Pseudospectra and the AdS/CFT correspondence

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Pseudospectra and the AdS/CFT correspondence

- Quasinormal modes
- AdS/CFT (vulgo Holography)
- QNMs in Holography
- Pseudospectra
- Pseudospectra of QNMs
- Summary and Outlook

Normal Modes

Eigen modes of string:

$$\frac{d^2\Phi(x)}{dx^2} + \lambda\Phi(x) = 0$$

Boundary conditions:

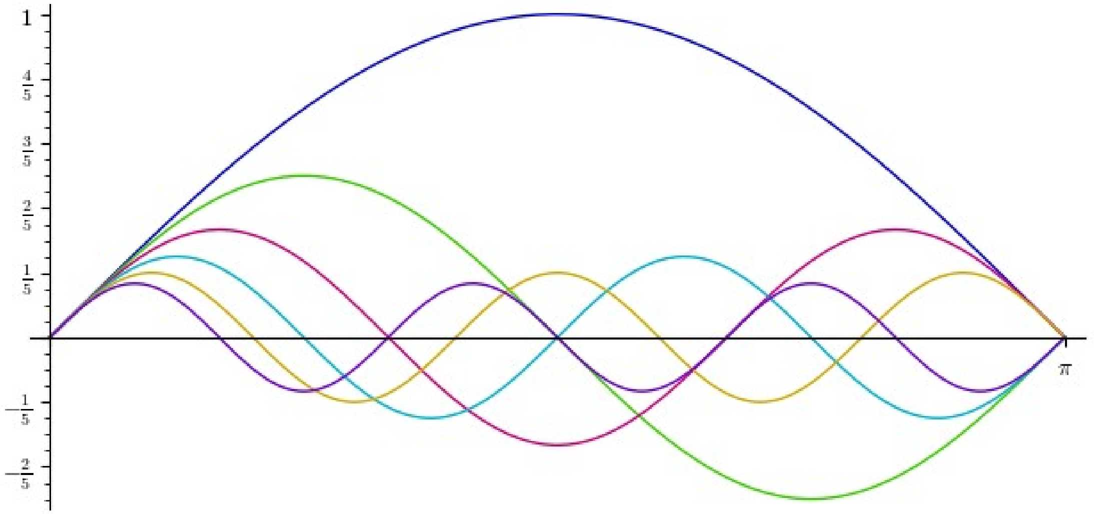
$$\Phi(0) = \Phi(\pi) = 0$$

Hermitian operator:

$$\mathcal{L} = \frac{d^2}{dx^2} \quad , \quad \mathcal{L}^\dagger = \mathcal{L}$$

$$\langle\Psi, \Phi\rangle = \int_0^\pi dx \bar{\Psi}\Phi$$

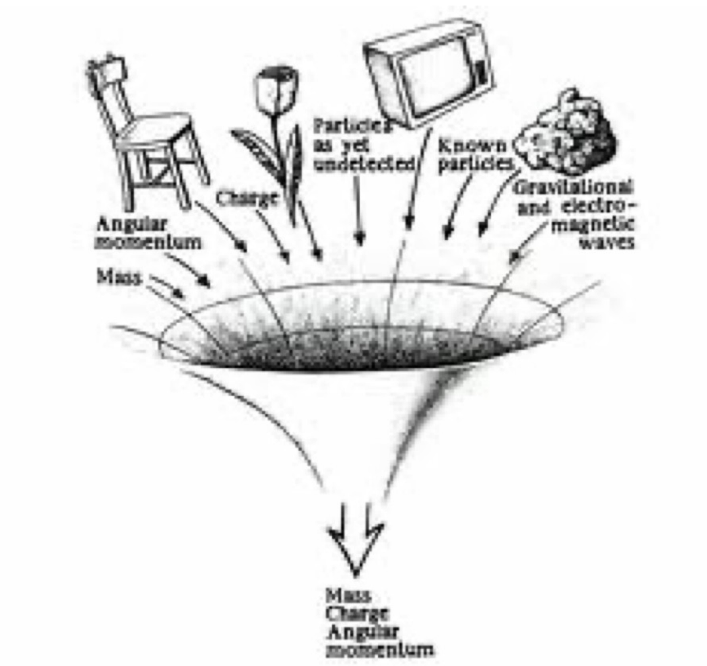
$$\langle\Psi, \mathcal{L}\Phi\rangle = \langle\mathcal{L}\Psi, \Phi\rangle$$



