\nu + A_\nu M_\mu\right) + 2\alpha_2 g_{\mu\nu}A_\rho M^\rho\,.$$

so that the generalized definition gives

$$\mathcal{A}^{n=4}_{s=1} = \frac{1}{360(4\pi)^2}\left[(90\zeta^2 - 54)C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} - (90\zeta^2 - 28)\mathcal{E}_4 + 60\Big(12\xi^2 + (\zeta-1)\zeta + \xi(6\zeta - 4)\Big)R^2\right].$$

- For a Dirac fermion there no surprises, $\hat{\beta}_{\text{tr}}^{s=1/2} = 1 - n$; $\beta^{s=1/2} = -1$.

- For a Weyl fermion, shall we use the modified definition? The same cancellation of the additional terms in the definition.

- In $n = 2$ we have

$$\mathcal{A}^{n=2} = \frac{(1 - 6\xi)N_0 + N_{1/2}}{24\pi} R.$$

Polyakov action?

Using the GSDW expansion,

$$\left\langle T_{\mu\nu}^{s=0,n=2} \right\rangle^{\star} = \frac{m^2 g_{\mu\nu}\left(2\gamma + 4\ln(m)\right)}{16\pi} + \frac{(1-10\xi+30\xi^2)}{120 m^2 \pi}\left(-g_{\mu\nu}\nabla^2 R + \nabla_\nu \nabla_\mu R - \frac{g_{\mu\nu}R^2}{4}\right) + \cdots .$$

Using the covariant perturbation theory (Barvinsky and Vilkovisky 1990, Franchino-Vinas, Netto, Shapiro and Zanusso 2019)

$$\Gamma_{\text{1-loop}}^{\text{CPT}} = -\frac{1}{96\pi}\int R \frac{1-12\xi+12\xi^2 \log\left(\frac{-\nabla^2}{m^2}\right)}{\nabla^2} R \sqrt{-g}\mathrm{d}^2 x \, .$$

Integrating our definition,

$$I^{\mathcal{A}} = -\frac{1}{96\pi}\int R \frac{1-6\xi}{\nabla^2} R \sqrt{-g}\mathrm{d}^2 x \, .$$

Outline

Chiral trace anomaly

- Nakayama [1201.3428]: what if a Pontryagin term ($P = \tilde{R}R$) ... (actually Wess–Zumino consistent)
- Bonora, Giaccari, Lima de Souza, +++ [arXiv:1403.2606, 1503.03326, 1703.10473, . . . , 2207.03279]:

  there is a P contribution to the chiral trace anomaly, which is purely imaginary ($\frac{\mathrm{i}}{180\,(4\pi)^2}\,\frac{15}{4}P$). Diagrammatic, heat-kernel (axial gravity).
- Bastianelli, Martelli, Broccoli [arXiv:1610.02304, 1911.02271, 2203.11668]: Pauli-Villars regularization, no P contribution.
- Abdallah, SF, Fröb [arXiv:2304.08939, arXiv:2101.11382]: diagrammatic, no P contribution.
- Duff [arXiv:2003.02688]: no computation.
- Discussion also on $F\tilde{F}$ contributions to trace anomaly (Bastianelli and Chiese 2023).
- Larue and Quevillon [arXiv: 2309.08670, +Zwicky 2024]: measure in the path integral.

Chiral trace anomaly

We consider a fermion in curved space,

$$S = -\int \bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi\sqrt{-g}\,\mathrm{d}^n x, \qquad (3)$$

where as usual we introduce the vielbein

$$\gamma^{\mu} \equiv e^{\mu}{}_{b}\gamma^{b},$$

and the covariant derivative involves the spin connection

$$\omega_{\mu\rho\sigma} = \eta_{ab}\left(e_{\sigma}{}^{a}\partial_{[\mu}e_{\rho]}{}^{b} - e_{\rho}{}^{a}\partial_{[\mu}e_{\sigma]}{}^{b} + e_{\mu}{}^{a}\partial_{[\sigma}e_{\rho]}{}^{b}\right).$$

The fermion is chiral:

$$\psi = \mathcal{P}_{+}\psi \equiv \frac{1}{2}(\mathbb{1} + \gamma_{*})\psi, \qquad \bar{\psi} = \bar{\psi}\mathcal{P}_{-} \equiv \frac{1}{2}\bar{\psi}(\mathbb{1} - \gamma_{*}).$$

How can we compute $\langle T^{\mu\nu}\rangle$?
Couple the left-handed fermion to gravity, right-handed as spectator.

The EM tensor is given by

$$T^{\mu\nu} = \frac{1}{2}\bar{\psi}\mathcal{P}_{-}\gamma^{(\mu}\overleftrightarrow{\nabla}^{\nu)}\mathcal{P}_{+}\psi + \frac{1}{2}g^{\mu\nu}\bar{\psi}\mathcal{P}_{-}\gamma^{\rho}\overleftrightarrow{\nabla}_{\rho}\mathcal{P}_{+}\psi,$$

which is classically traceless (on-shell), since the action is scale invariant.

Chiral trace anomaly

The simplest thing we know to do is to perform an expansion around a given metric

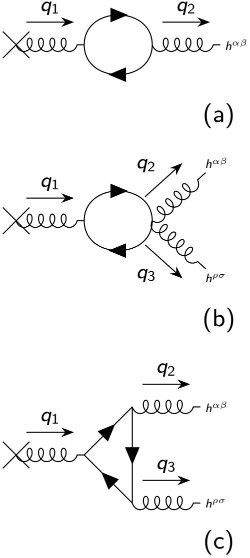
$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

$$\Downarrow$$

$$e^{\mu}{}_{a} = e^{(0)\,\rho}{}_{a}\left(\eta^{\mu}{}_{\rho} - \frac{1}{2}\kappa h^{\mu}{}_{\rho} + \frac{3}{8}\kappa^2 h^{\mu\sigma}h_{\sigma\rho}\right) + \mathcal{O}\left(\kappa^3\right),$$

$$g^{\mu\nu} = \cdots,$$

$$\langle T^{\mu\nu}(x)\rangle_g = \frac{\left\langle T^{\mu\nu}(x)\exp\left[\mathrm{i}\left(\kappa S_{(1)} + \kappa^2 S_{(2)}\right)\right]\right\rangle}{\left\langle \exp\left[\mathrm{i}\left(\kappa S_{(1)} + \kappa^2 S_{(2)}\right)\right]\right\rangle} + \mathcal{O}\left(\kappa^3\right)$$
$$= \langle T^{\mu\nu}(x)\rangle_{(0)} + \kappa\,\langle T^{\mu\nu}(x)\rangle_{(1)} + \kappa^2\,\langle T^{\mu\nu}(x)\rangle_{(2)} + \mathcal{O}\left(\kappa^3\right).$$



Dimensional dependent properties

Suppose we define $\gamma_*^{(4)} = -\frac{\mathrm{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ in $n = 4$ dimensions. Then

$$\begin{aligned} \mathrm{tr}\left(\gamma_*^{(4)}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\right) &= -\mathrm{i}\,\epsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}\mathbb{1}, \\ \{\gamma_\mu, \gamma_*^{(4)}\} &= 0. \end{aligned} \tag{4}$$

Consider now γ_* in an arbitrary n with the same anticommutators + ciclicity of the trace:

$$n\,\mathrm{tr}\gamma_* = \mathrm{tr}\left(\gamma_*\gamma^\alpha\gamma_\alpha\right) = -\mathrm{tr}\left(\gamma^\alpha\gamma_*\gamma_\alpha\right) = -\mathrm{tr}\left(\gamma_*\gamma_\alpha\gamma^\alpha\right) = -n\,\mathrm{tr}\gamma_* \Longrightarrow n\,\mathrm{tr}\gamma_* = 0. \tag{5}$$

Mutatis mutandis...

$$\begin{aligned} n(n-2)\mathrm{tr}\left(\gamma_*\gamma_\mu\gamma_\nu\right) &= 0, \\ n(n-2)(n-4)\mathrm{tr}\left(\gamma_*\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\right) &= 0, \\ &\vdots \end{aligned} \tag{6}$$

which conflicts with formula (4)!

with

(18) $$A = -\frac{e^2}{(4\pi)^{\nu/2}} \cdot m^{\nu-3} \cdot \Gamma\left(2-\frac{\nu}{2}\right)\frac{\nu-1}{\nu-3},$$

(19) $$B = -\frac{e^2}{(4\pi)^{\nu/2}} \cdot m^{\nu-4} \cdot \Gamma\left(2-\frac{\nu}{2}\right)\cdot\frac{\nu-1}{\nu-3}$$

and

(20) $$\Sigma_f(p,\nu) = \frac{e^2}{(4\pi)^{\nu/2}} \cdot m^{\nu-4} \cdot \Gamma\left(2-\frac{\nu}{2}\right)\cdot$$

$$\cdot\left\{\left[(2-\nu)\frac{m-i\gamma\cdot p}{m^2\varrho} - 4\frac{m-i\gamma\cdot p}{m^2\varrho^2} + \frac{2}{\ldots}\right]\ldots\right.$$

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Dimensional Renormalization: The Number of Dimensions as a Regularizing Parameter.