$$\Phi_n(x) = \sin(nx)$$
$$\lambda = n^2$$

Quasi-Normal Modes

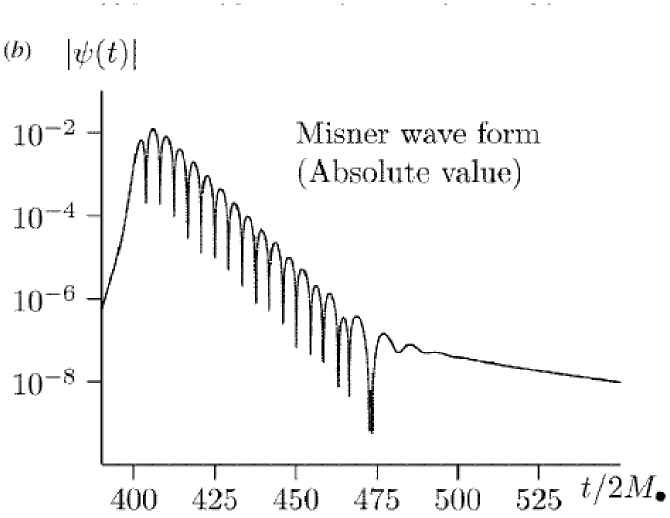
- Black Hole



- ➔ Black Holes no Hair
- ➔ Swallow everything
- ➔ Fate of a perturbation
  - ➔ Either fall into BH
  - ➔ Or radiate off to infinity
  - ➔ Perturbation eventually dies off

Quasi-Normal Modes

Numerical simulations:



[Nollert, CQG 16, R195]
[Price, Pulin, PRL 72,3297]

Characteristic “ringdown” frequency

Quasi-Normal Modes

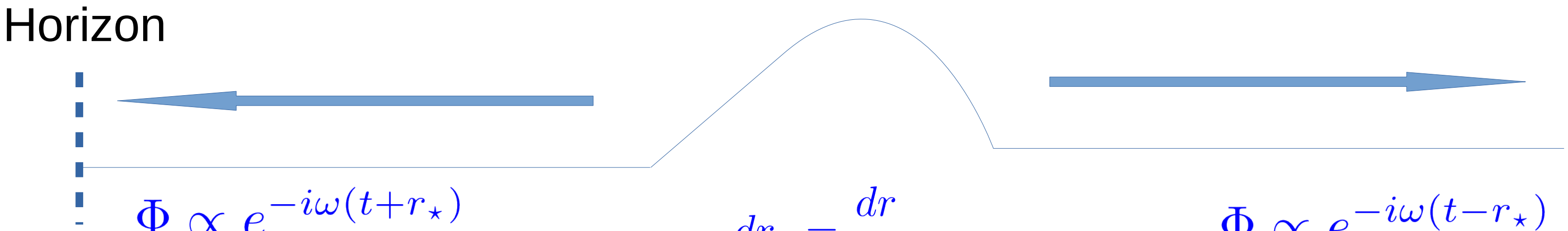
How to compute Quasi Normal Modes:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

$$\Phi = \phi(r) e^{-i\omega t} Y_{lm}(\Omega)$$

$$\phi'' + \frac{f'}{f}\phi + \left(\frac{\omega^2}{f^2} - \frac{l(l+1)}{r^2 f} + \frac{f'}{rf}\right)\phi = 0$$

"Outgoing" boundary conditions:



[Vishveshwara]
[Press]

$$\Phi \propto e^{-i\omega(t+r_\star)} \qquad dr_* = \frac{dr}{f} \qquad \Phi \propto e^{-i\omega(t-r_\star)}$$