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Universidad Nacional de La Plata
Consejo Nacional de Investigaciones Científicas y Técnicas - La Plata

(ricevuto l'8 Febbraio 1972)

Summary. — We perform an analytic extension of quantum electrodynamics matrix elements as (analytic) functions of the number of dimensions of space (ν). The usual divergences appear as poles for ν integer. The renormalization of those matrix elements (for ν arbitrary) leads to expressions which are free of ultraviolet divergences for ν equal to 4. This shows that ν can be used as an analytic regularizing parameter with advantages over the usual analytic regularization method. In particular, gauge invariance is mantained for any ν.

1. – Introduction.

In a previous paper ([1]) we pointed out the possibility of studying the struc- ... of the number of dimensions. We ...

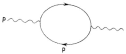

Fig. 8.

$$\int d_n p \frac{\mathrm{Tr}\,[\gamma^\mu(i\gamma(p+k)+m)\gamma^\nu(i\gamma p+m)]}{((p+k)^2+m^2)(p^2+m^2)}.$$

The trace may be evaluated using (33), (34) and (35). Taking denominators together one obtains:

$$4\int_0^1 dx \int d_n p \frac{\delta_{\mu\nu}(m^2+p^2+pk) - 2p_\mu p_\nu - p_\mu k_\nu - k_\mu p_\nu}{(p^2+2pkx+k^2x+m^2)^2}.$$

Using the equations of appendix A:

$$= \frac{4i\pi^{\frac{1}{2}n}}{\Gamma(2)}\Gamma(2-\tfrac{1}{2}n)\int_0^1 dx \frac{2x(1-x)(k_\mu k_\nu - \delta_{\mu\nu}k^2)}{(m^2+k^2x(1-x))^{2-\frac{1}{2}n}}$$

which is manifestly gauge invariant.

6. LIMITATIONS OF THE METHOD

The method fails if in the Ward identities there appear quantities that have the desired properties only in four dimensional space. An example is the completely antisymmetric tensor $\epsilon_{\mu\nu\alpha\beta}$. If the particular properties of this tensor are vital for the Ward identities to hold our method will fail because we cannot generalize $\epsilon_{\mu\nu\alpha\beta}$ to a tensor satisfying the required properties for non-integer n. Similarly for γ^5. One can write:

$$\gamma^5 = \frac{1}{4!}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta,$$

insert this whenever γ^5 occurs and take the ϵ-tensor outside of the expression to be generalized to non-integer n. However, if we are dealing with Ward identities that rely on

$$\{\gamma^5, \gamma^\alpha\} = 0 \qquad \text{for} \qquad \alpha = 1, \ldots, n$$

$$\mathrm{Tr}\,(\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = 4\epsilon_{\mu\nu\alpha\beta}$$

...ories of charged massive vector bosons. ... to formulate renormalizable ... particles, and in order to establish unitarity and causality ... The derived Feynman rules involve ... tities are needed. The necessary combi... the treatment of ...

Chiral anomalies

Do we need to consider γ_*? The standard model is written in terms of chiral fermions.

Helpful to find a solution to this γ_* problem:

- give up the ciclicity of the trace (Kreimer+).
- give up $\{\gamma_\mu, \gamma_*\} = 0$
  - give up all (Thompson-Yu);
  - keep the first four (Breitenlohner-Maison);

BM scheme

- n-dimensional usual metric $\eta_{\mu\nu}$, γ_μ, p_μ, $\cdots$.
- decompose them into a four-dimensional part ($\bar{X}$) and an $(n-4)$-dimensional part ($\hat{X}$):

$$\eta_{\mu\nu} = \bar{\eta}_{\mu\nu} + \hat{\eta}_{\mu\nu}, \quad \gamma_\mu = \bar{\gamma}_\mu + \hat{\gamma}_\mu, \quad p_\mu = \bar{p}_\mu + \hat{p}_\mu, \ldots$$
$$\eta_\mu{}^\nu \hat{\eta}_{\nu\rho} = \hat{\eta}_{\mu\nu}\hat{\eta}^\nu{}_\rho = \hat{\eta}_{\mu\rho}, \quad \hat{\eta}_{\mu\nu} = \hat{\eta}_{\nu\mu},$$
$$\bar{\eta}^{\mu\nu}\hat{\eta}_{\nu\rho} = 0, \quad \bar{\eta}_\mu{}^\nu p_\nu = \bar{p}_\mu, \quad \hat{\eta}_\mu{}^\nu \gamma_\nu = \hat{\gamma}_\mu, \cdots$$

- $\epsilon_{\mu\nu\rho\sigma}$ is a purely four-dimensional object: $\epsilon_{\mu\nu\rho\sigma} = \bar{\epsilon}_{\mu\nu\rho\sigma}$, $\hat{\eta}^{\alpha\mu}\epsilon_{\mu\nu\rho\sigma} = 0$.
- γ_* the same as in four dimensions,

$$\gamma_* \equiv -\frac{\mathrm{i}}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma.$$

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$$\eta_\mu{}^\nu \hat{\eta}_{\nu\rho} = \hat{\eta}_{\mu\nu}\hat{\eta}^\nu{}_\rho = \hat{\eta}_{\mu\rho}, \quad \hat{\eta}_{\mu\nu} = \hat{\eta}_{\nu\mu},$$

$$\bar{\eta}^{\mu\nu}\hat{\eta}_{\nu\rho} = 0, \quad \bar{\eta}_\mu{}^\nu p_\nu = \bar{p}_\mu, \quad \hat{\eta}_\mu{}^\nu \gamma_\nu = \hat{\gamma}_\mu, \cdots$$

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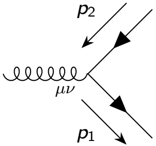
$$\eta_\mu{}^\nu \hat{\eta}_{\nu\rho} = \hat{\eta}_{\mu\nu} \hat{\eta}^\nu{}_\rho = \hat{\eta}_{\mu\rho}, \quad \hat{\eta}_{\mu\nu} = \hat{\eta}_{\nu\mu},$$
$$\bar{\eta}^{\mu\nu} \hat{\eta}_{\nu\rho} = 0, \quad \bar{\eta}_\mu{}^\nu p_\nu = \bar{p}_\mu, \quad \hat{\eta}_\mu{}^\nu \gamma_\nu = \hat{\gamma}_\mu, \cdots$$

- $\epsilon_{\mu\nu\rho\sigma}$ is a purely four-dimensional object: $\epsilon_{\mu\nu\rho\sigma} = \bar{\epsilon}_{\mu\nu\rho\sigma}$, $\hat{\eta}^{\alpha\mu} \epsilon_{\mu\nu\rho\sigma} = 0$.
- γ_* the same as in four dimensions,

$$\gamma_* \equiv -\frac{\mathrm{i}}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma .$$

Chiral trace anomaly

It is easy to write down the computations that should be done...

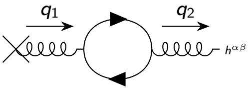


$$= \frac{\mathrm{i}}{8}\mathcal{P}_- \left[(p_1 + p_2)_\mu \gamma_\nu + (p_1 + p_2)_\nu \gamma_\mu\right] \mathcal{P}_+ \, ,$$

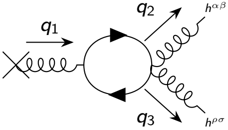
$$\mathcal{I}^{\mu_1 \cdots \mu_m}(p) \equiv \int \frac{q^{\mu_1} \cdots q^{\mu_m}}{(q^2 - \mathrm{i}0)[(q+p)^2 - \mathrm{i}0]} \mathrm{d}^n q,$$

$$\mathrm{tr}\left[\gamma^\mu \mathcal{P}_+ \gamma^\tau \mathcal{P}_- \gamma^\sigma \mathcal{P}_+ \gamma^\delta \mathcal{P}_- \gamma^\alpha \mathcal{P}_+ \gamma^\lambda \mathcal{P}_-\right] \, ,$$

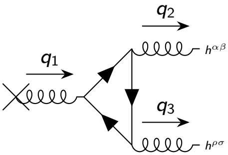
$$\mathcal{I}^{\mu_1 \cdots \mu_m}(p, k) \equiv \int \frac{q^{\mu_1} \cdots q^{\mu_m}}{(q^2 - \mathrm{i}0)[(q-k)^2 - \mathrm{i}0][(q-p)^2 - \mathrm{i}0]} \mathrm{d}^n q.$$



(a)



(b)



(c)

$$
\begin{aligned}
\mathcal{I}^{\mu\nu\rho\sigma\alpha\beta}(p,k) =\ & \frac{1}{192}\frac{\mathrm{i}}{(4\pi)^2}\eta^{(\mu\nu}\eta^{\rho\sigma}\eta^{\alpha\beta)}\Big[3(k^2)^2 - 6k^2\Big(pk\Big) + 4\Big(pk\Big)^2 + 5k^2p^2 - 6\Big(pk\Big)p^2 + 3(p^2)^2\Big]\Big(\mathcal{D} + \frac{3}{2}\Big) \\
-\ & \frac{1}{16}\frac{\mathrm{i}}{(4\pi)^2}\eta^{(\mu\nu}\eta^{\rho\sigma}\Big[p^\alpha p^{\beta)}\Big[3p^2 + 2k^2 - 3\Big(pk\Big)\Big] + k^\alpha k^{\beta)}\Big[2p^2 + 3k^2 - 3\Big(pk\Big)\Big] + p^\alpha k^{\beta)}[3p^2 + 3k^2 - 4(p\kappa)]\Big](\mathcal{D} + 1) \\
+\ & \frac{1}{4}\frac{\mathrm{i}}{(4\pi)^2}\eta^{(\mu\nu}\Big(p^a p^\beta p^\rho p^{\sigma)} + p^\alpha p^\beta p^\rho k^{\sigma)} + p^\alpha p^\beta k^\rho k^{\sigma)} + p^\alpha k^\beta k^\rho k^{\sigma)} + k^\alpha k^\beta k^\rho k^{\sigma)}\Big)\mathcal{D} \\
-\ & \frac{15}{16}\frac{\mathrm{i}}{(4\pi)^2}\eta^{(\mu\nu}\eta^{\rho\sigma}\eta^{\alpha\beta)}\Big[(p^2)^2(G_{04}(p,k) + G_{02}(p,k) - 2G_{03}(p,k)) + \Big(k^2\Big)^2(G_{40}(p,k) + G_{20}(p,k) - 2G_{30}(p,k)) + 4p^2(pk)\Big(G_{13}(p,k) \\
& - G_{12}(p,k)\Big) + 4k^2(pk)(G_{31}(p,k) - G_{21}(p,k)) + 2p^2k^2\Big(G_{11}(p,k) - G_{12}(p,k) + G_{22}(p,k) - G_{21}(p,k)\Big)\Big] \\
+\ & \frac{45}{2}\frac{\mathrm{i}}{(4\pi)^2}\eta^{(\mu\nu}\eta^{\rho\sigma}p^\alpha k^{\beta)}\Big[p^2(G_{12}(p,k) - G_{13}(p,k)) + k^2\Big(G_{21}(p,k) - G_{31}(p,k)\Big) - 2(pk)G_{22}(p,k)\Big] \\
+\ & \frac{45}{4}\frac{\mathrm{i}}{(4\pi)^2}\eta^{(\mu\nu}\eta^{\rho\sigma}k^\alpha k^{\beta)}\Big[p^2(G_{21}(p,k) - G_{22}(p,k)) + k^2\Big(G_{30}(p,k) - G_{40}(p,k)\Big) - 2(pk)G_{31}(p,k)\Big] \\
+\ & \frac{45}{4}\frac{\mathrm{i}}{(4\pi)^2}\eta^{(\mu\nu}\eta^{\rho\sigma}p^\alpha p^{\beta)}\Big[p^2(G_{03}(p,k) - G_{04}(p,k)) + k^2\Big(G_{12}(p,k) - G_{22}(p,k)\Big) - 2(pk)G_{13}(p,k)\Big] \\
-\ & \frac{15}{2}\frac{\mathrm{i}}{(4\pi)^2}\eta^{(\mu\nu}\Big[p^\rho p^\sigma p^\alpha p^{\beta)}G_{04}(p,k) + 4p^\rho p^\sigma p^\alpha k^{\beta)}G_{13}(p,k) + 6p^\rho p^\sigma k^\alpha k^{\beta)}G_{22}(p,k) + 4p^\rho k^\sigma k^\alpha k^{\beta)}G_{31}(p,k) + k^\rho k^\sigma k^\alpha k^{\beta)}G_{40}(p,k)\Big] \\
+\ & \frac{\mathrm{i}}{(4\pi)^2}\Big[p^\mu p^\nu p^\rho p^\sigma p^\alpha p^\beta F_{06}(p,k) + 6p^{(\mu}p^\nu p^\rho p^\sigma p^\alpha k^{\beta)}F_{15}(p,k) + 15p^{(\mu}p^\nu p^\rho p^\sigma k^\alpha k^{\beta)}F_{24}(p,k) + 20p^{(\mu}p^\nu p^\rho k^\sigma k^\alpha k^{\beta)}F_{33}(p,k) \\
& + 15p^{(\mu}p^\nu k^\rho k^\sigma k^\alpha k^{\beta)}F_{42}(p,k) + 6p^{(\mu}k^\nu k^\rho k^\sigma k^\alpha k^{\beta)}F_{51}(p,k) + k^\mu k^\nu k^\rho k^\sigma k^\alpha k^\beta F_{60}(p,k)\Big],
\end{aligned}
$$

where F_{ij} and G_{ij} are functions of k and p which satisfy several nontrivial relations.