Quasi-Normal Modes

Leaky boundary conditions lead to complex frequencies

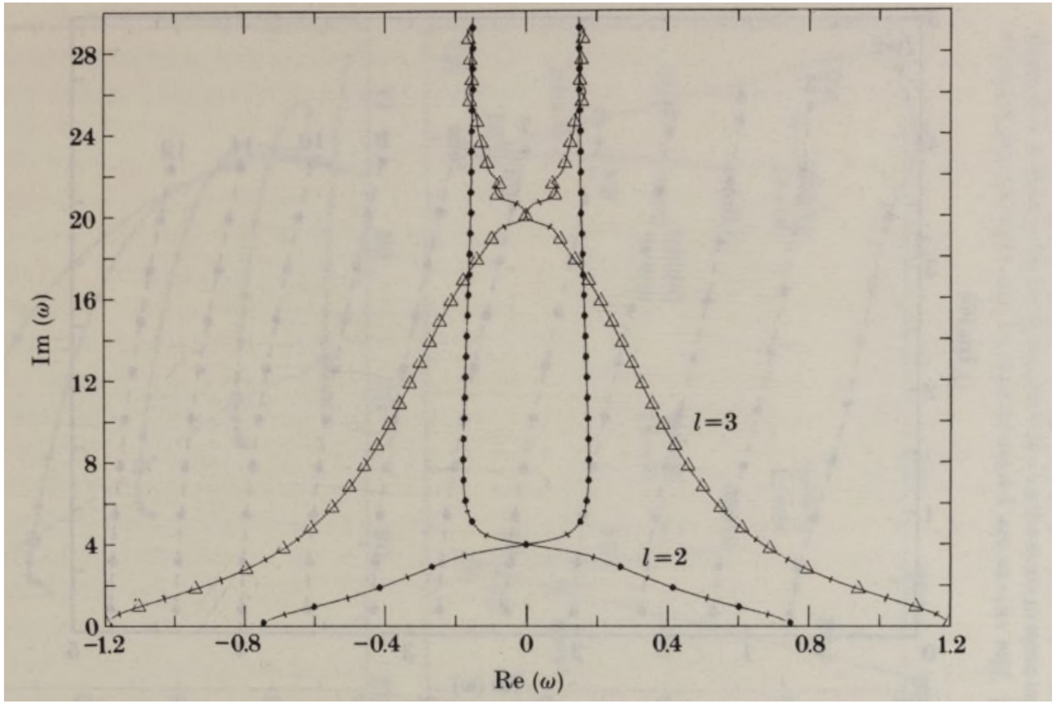
$$\omega_n = \Omega_n - i\Gamma_n$$

Oscillation

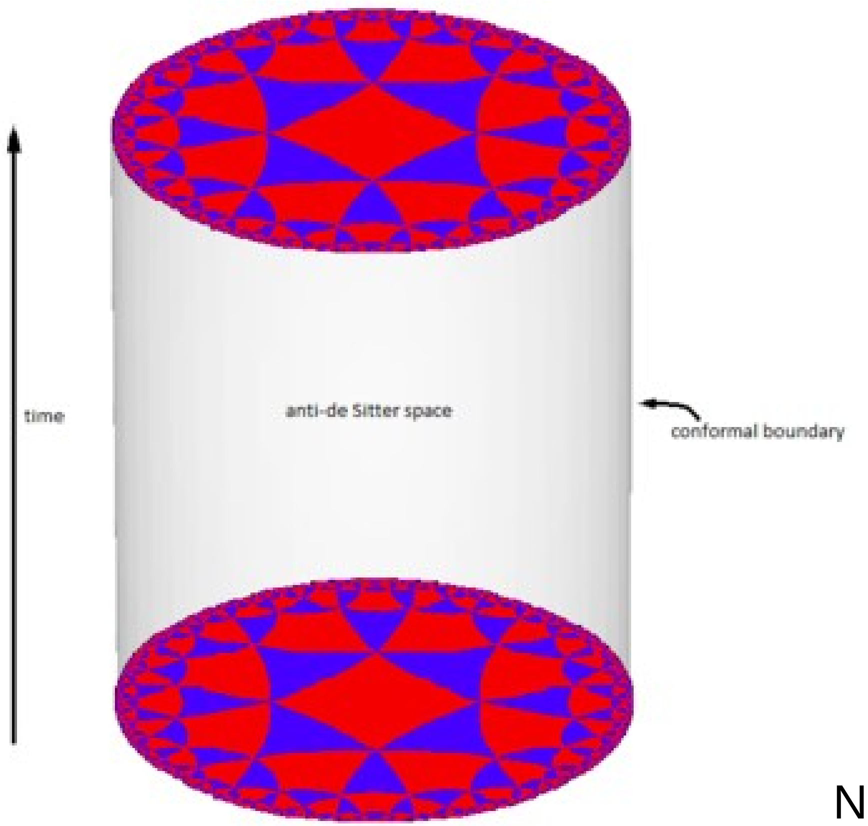
Damping

“Black Hole spectroscopy”

[Berti, Cardoso, Will, PRD 73 (2006)]



Holography



Gravity in asymptotically AdS = QFT

Holographic Dictionary	
Gravity	Quantum Field Theory
Metric	*Energy Momentum Tensor*
Gauge field	*Conserved current*
Scalar field	*Scalar operator*

Notice: Symmetry in QFT = gauge principle in AdS

Holographic QNMs

Planar AdS Black hole: $$ds^2 = \frac{r^2}{L^2}\left(-\left(1-\pi^4T^4/r^4\right)dt^2 + dx^2\right) + \frac{L^2 dr^2}{r^2\left(1-\frac{\pi^4T^4}{r^4}\right)}$$