Chiral trace anomaly

You'd better use tensorial simplifications:

$$\mathcal{I}^{\mu_1\cdots\mu_m}(p,k) = 2\int_0^1\int_0^{1-y}\int \frac{(q+yp+xk)^{\mu_1}\cdots}{(q^2+M_E-\mathrm{i}0)^3}\mathrm{d}^n q\,\mathrm{d}x\,\mathrm{d}y.$$

$$M_E \equiv y(1-y)p^2 + x(1-x)k^2 - 2xy(p\cdot k)\,.$$

Because of symmetry, we can substitute

$$\begin{aligned} q^\mu q^\nu &\to \frac{1}{n}\eta^{\mu\nu}q^2, \\ q^\mu q^\nu q^\rho q^\sigma &\to \frac{3}{n(n+2)}\eta^{(\mu\nu}\eta^{\rho\sigma)}q^4, \\ &\vdots \end{aligned} \tag{7}$$

$\Downarrow$ the traceless part of $\eta^{(\mu\nu}\cdots$ vanishes

$$[\mathcal{I}^{\mu\cdots}(p,k)]_{\mathrm{trless}} = 2\int_0^1\int_0^{1-y}\int \frac{[(yp+xk)^{\mu}\cdots]_{\mathrm{trless}}}{(q^2+M_E-\mathrm{i}0)^3}\mathrm{d}^n q\,\mathrm{d}x\,\mathrm{d}y$$

$$
\begin{aligned}
\mathcal{I}^{\mu\nu\rho\sigma\alpha\beta}(p,k) = {} & \Big[\mathcal{I}^{\mu\nu\rho\sigma\alpha\beta}(p,k)\Big]_{\rm trless} \\
& + \frac{15\mathcal{I}(p-k)}{16(-1+n)(1+n)(8+n)}\eta^{(\rho\sigma} \\
& \Big[(2+n)(4+n)k^\alpha k^\beta k^\mu k^{\nu)} + 4(-4+n^2)k^\alpha k^\beta k^\mu p^{\nu)} + 6(-2+n)nk^\alpha k^\beta p^\mu p^{\nu)} \\
& + 4(-4+n^2)k^\alpha p^\beta p^\mu p^{\nu)} + (2+n)(4+n)p^\alpha p^\beta p^\mu p^{\nu)}\Big] \\
& - \frac{45\mathcal{I}(p-k)}{8(-1+n)(1+n)(6+n)(8+n)}\eta^{(\mu\nu}\eta^{\rho\sigma} \\
& \Big[2k^\alpha p^{\beta)}\Big(\Big(-2+n(4+n)\Big)k^2 - 10n(p\cdot k) + \Big(-2+n(4+n)\Big)p^2\Big) \\
& + k^\alpha k^{\beta)}\Big((2+n)\Big((6+n)k^2 - 10(p\cdot k)\Big) + \Big(8+n(4+n)\Big)p^2\Big) \\
& + p^\alpha p^{\beta)}\Big(\Big(8+n(4+n)\Big)k^2 + (2+n)\Big(-10(p\cdot k) + (6+n)p^2\Big)\Big)\Big] \\
& + \frac{15\mathcal{I}(p-k)}{16(-1+n)(1+n)(4+n)(6+n)(8+n)}\eta^{(\mu\nu}\eta^{\rho\sigma}\eta^{\alpha\beta)} \\
& \Big[3(4+n)(6+n)k^4 + 4\Big(48+n(4+n)\Big)(p\cdot k)^2 - 60(4+n)(p\cdot k)p^2 \\
& + 3(4+n)(6+n)p^4 + 2k^2\Big(-30(4+n)(p\cdot k) + \Big(64+n(22+3n)\Big)p^2\Big)\Big]
\end{aligned}
$$

A couple of contributions of this type...

- compute spinorial factors $(\gamma^\mu\cdots)$;
- renormalize;
- check divergence (diffeo anomaly);
- compute the trace;
- identify geometrical invariants (which are written as expansions in $h^{\mu\nu}$).

Chiral trace anomaly

Our results are exactly half of the trace anomaly for the Dirac spinor

$$(g_{\mu\nu}\langle T^{\mu\nu}\rangle)^{\text{ren}}_{(2)}(x) = \frac{1}{16\cdot 45(4\pi)^2}\left(-11\mathcal{E}_4 + 18C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + 12\nabla^2 R\right)_{(2)},$$
$$(\nabla_\mu \langle T^{\mu\nu}\rangle)^{\text{ren}}_{(2)}(x) = 0,$$

in terms of the Weyl tensor $C_{\mu\nu\rho\sigma}$ and the four-dimensional Euler density E_4, which in four dimensions satisfy

$$C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 2R^{\mu\nu}R_{\mu\nu} + \frac{1}{3}R^2,$$
$$E_4 = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2.$$

What about the parity-odd contribution?

The parity-odd contribution

Two-point contributions, $\left\langle T_*^2 \right\rangle$:

$$\epsilon^{\delta\mu\sigma\tau} \times \text{symmetric in external momenta} = 0 \tag{8}$$

The $\langle T_* J \rangle$ contribution:

$$(2\epsilon^{\alpha\beta\gamma(\tau}\hat{\eta}^{\lambda)\mu} - \epsilon^{\alpha\beta\gamma\mu}\hat{\eta}^{\lambda\tau}) \times \delta^{\nu}{}_{[\tau} p_{\lambda]} p^2 = 0 \tag{9}$$

The three-point contribution, $\left\langle T_*^3 \right\rangle$:

$$\epsilon^{[\delta\mu\sigma\tau}\hat{\eta}^{\alpha]\lambda} \times \text{something} = 0 \tag{10}$$

Dimensionally dependent identities [arXiv:gr-qc/0105066v1, Edgar & Höglund]. The Cayley-Hamilton theorem for an $n \times n$ matrix ("*every square matrix over a commutative ring satisfies its own characteristic equation*"):

$$M^{c_1}{}_{[c_1} M^{c_2}{}_{c_2} \cdots M^{c_n}{}_{c_n} \delta^{b}{}_{a]} = 0 \tag{11}$$

Outline

Outlook

- We have obtained no parity-odd contribution to the chiral trace anomaly (BM scheme/Feynman diagrams, Duff's definition of the anomaly).

- Cosmology/astrophysics and (non-)conformal fields (inflation, baryogenesis, ...).

- Cancellation of anomalies in supergravity (a_2 HK coefficient, Meissner and Nicolai, 2017).

- RG flow of the generalized anomaly and c-theorem?

$$\vdots$$

Backup slides

Some current discussions

- For a CFT in $n = 4$, the correlator of three stress-energy tensors is necessarily parity-even (Stanev, arXiv: 1206.5639)

$$\langle T_{\mu\nu}(x_1) T_{\rho\sigma}(x_2) T_{\alpha\beta}(x_3) \rangle = (\text{parity-even term}) + 0 \times \epsilon_{\bullet\bullet\bullet\bullet}, \tag{12}$$

  but this is for the "regular" contribution. One may still have contributions at coincident points (that should be seen in our computation).

- The folklore says that the definition of the conformal anomaly for nonconformal theories is

$$\mathcal{A} = g^{\mu\nu} \langle T_{\mu\nu} \rangle - \langle g^{\mu\nu} T_{\mu\nu} \rangle. \tag{13}$$

  Why should we employ it for a conformal theory?

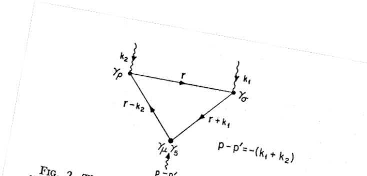
Anomalies

Consider a massless fermion, coupled to an *external* EM field A_μ

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi + eJ_V^\mu A_\mu, \quad J_V^\mu = \bar{\psi}\gamma^\mu\psi, \quad J_A^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi \tag{14}$$

The system has two global symmetries: $\psi \to e^{i\alpha}\psi$, $e^{i\alpha\gamma_5}\psi$, so that classically (on-shell) $\partial_\mu J^\mu_{A,V} = 0$.

$$\begin{aligned}\langle\partial_\mu J_A^\mu(x)\rangle &\sim \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\partial_\mu J_A^\mu e^{i\int i\bar{\psi}\partial\!\!\!/\psi + eJ_V^\mu A_\mu} \\ &= -\frac{e^2}{2}\partial_{x^\mu}\iint \mathrm{d}y\mathrm{d}z\langle J_A^\mu(x)J_V^\nu(y)J_V^\sigma(z)\rangle A_\nu(z)A_\sigma(y) + \cdots \\ &= -\frac{e^2}{16\pi^2}\epsilon^{\alpha\beta\sigma\lambda}F_{\alpha\beta}F_{\sigma\lambda}.\end{aligned} \tag{15}$$



STEPHEN L. ADI

term is

$$r\left\{\sum_{k=1}^{2n}\prod_{j=1}^{k-1}\left[\gamma^{(j)}\frac{1}{r+p_j-m_0}\right]\gamma^{(k)}\frac{1}{r+p_k-m_0}\gamma_\mu\gamma_5\right.$$

$$\frac{1}{r+p_k+p'-p-m_0}\prod_{i=k+1}^{2n}\left[\gamma^{(i)}\frac{1}{r+p_i+p'-p-m_0}\right]\Big\}\times(\cdots). \quad (12)$$

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plying by $(p-p')^\mu$ and using Eq. (9) gives

$$\mathrm{Tr}\left\{\sum_{k=1}^{2n}\prod_{j=1}^{k-1}\left[\gamma^{(j)}\frac{1}{r+p_j-m_0}\right]\gamma^{(k)}\frac{1}{r+p_k-m_0}2m_0\gamma_5\right.$$

$$\frac{1}{r+p_k+p'-p-m_0}\prod_{i=k+1}^{2n}\left[\gamma^{(i)}\frac{1}{r+p_i+p'-p-m_0}\right]\Big\}$$

$$(\cdots)+\int d^4r\,\mathrm{tr}\left\{\gamma_5\prod_{j=1}^{2n}\left[\gamma^{(j)}\frac{1}{r+p_j-m_0}\right]\right.$$

$$-\gamma_5\prod_{j=1}^{2n}\left[\gamma^{(j)}\frac{1}{r+p_j+p'-p-m_0}\right]\Big\}(\cdots). \quad (13)$$

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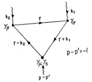

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PHYSICAL REVIEW VOLUME 177, NUMBER 5 25 JANUARY 19

Axial-Vector Vertex in Spinor Electrodynamics

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(Received 24 September 1968)

Working within the framework of perturbation theory, we show that the axial-vector vertex in spinor electrodynamics has anomalous properties which disagree with those found by the formal manipulation of field equations. Specifically, because of the presence of closed-loop "triangle diagrams," the divergence of the axial-vector current is not the usual expression calculated from the field equations, and the axial-vector current does not satisfy the usual Ward identity. One consequence is that, even after the external-line wave-function renormalizations are made, the axial-vector vertex is still divergent in fourth- (and higher-) order perturbation theory. A corollary is that the radiative corrections to $\nu_l l$ elastic scattering in the local current-current theory diverge in fourth (and higher) order. A second consequence is that, in massless electrodynamics, despite the fact that the theory is invariant under γ_5 transformations, the axial-vector current is not conserved. In an Appendix we demonstrate the uniqueness of the triangle diagrams, and discuss a possible connection between our results and the $\pi^0\to 2\gamma$ and $\eta\to 2\gamma$ decays. In particular, we argue that as a result of triangle diagrams, the equations expressing partial conservation of axial-vector current (PCAC) for the neutral members of the axial-vector-current octet must be modified in a well-defined manner, which completely alters the PCAC predictions for the π^0 and the η two-photon decays.

INTRODUCTION

THE axial-vector vertex in spinor electrodynamics is of interest because of its connections (i) with radiative corrections to $\nu_l l$ scattering and (ii) with the γ_5 invariance of massless electrodynamics. We will show in this paper, within the framework of perturbation theory, that the axial-vector vertex has anomalous properties which disagree with those found by the formal manipulation of field equations. In particular, because of the presence of closed-loop "triangle diagrams," the divergence of the axial-vector current is not the usual expression calculated from the field equations, and the axial-vector current does not satisfy the usual W...

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I. AXIAL CURRENT DIVERGENCE AND WARD IDENTITY

We work in the usual spinor electrodynamics, described by the Lagrangian density[1]

$$\mathcal{L}(x)=\bar\psi(x)(i\gamma\cdot\Box-m_0)\psi(x)-\tfrac14F_{\mu\nu}(x)F^{\mu\nu}(x)$$
$$-:e_0\bar\psi(x)\gamma_\mu\psi(x)A^\mu(x):, \quad (1)$$

$$\partial A_\mu(x)\quad \partial A_\nu(x)$$

easily be calculated to be

$$\frac{\partial}{\partial x_\mu}j_\mu^5(x)=2im_0j^5(x). \quad (4)$$

From Eqs. (3) and (4), we obtain the usual axial-vector Ward identity

$$(p-p')^\mu\Gamma_\mu^5(p,p')=2m_0\Gamma^5(p,p')$$
$$+S_F'(p)^{-1}\gamma_5+\gamma_5S_F'(p')^{-1}. \quad (5)$$

Our task in this section is to see whether Eqs. (4) and (5), which we have formally derived from the field equations, actually hold in perturbation theory.

To this end, let us rederive Eq. (5) in perturbation theory. It is convenient to write

$$\Gamma_\mu^5=\gamma_\mu\gamma_5+\Lambda_\mu^5,$$
$$\Gamma^5=\gamma_5+\Lambda^5,$$
$$S_F'(p)^{-1}=p-m_0-\Sigma(p), \quad (6)$$

where the vertex corrections Λ_μ^5 and Λ^5 and the proper self-energy part $\Sigma(p)$ are calculated using $(p-m_0)^{-1}$ as the free propagator. (Use of the bare mass $m_0=m-\delta m$ in the free propagator automatically includes the mass-renormalization counter terms.) In terms of Λ_μ^5, Λ^5, and Σ, Eq. (5) becomes

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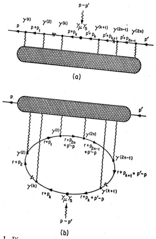

FIG. 1. Diagrams contributing to the axial-vector vertex. (a) The axial-vector vertex is attached ...

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A PCAC Puzzle: $\pi^0\to\gamma\gamma$ in the σ-Model.

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1. – Introducti

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where p a their mas intereste above g vation. $= T^{\nu\mu}$(

Since we are working to lowest order in electromagnetism F_1-F_3 possess a dynamical singularity at $k^2=0$, and therefore we recover th $T(k^2)=Ok^2$.

3. – Perturbation-theory argument.

The formal reasoning of the previous Section should be verifiable by plicit perturbation calculation in a model with PCAC and gauge invari We consider the σ-model interacting with the electromagnetic field a_μ, o ting the charged pion and the neutron fields, which are not necessary for problem. The Lagrange density is [8]

$$(3.1)\qquad \mathcal{L}=-\frac12\partial_\nu a_\mu\partial^\nu a^\mu+\bar\psi[i(\gamma\cdot\partial)+e(\gamma\cdot a)-m+g(\sigma+\varphi\gamma_5)]\psi+$$
$$+\frac12(\partial\varphi)^2+\frac12(\partial\sigma)^2-\frac12\mu^2\varphi^2-\frac12\left(\mu^2+\frac{2\lambda}{f}\right)\sigma^2-\lambda[(\sigma^2\ldots$$

Here $f=g/\ldots$

term is

$$\mathrm{tr}\left\{\sum_{k=1}^{2n}\prod_{j=1}^{k-1}\left[\gamma^{(j)}\frac{1}{r+p_j-m_0}\right]\gamma^{(k)}\frac{1}{r+p_k-m_0}\gamma_\mu\gamma_5 \frac{1}{r+p_k+p'-p-m_0}\prod_{j=k+1}^{2n}\left[\gamma^{(j)}\frac{1}{r+p_j+p'-p-m_0}\right]\right\}\times(\cdots). \quad (12)$$

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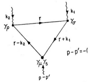

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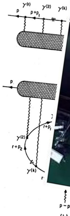

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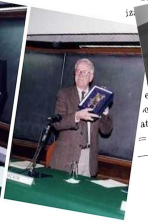

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Here $f=\cdots$

Inflationary application

From Shapiro et al. 1998

$$S = -\frac{1}{\kappa^2}\int \mathrm{d}^4x\sqrt{-g}R + \Gamma[g], \tag{16}$$

$$\frac{a''''}{a} - \frac{4a'a''' + 3a''^2}{a^2} + 2\left(3 - \frac{2b}{c}\right)\frac{a''a'^2}{a^3} + \frac{4b}{c}\frac{a'^4}{a^4} - \frac{2}{c}M_{\mathrm{Pl}}^2 a''a = 0 \tag{17}$$

b, c, correspond to E and $\Box R$.