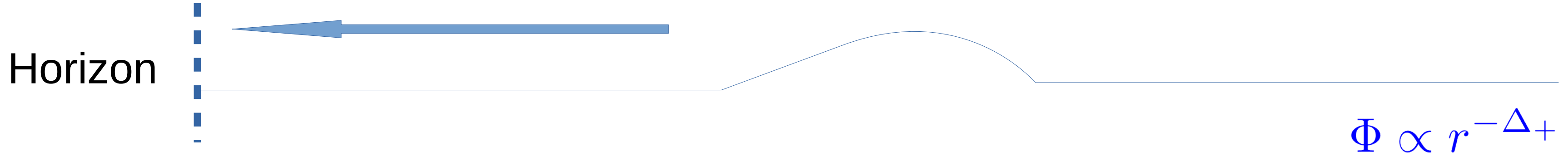
$$\Phi(r,t,\vec{x}) = e^{-i\omega t + i\vec{k}.\vec{x}}\phi_{\omega,\vec{k}}(r)$$

Boundary condition Horizon: $$\phi_{\omega,\vec{k}} \propto e^{-i\omega(t+r_*)}$$

Boundary condition boundary: $$\phi \approx A(\omega,\vec{k})r^{-\Delta_-}(1+\dots) + B(\omega,\vec{k})r^{-\Delta_+}(1+\dots)$$

Retarded Green's function: $$G_R(\omega,\vec{k}) = K\frac{B(\omega,\vec{k})}{A(\omega,\vec{k})}$$

[Horowitz, Hubeny],
[Birmingham, Sachs, Solodhukin]
[Kovtun, Son, Starinets]



Holographic QNMs

Example: Scalar field in BTZ black hole

$$G_R(\omega,k) = \frac{(\omega^2-k^2)}{4\pi^2}\left[\psi\left(1-i\frac{\omega-k}{4\pi T}\right)+\psi\left(1-i\frac{\omega+k}{4\pi T}\right)\right]$$

$$\omega_n = \pm k - i4(n+1)$$ Exact spectrum of QNMs !

In general no exact solution, e.g. scalar in AdS_5:

$$\phi'' + \left(\frac{5}{r}+\frac{f'}{f}\right)\phi' + \frac{\omega^2 - f^2\vec{k}}{r^2 f}\phi = 0$$

"Christmas tree"

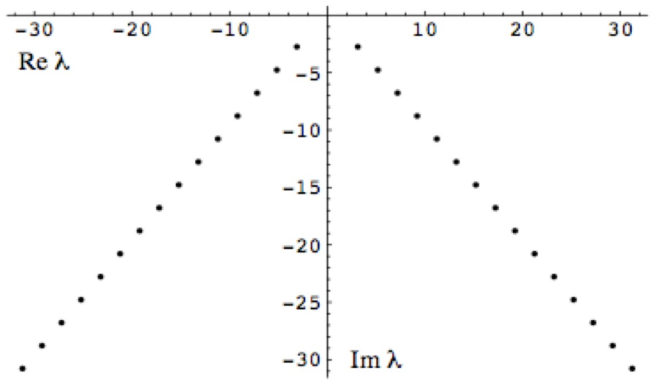


Fig. 1: The lowest 15 quasinormal frequencies in the complex λ-plane for $q=0$.

[Starinets]

Holographic QNMs

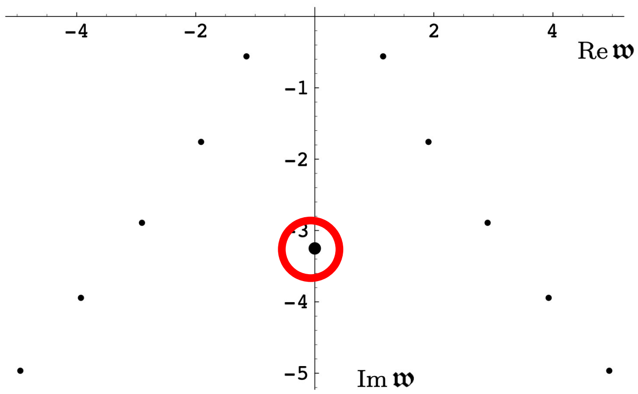
Gauge fields: new ingredient gauge symmetry: conserved current $\partial_\mu J^\mu = 0$

$$\frac{d}{dt}Q = 0 \qquad\qquad Q = \int d^3x J^0$$

2 channels:

- Transverse is like scalar
- Longitudinal new: diffusion

[Son, Starinets]