Cohomology

Represent the "BRST" coboundary operator using the infinitesimal anticommuting parameter χ, acting on the space of metric tensors $\times$ matter (Bonora et al., PLB 1983) (notice change of convention in Weyl weights)

$$\Theta := \Theta_g + \Theta_\phi, \tag{18}$$

$$\Theta_g := \int \mathrm{d}^4 x 2\chi g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}}, \tag{19}$$

$$\Theta_\phi := -\int \mathrm{d}^4 x \chi \phi \frac{\delta}{\delta \phi}. \tag{20}$$

Then, acting on the classical action $S_{\rm cl}$ and the effective action Γ,

$$\Theta S_{\rm cl} = 0, \tag{21}$$

$$\Theta \Gamma = \Delta. \tag{22}$$

Given that $\Theta^2 = 0$, it should be $\Theta\Delta = 0$; we can thus write $\Delta = \Theta\omega + \bar{\Delta}$, such that no $\bar{\omega}$ satisfying $\bar{\Delta} = \Theta\bar{\omega}$ exists.

$\bar{\Delta}$ is the anomaly.

Baryogenesis

Particle-antiparticle asymmetry in our universe is broken ($\eta_B = \frac{n_B - n_{\bar{B}}}{n_\gamma} \sim 10^{-10}$).

A possibility is that it could be related to an electroweak anomaly. Original idea by Kuzmin, Rubakov and Shaposhnikov (1985); still new proposals as QCD baryogenesis [1911.01432v2 [hep-ph]]:

$$\begin{aligned} \partial_\mu J_B^\mu &= C G_{\mu\nu} \tilde{G}^{\mu\nu} \\ &\sim C \partial_\mu K^\mu , \end{aligned}$$

with

$$K^\mu = \epsilon^{\mu\nu\alpha\beta} \left(A_\nu \partial_\alpha A_\beta + \frac{2}{3} ig A_\nu A_\alpha A_\beta \right) .$$

By considering transitions among different vacua of $G_{\mu\nu}$, one is able to change the baryonic number.

Wess-Zumino consistency conditions

Consider the variation of the W action functional (connected vacuum functional). Consider two transformations X, Y.

$$(\delta_X \delta_Y - \delta_Y \delta_X) W = \delta_{[X,Y]} W. \tag{23}$$

BRST way: be $a^{k,q}$ a local q-form of ghost number k. The WZ consistency condition is

$$s a^{k,q} + \mathrm{d} a^{k+1,q-1} = 0. \tag{24}$$

Descent equation techniques [hep-th/9505173, Barnich, Brndt, Henneaux],

$$a^{k,q} \neq s b^{k-1,q} + \mathrm{d} b^{k,q-1}, \tag{25}$$

then also $a^{k+1,q-1}$ is also a solution of the WZ consistency condition.

Induced inflation

$$\Gamma_{\mathrm{ind}} = S_c[\bar{g}_{\mu\nu}] + \int \mathrm{d}^4x\sqrt{-\bar{g}}\{\sigma(w\bar{C}^2 + \beta\bar{F}^2) + b\sigma(\bar{E} - \frac{2}{3}\Box\bar{R}) + 2b\sigma\bar{\Delta}_4\sigma\} - \frac{3c+2b}{36}\int \mathrm{d}^4x\sqrt{-g}R^2, \quad (26)$$

$g_{\mu\nu} = e^{2\sigma}\bar{g}_{\mu\nu}$, S_c conformal invariant "integration constant", Δ_4 Paneitz operator

$$\Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla\nu - \frac{2}{3}R\Box + \frac{1}{3}(\nabla^\mu R)\nabla_\mu. \quad (27)$$

Expansion of $\langle T^{\mu\nu} \rangle$

Define

$$\Psi^{\mu\nu} \equiv \bar{\psi}\mathcal{P}_- \gamma^\mu \mathcal{P}_+ \partial^\nu \psi - \partial^\nu \bar{\psi}\mathcal{P}_- \gamma^\mu \mathcal{P}_+ \psi, \tag{28}$$

$$J^{\mu\nu\rho} \equiv \bar{\psi}\mathcal{P}_- \gamma^{\mu\nu\rho} \mathcal{P}_+ \psi, \tag{29}$$

as well as the traces $\Psi \equiv \Psi^\mu{}_\mu$ and $h \equiv h^\mu{}_\mu$. In this way, we obtain the coefficients in the expansion of the stress tensor,

$$T^{\mu\nu}_{(0)} = \frac{1}{2}\left[\Psi^{(\mu\nu)} - \Psi\eta^{\mu\nu}\right], \tag{30}$$

$$T^{\mu\nu}_{(1)} = \frac{1}{4}\left[2\Psi h^{\mu\nu} + h_{\alpha\beta}\Psi^{\alpha\beta}\eta^{\mu\nu} - 2h^{\alpha(\mu}\Psi^{\nu)}{}_\alpha - h^{\alpha(\mu}\Psi_\alpha{}^{\nu)} - J^{\alpha\beta(\mu}\partial_\alpha h^{\nu)}{}_\beta\right], \tag{31}$$

$$\begin{aligned} T^{\mu\nu}_{(2)} = \frac{1}{16}\Big[& 4h^{\rho(\mu}h^{\nu)\sigma} - 4h_{\rho\sigma}h^{\mu\nu} + 8h^{\alpha(\mu}\left(\delta^{\nu)}_\rho h_{\alpha\sigma} - h^{\nu)}{}_\alpha \eta_{\rho\sigma}\right) \\ & + 3h_{\alpha\rho}\left(h^{\alpha(\mu}\delta^{\nu)}_\sigma - h_\sigma{}^\alpha \eta^{\mu\nu}\right)\Big]\Psi^{\rho\sigma} + \frac{1}{16}\Big[\left(4h^{\alpha(\mu}\eta^{\nu)\rho} - \eta^{\mu\nu}h^{\alpha\rho}\right)\partial^\sigma h_\alpha{}^\tau \\ & + 2h^{\mu\rho}\partial^\sigma h^{\nu\tau} - h^{\alpha\sigma}\eta^{\rho(\mu}\left(\partial^{\nu)}h_\alpha{}^\tau - 2\partial_\alpha h^{\nu)}{}_\tau + 2\partial_\tau h^{\nu)}{}_\alpha\right)\Big]J_{\rho\sigma\tau}. \end{aligned} \tag{32}$$

From this, one can easily derive the following identities [**Breitenlohner:1977hr**]:

$$\{\gamma_\mu, \hat{\gamma}_\nu\} = \{\hat{\gamma}_\mu, \hat{\gamma}_\nu\} = 2\hat{\eta}_{\mu\nu}\mathbb{1}, \tag{33}$$

$$\hat{\eta}_\mu{}^\mu = n - 4, \tag{34}$$

$$\{\gamma_\mu, \gamma_*\} = \{\hat{\gamma}_\mu, \gamma_*\} = 2\hat{\gamma}_\mu\gamma_*, \tag{35}$$

$$[\gamma_\mu, \gamma_*] = [\bar{\gamma}_\mu, \gamma_*] = 2\bar{\gamma}_\mu\gamma_*, \tag{36}$$

$$\gamma_*^2 = \mathbb{1}, \tag{37}$$

$$\mathcal{P}_\pm\gamma^\mu\mathcal{P}_\mp = \mathcal{P}_\pm\bar{\gamma}^\mu, \tag{38}$$

where the last formula is a consequence of the definition of the projectors.

The action

the action,

$$S_{(0)} = -\frac{1}{2}\int \Psi \mathrm{d}^n x\,, \tag{39}$$

$$S_{(1)} = \frac{1}{4}\int \left(h_{\alpha\beta}\Psi^{\alpha\beta} - h\Psi\right)\mathrm{d}^n x\,, \tag{40}$$

$$S_{(2)} = \frac{1}{16}\int \left[\left(2hh_{\alpha\beta} - 3h_{\alpha}{}^{\delta}h_{\beta\delta}\right)\Psi^{\alpha\beta} + \left(2h_{\alpha\beta}h^{\alpha\beta} - h^2\right)\Psi + h_{\alpha}{}^{\delta}\partial_{\gamma}h_{\beta\delta}J^{\alpha\beta\gamma}\right]\mathrm{d}^n x\,. \tag{41}$$

BM scheme

The spinorial contribution at $\mathcal{O}(\kappa)$:

$$\mathrm{tr}\left(\mathcal{P}_+\gamma^\tau\mathcal{P}_-\gamma^\mu\mathcal{P}_+\gamma^\lambda\mathcal{P}_-\gamma^\rho\right) = 2\left(\bar{\eta}^{\tau\mu}\bar{\eta}^{\lambda\rho} - \bar{\eta}^{\tau\lambda}\bar{\eta}^{\mu\rho} + \bar{\eta}^{\tau\rho}\bar{\eta}^{\mu\lambda}\right) - 2\mathrm{i}\epsilon^{\tau\mu\lambda\rho}\,, \tag{42}$$

The regularized contribution to $\langle T\rangle$:

$$\begin{aligned}\langle T^{\mu\nu}(x)\rangle^{\mathrm{reg}}_{(1)} = \frac{\mathrm{i}}{8(n^2-1)}\iint \mathcal{I}(p)\Big[&\bar{\Pi}^{\mu\nu}(p)\bar{\Pi}^{\rho\sigma}(p) - (n-1)\bar{\Pi}^{\rho(\mu}(p)\bar{\Pi}^{\nu)\sigma}(p) \\ &- \frac{n-4}{2}p^2\left(\bar{\eta}^{\rho(\mu}\bar{\Pi}^{\nu)\sigma}(p) + \bar{\Pi}^{\mu\nu}(p)\bar{\eta}^{\rho\sigma} + \eta^{\mu\nu}\bar{\Pi}^{\rho\sigma}(p) + 3\eta^{\mu\nu}\bar{\eta}^{\rho\sigma}p^2\right)\Big] \\ &\times \mathrm{e}^{\mathrm{i}p(x-y)}\,h_{\rho\sigma}(y)\,\mathrm{d}^4p\mathrm{d}^4y\,,\end{aligned} \tag{43}$$

BM scheme

$$\begin{aligned}\langle T^{\mu\nu}(x)\rangle^{\text{ren}}_{(1)} &= \langle T^{\mu\nu}(x)\rangle^{\text{reg}}_{(1)} - \langle T^{\mu\nu}(x)\rangle^{\text{div}}_{(1)} \\ &= -\frac{1}{120\cdot 16\pi^2}\iint\left[\frac{46}{15} - \ln\left(\frac{p^2 - \mathrm{i}0}{\mu^2}\right)\right] \\ &\qquad \times\left[\bar{\Pi}^{\mu\nu}(p)\bar{\Pi}^{\rho\sigma}(p) - 3\bar{\Pi}^{\mu(\rho}(p)\bar{\Pi}^{\sigma)\nu}(p)\right]\mathrm{e}^{\mathrm{i}p(x-y)[}4]p h_{\rho\sigma}(y)\mathrm{d}^4 y \\ &\quad - \frac{1}{120\cdot 16\pi^2}\iint\left[p^2\left[\bar{\eta}^{\rho(\mu}\bar{\Pi}^{\nu)\sigma}(p) + \bar{\Pi}^{\mu\nu}(p)\bar{\eta}^{\rho\sigma} + \bar{\eta}^{\mu\nu}\bar{\Pi}^{\rho\sigma}(p) + 3\bar{\eta}^{\mu\nu}\bar{\eta}^{\rho\sigma}p^2\right]\right. \\ &\qquad \left. + 2\bar{\Pi}^{\rho(\mu}(p)\bar{\Pi}^{\nu)\sigma}(p)\right]\mathrm{e}^{\mathrm{i}p(x-y)[}4]p h_{\rho\sigma}(y)\mathrm{d}^4 y\,,\end{aligned} \tag{44}$$

First line conserved and traceless. Second line is local and can be removed by a counterterm (not covariant):

$$\begin{aligned}&2\partial^\mu\partial^\nu\partial^\rho\partial^\sigma h_{\rho\sigma}(x) + \partial^\mu\partial^\nu\partial^2 h(x) - 3\partial^2\partial^{(\mu}\partial_\rho h^{\nu)\rho}(x) + \partial^2\left[\bar{\eta}^{\mu\nu}\partial^\rho\partial^\sigma h_{\rho\sigma}(x) + \partial^2 h^{\mu\nu}(x) + \bar{\eta}^{\mu\nu}\partial^2 h(x)\right] \\ &= \frac{1}{2}\frac{\delta}{\delta h_{\mu\nu}(x)}\int\left[\partial^2 h^{\rho\sigma}\partial^2 h_{\rho\sigma} + \left(\partial^2 h\right)^2 + 3\partial_\rho h^{\rho\alpha}\partial^2\partial^\sigma h_{\sigma\alpha} + 2\partial^2 h\partial_\rho\partial_\sigma h^{\rho\sigma} + 2\left(\partial_\rho\partial_\sigma h^{\rho\sigma}\right)^2\right]\mathrm{d}^n y\,,\end{aligned} \tag{45}$$

BM scheme

Modification in the parity-even sector:

$$\mathrm{tr}\left(\mathcal{P}_+\gamma^\tau\mathcal{P}_-\gamma^\mu\mathcal{P}_+\gamma^\lambda\mathcal{P}_-\gamma^\rho\right) = 2\left(\eta^{\tau\mu}\eta^{\lambda\rho} - \eta^{\tau\lambda}\eta^{\mu\rho} + \eta^{\tau\rho}\eta^{\mu\lambda}\right) - 2\mathrm{i}\epsilon^{\tau\mu\lambda\rho}\,, \tag{46}$$

Nonconformal theories

General definition of anomaly

$$\mathcal{A} \equiv \lim_{n \to 4} \left(g^{\mu\nu} \langle T_{\mu\nu}(x) \rangle - \langle g^{\mu\nu} T_{\mu\nu}(x) \rangle \right), \tag{47}$$

Renormalization

On the existence of a counterterm to cancel divergent term in $\langle T \rangle$:

$$\begin{aligned}\langle T^{\mu\nu}(x)\rangle^{\text{div}} = \frac{\mathcal{N}}{1440(4\pi)^2}\Big[&8g^{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + 56R^{\mu\alpha}R^{\nu}_{\alpha} + 20R^{\mu\nu}R - 5g^{\mu\nu}R^2 \\ &+ 7g^{\mu\nu}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 88R_{\alpha\beta}R^{\mu\alpha\nu\beta} - 28R^{\mu\alpha\beta\gamma}R^{\nu}{}_{\alpha\beta\gamma} \\ &- 72\nabla^2R^{\mu\nu} + 12g^{\mu\nu}\nabla^2R + 24\nabla^{\mu}\nabla^{\nu}R\Big] + \mathcal{O}\left(\kappa^3\right)\,, \\ = \frac{\mathcal{N}}{720(4\pi)^2}\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}(x)}\int&\left(7R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} + 8R^{\mu\nu}R_{\mu\nu} - 5R^2\right)\sqrt{-g}\,\mathrm{d}^n x\,.\end{aligned} \tag{48}$$

$$\mathcal{N} \equiv -\frac{2}{n-4} - \gamma - \log\left(\frac{\mu^2}{4\pi}\right)\,. \tag{49}$$