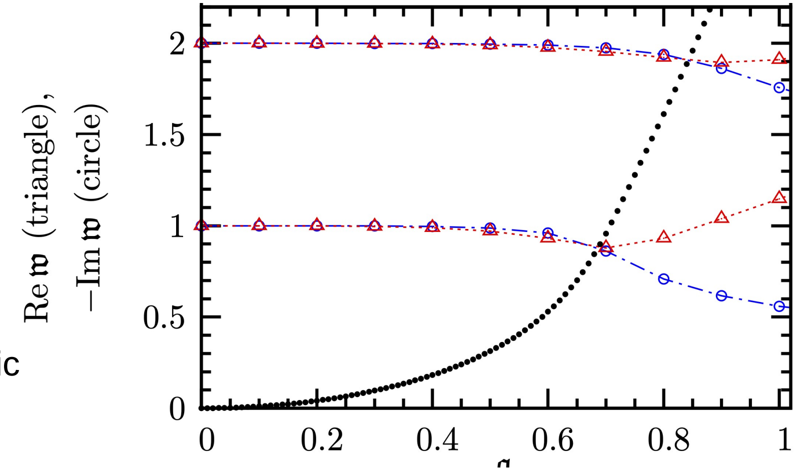


$$\omega_{\mathrm{hydro}}(k) = -iDk^2$$

$$\omega = i\sum_{n=1}^{\infty} a_i k^{2n}$$

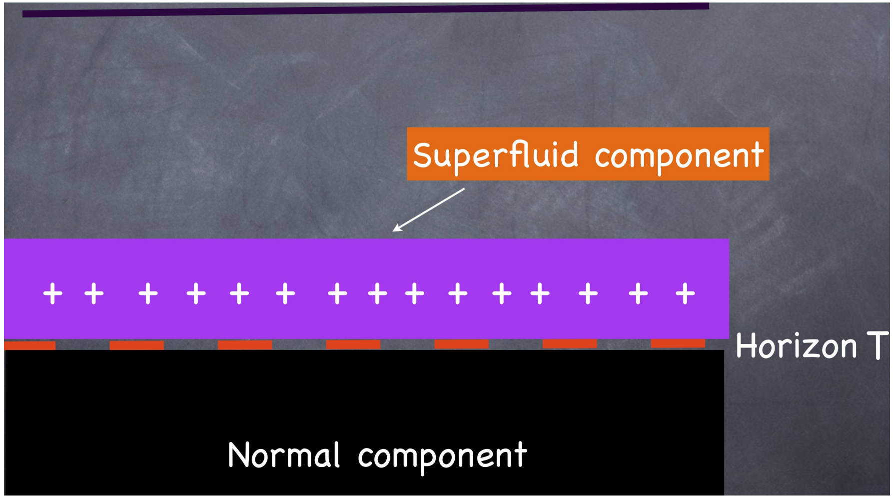
Problematic Causality!

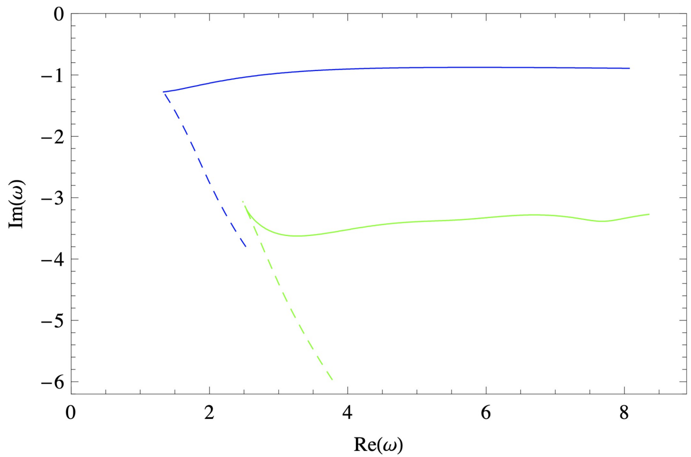
[Amado, Hoyos, K.L., Montero]

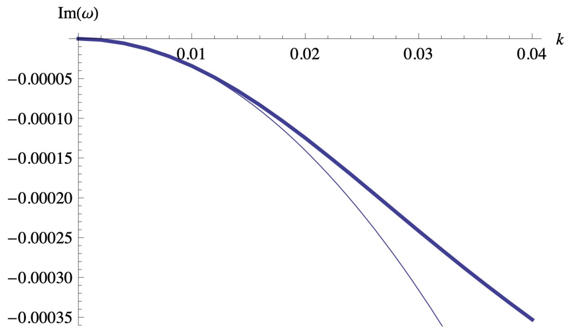


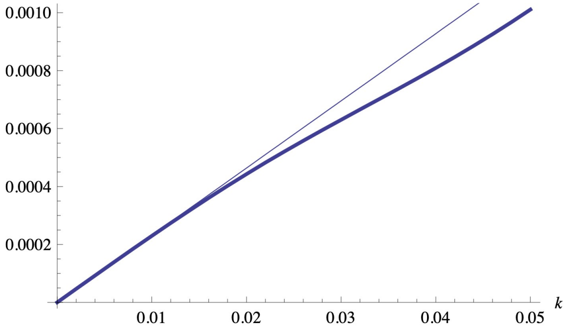
Holographic QNMs

Broken phase with scalar condensation T>T$_c$









$$\omega = \pm v_s^{(2)}(T) - i\gamma(T)k^2 + O(k^3)$$

Second sound!