F integrals

$$F_{ab}(k,p) = \int_0^1 \int_0^{1-y} \frac{x^a y^b}{x(1-x)k^2 + y(1-y)p^2 - 2xy(p\cdot k) - \mathrm{i}0} \mathrm{d}x\mathrm{d}y\,, \tag{50}$$

$$\begin{aligned} &k^2 F_{a,b-1} - k^2 F_{a+1,b-1} + p^2 F_{a-1,b} - p^2 F_{a-1,b+1} - 2(p\cdot k)F_{ab} \\ &= \int_0^1 \int_0^{1-y} x^{a-1} y^{b-1} \mathrm{d}x\mathrm{d}y = \frac{(a-1)!(b-1)!}{(a+b)!}\,, \end{aligned} \tag{51}$$

$$(a+1)p^2\left(2F_{a,b+2} - F_{a,b+1}\right) = (b+1)k^2\left(2F_{a+2,b} - F_{a+1,b}\right) \tag{52}$$

$$+ (b-a)\left[\frac{a!b!}{(a+b+2)!} + 2(p\cdot k)F_{a+1,b+1}\right]\,, \tag{53}$$

$$(a+b+2)k^2 F_{a+2,b} - (a+1)k^2 F_{a+1,b} = (a+b+2)p^2 F_{a,b+2} - (b+1)p^2 F_{a,b+1}\,. \tag{54}$$

Expansion of geometrical objects

$$\Gamma^{\rho}{}_{\mu\nu} = \kappa\left[\partial_{(\mu}h^{\rho}{}_{\nu)} - \frac{1}{2}\partial^{\rho}h_{\mu\nu}\right] + \kappa^2\left[\frac{1}{2}h^{\rho\alpha}\partial_{\alpha}h_{\mu\nu} - h^{\rho\alpha}\partial_{(\mu}h_{\nu)\alpha}\right] + \mathcal{O}\left(\kappa^3\right) , \tag{55}$$

$$\begin{aligned} R_{\rho\sigma\mu\nu} = {} & \kappa\left[-\partial_{\mu}\partial_{[\rho}h_{\sigma]\nu} + \partial_{\nu}\partial_{[\rho}h_{\sigma]\mu}\right] + \kappa^2\Big[-\frac{1}{2}\partial_{[\rho}h_{|\mu|}{}^{\alpha}\partial_{\sigma]}h_{\nu\alpha} - \frac{1}{2}\partial_{\mu}h_{[\rho}{}^{\alpha}\partial_{\sigma]}h_{\nu\alpha} \\ & + \frac{1}{2}\partial_{\nu}h_{[\rho}{}^{\alpha}\partial_{\sigma]}h_{\mu\alpha} - \frac{1}{2}\partial_{\mu}h_{[\rho}{}^{\alpha}\partial_{|\nu|}h_{\sigma]\alpha} + \frac{1}{2}\partial^{\alpha}h_{\mu[\rho}\partial_{\sigma]}h_{\nu\alpha} - \frac{1}{2}\partial^{\alpha}h_{\nu[\rho}\partial_{\sigma]}h_{\mu\alpha} \\ & + \frac{1}{2}\partial_{\mu}h_{\alpha[\rho}\partial^{\alpha}h_{\sigma]\nu} - \frac{1}{2}\partial_{\nu}h_{\alpha[\rho}\partial^{\alpha}h_{\sigma]\mu} - \frac{1}{2}\partial_{\alpha}h_{\mu[\rho}\partial^{\alpha}h_{\sigma]\nu}\Big] + \mathcal{O}\left(\kappa^3\right) , \end{aligned} \tag{56}$$

$$\begin{aligned} R_{\mu\nu} = {} & \kappa\left[\partial^{\alpha}\partial_{(\mu}h_{\nu)\alpha} - \frac{1}{2}\partial^2 h_{\mu\nu} - \frac{1}{2}\partial_{\mu}\partial_{\nu}h\right] + \kappa^2\Big[-\frac{1}{4}\partial_{\alpha}h\partial^{\alpha}h_{\mu\nu} + \frac{1}{2}\partial^{\alpha}h_{\mu\nu}\partial_{\beta}h_{\alpha}{}^{\beta} \\ & + \frac{1}{2}h^{\alpha\beta}\partial_{\alpha}\partial_{\beta}h_{\mu\nu} + \frac{1}{2}h^{\alpha\beta}\partial_{\mu}\partial_{\nu}h_{\alpha\beta} - \frac{1}{2}\partial^{\alpha}h_{\beta(\mu}\partial^{\beta}h_{\nu)\alpha} + \frac{1}{2}\partial_{\alpha}h_{(\mu}{}^{\beta}\partial^{\alpha}h_{\nu)\beta} \\ & - h^{\alpha\beta}\partial_{\alpha}\partial_{(\mu}h_{\nu)\beta} - \partial_{(\mu}h_{\nu)}{}^{\alpha}\partial^{\beta}h_{\alpha\beta} + \frac{1}{4}\partial_{(\mu}h^{\alpha\beta}\partial_{\nu)}h_{\alpha\beta} + \frac{1}{2}\partial_{(\mu}h_{\nu)}{}^{\alpha}\partial_{\alpha}h\Big] + \mathcal{O}\left(\kappa^3\right) , \end{aligned} \tag{57}$$

Expansion of geometrical objects

$$
\begin{aligned}
R = \kappa \left[\partial_\alpha \partial_\beta h^{\alpha\beta} - \partial^2 h\right] + \kappa^2 \Big[& h^{\alpha\beta} \partial^2 h_{\alpha\beta} - \frac{1}{4} \partial_\alpha h \partial^\alpha h + \partial^\alpha h \partial_\beta h_\alpha{}^\beta + h^{\alpha\beta} \partial_\alpha \partial_\beta h \\
& - \partial_\alpha h^{\alpha\beta} \partial_\gamma h_\beta{}^\gamma - 2 h^{\alpha\beta} \partial_\beta \partial_\gamma h_\alpha{}^\gamma - \frac{1}{2} \partial_\beta h_{\alpha\gamma} \partial^\gamma h^{\alpha\beta} + \frac{3}{4} \partial_\gamma h_{\alpha\beta} \partial^\gamma h^{\alpha\beta} \Big] + \mathcal{O}\left(\kappa^3\right) ,
\end{aligned} \tag{58}
$$

$$
R^2 = \kappa^2 \left[\partial_\alpha \partial_\beta h^{\alpha\beta} - \partial^2 h\right]^2 + \mathcal{O}\left(\kappa^3\right) , \tag{59}
$$

$$
R^{\mu\nu} R_{\mu\nu} = \frac{1}{4} \kappa^2 \left[2 \partial^\alpha \partial_{(\mu} h_{\nu)\alpha} - \partial^2 h_{\mu\nu} - \partial_\mu \partial_\nu h\right] \left[2 \partial_\beta \partial^\mu h^{\nu\beta} - \partial^2 h^{\mu\nu} - \partial^\mu \partial^\nu h\right] + \mathcal{O}\left(\kappa^3\right) , \tag{60}
$$

$$
R_{\rho\sigma\mu\nu} R^{\rho\sigma\mu\nu} = 4 \kappa^2 \partial_\rho \partial_{[\mu} h_{\nu]\sigma} \partial^\mu \partial^{[\rho} h^{\sigma]\nu} + \mathcal{O}\left(\kappa^3\right) . \tag{61}
$$

Conventions

In the following we will employ the conventions of [**Freedman:2011hp**], in which the metric is $(-,+,+,\cdots)$, the gamma matrices fulfill the usual Clifford algebra, i.e., $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$, and $\bar{\psi} = \mathrm{i}\psi^*\gamma^0$. We consistently work in n dimensions in order to employ dimensional regularization, and use the Breitenlohner-Maison scheme [**Breitenlohner:1977hr**] for the definition of the chiral matrix γ_* in n dimensions. The Riemann tensor is $R^\sigma{}_{\mu\rho\nu} = \partial_\rho\Gamma^\sigma_{\mu\nu} + \cdots$, and the Ricci tensor is obtained as $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$. We use geometric units $c = \hbar = 1$ and the totally antisymmetric symbol normalized to $\epsilon_{0123} = 1$. We denote (idempotent) symmetrization of indices by parentheses, e.g., $v^{(a}w^{b)} = \frac{1}{2}\left(v^a w^b + v^b w^a\right)$, and antisymmetrization by brackets, e.g., $v^{[a}w^{b]} = \frac{1}{2}\left(v^a w^b - v^b w^a\right)$.