Holographic QNMs

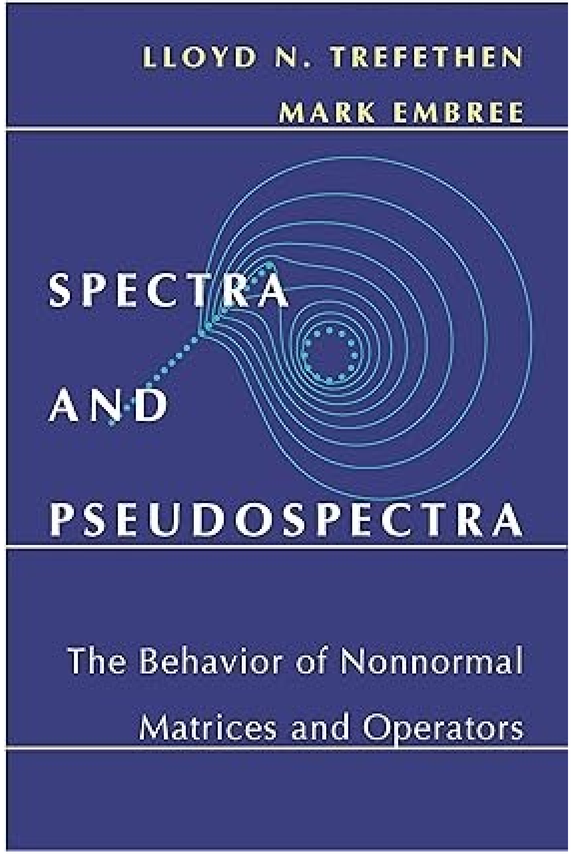
Quasinormal Modes are an essential ingredient of the Gauge/Gravity duality !

Pseudospectra

- QNMs are eigenvectors of non-Hermitian operators
- No spectral theorem $\mathcal{O} \neq \sum_n |n\rangle \lambda_n \langle n|$
- QNMs are not complete (only late time)
- Eigenfunctions are singular in Schwarzschild coords
- Robustness and physical significance in question

Near Horizon $\phi(r) = e^{-i\omega r_*} = e^{-i\Omega r_* - \Gamma r_*} \quad , \quad r_* \to -\infty$

No Hilbert space interpretation



Pseudospectra

Resolvent: $\mathcal{R}(\mathcal{L}, z) = (\mathcal{L} - z)^{-1}$

Spectrum: $\sigma(\mathcal{L}) = \{z \in \mathbb{C} : \text{"}\mathcal{R}(\mathcal{L}, z) = \infty\text{"}\}$

Eigenvalues: $\mathcal{L} u_n = \lambda_n u_n$

Operator norm: $||\mathcal{L}|| = \sup_{u \in H} \frac{||\mathcal{L}u||}{||u||}$

Definitions of Pseudospectra:

1) Resolvent norm $\sigma_\epsilon(\mathcal{L}) = \{z \in \mathbb{C} \ : \ ||\mathcal{R}(\mathcal{L}, z)|| > 1/\epsilon\}$

2) Perturbation $\sigma_\epsilon(\mathcal{L}) = \{z \in \mathbb{C}, \exists \delta\mathcal{L}, ||\delta\mathcal{L}|| < \epsilon : z \in \sigma(\mathcal{L} + \delta L)\}$

3) Pseudo eigenvector $\sigma_\epsilon(\mathcal{L}) = \{z \in \mathbb{C}, \exists u^\epsilon : ||(\mathcal{L} - z)u^\epsilon|| < \epsilon ||u^\epsilon||\}$

Theorem: The 3 definitions are equivalent

Pseudospectra

Condition number: $$\kappa_i = \frac{||v_i||||u_i||}{|\langle v_i, u_i \rangle|}$$

Right eigenvector: $\mathcal{L} u_i = \lambda_i u_i$

Left eigenvector: $\mathcal{L}^\dagger v_i = \lambda_i^* v_i$

Perturbation: $||\delta\mathcal{L}|| = \epsilon$

Perturbed eigenvalue: $(\mathcal{L} + \delta\mathcal{L}) u_i(\epsilon) = \lambda(\epsilon) u_i(\epsilon)$

$$|\lambda(\epsilon)_i - \lambda_i| \leq \epsilon\kappa_i$$

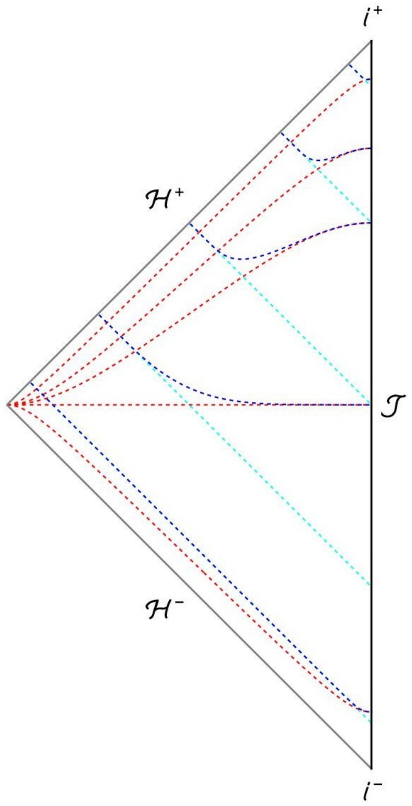
Def "small": Let d_{min} be the minimal distance between disconnected regions in the spectrum.

$\delta\mathcal{L}$, $||\delta\mathcal{L}|| = \epsilon$ is small if

$$\frac{\epsilon}{d_{min}} \ll 1$$

Pseudospectra

How to deal with the QNM problems: chose better coordinates!



Schwarzschild coordinates (worst)

$$ds^2 = r^2\left[-f(r)dt^2 + d\vec{x}^2\right] + \frac{dr^2}{r^2 f(r)}$$

$$\phi(r) \propto (r - r_h)^{-i\omega/2}$$

Infalling Eddington-Finkelstein

$$dv = dt + \frac{dr}{r^2 f(r)}$$

$$ds^2 = r^2\left[-f(r)dv^2 + d\vec{x}^2\right] + 2dvdr$$

$\phi(r)$ regular at $r = r_h$

"Regular"

$$\tau = v - (1 - r_h/r)$$

$$ds^2 = r^2\left[-fd\tau^2 + d\vec{x}^2\right] + 2(1-f)d\tau dr + (2-f)\frac{dr^2}{r^2}$$

[Warnick]

Pseudospectra

We need a physically motivated norm: **Energy** !

Energycurrent: $J = t^\mu T_{\mu\nu} dx^\mu$ $E[\Phi] = \int_{\Sigma_t} \star J$

Schwarzschild:	$E = \frac{1}{2}\int d^3x dr \left[r^2 f (\Phi')^2 + \frac{(\partial_t \Phi)^2}{r^2 f} + m^2 \Phi^2 + \frac{(\vec{\partial}\Phi)^2}{r^2} \right]$	$\frac{dE}{dt} = 0$
Infalling EF:	$E = \frac{1}{2}\int d^3x dr r^3 \left[r^2 f (\Phi')^2 + m^2 \Phi^2 + \frac{(\vec{\partial}\Phi)^2}{r^2} \right]$	$\frac{dE}{dv} = -r_h^3 \int_{r=r_h} (\partial_v \Phi)^2$
Regular:	$E = \frac{1}{2}\int d^3x dr r^3 \left[r^2 f (\Phi')^2 + m^2 \Phi^2 + \frac{(\vec{\partial}\Phi)^2}{r^2} + (2-f)(\partial_\tau \Phi)^2 \right]$	$\frac{dE}{d\tau} = -r_h^3 \int_{r=r_h} (\partial_\tau \Phi)^2$

Pseudospectra

- We want to re-write the wave equation as a standard eigenvalue problem
- Only possible in regular coordinates

$$\psi = \partial_\tau \Phi \qquad \Psi = \begin{pmatrix} \Phi \\ \psi \end{pmatrix} \qquad \mathcal{L} = i \begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix} \quad \Rightarrow \quad \mathcal{L}\Psi = \omega \Psi$$

- Compactify radial coordinate $\rho = 1 - \frac{r_h}{r}$

$$L_1 = [f(\rho) - 2]^{-1} \left[\frac{m^2 l^2}{(1-\rho)^2} + \mathbf{q}^2 - (1-\rho)^3 \left(\frac{f(\rho)}{(1-\rho)^3} \right)' \partial_\rho - f(\rho)\partial_\rho^2 \right]$$

$$L_2 = [f(\rho) - 2]^{-1} \left[(1-\rho)^3 \left(\frac{f(\rho) - 1}{(1-\rho)^3} \right)' + 2\,(f(\rho) - 1)\,\partial_\rho \right]$$

- Adjoint operator in energy norm $\mathcal{L}^\dagger = \mathcal{L} + \begin{pmatrix} 0 & 0 \\ 0 & -i\,\delta(\rho) \end{pmatrix}$ $\qquad \frac{d}{d\tau} E = -\bar{\psi}\psi|_{\rho=0}$

AdS in IEF [Cownden, Pantelidou, Zilhao] e-Print: 2312.08352

Pseudospectra

- No exact solutions → numerical methods
- Pseudospectral methods
- Chebyshev polynomials for interpolation

$$F(\rho) \approx \sum_{n=0}^{N} c_n T_n(\rho)$$

$$F(\rho_j) = \sum_{n=0}^{N} c_n T_n(\rho_j) \quad , \quad \rho_j = \frac{1}{2}\left(1 - \cos\left(\frac{j\pi}{N}\right)\right) \quad , \quad j = 0 \ldots N$$

- Differential operator becomes a (N+1)x(N+1) matrix D $F'(\rho_j) = D_{jk} F(\rho_j)$
- Boundary conditions: delete rows and columns corresponding to $\rho = 1$
  regularity corresponds to no boundary condition at $\rho = 0$
- Resolvent norm becomes maximal svd $||\mathcal{L} - \omega \mathbf{1}|| \approx \mathrm{inf(svd)}$
- Energy norm becomes a metric 2Nx2N matrix $E \approx \bar{u}_k^* G_E^{km} u_m \quad , \quad u \approx (\phi(\rho_j), \psi(\rho_j))^T$

Pseudospectra

A toy example: $A = \begin{pmatrix} -1 & 0 \\ -50 & -2 \end{pmatrix}$ Eigenvalues: $\lambda_1 = -1 \;\; , \;\; \lambda_2 = -2$

1. ℓ_2 norm: $||u|| = [\bar{u}.u]^{1/2}$

$$\kappa_1 = \kappa_2 = \sqrt{2501} \approx 50$$

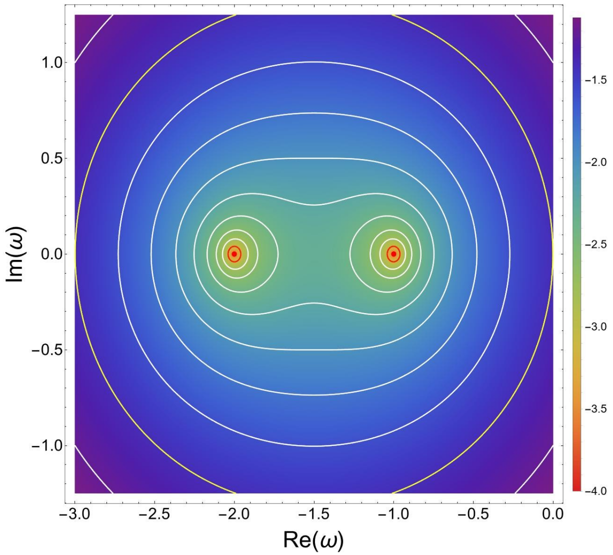
1. G-norm: $||u||_G = [\bar{u}.G.u]^{1/2}$

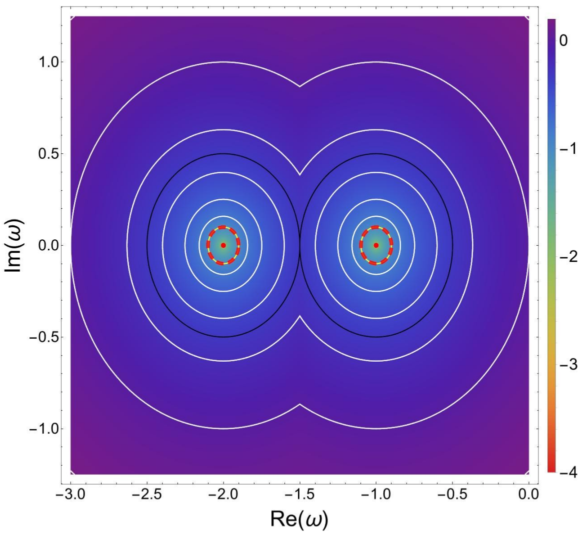
$$A^\dagger = \left[(G.A.G^{-1})^T\right]^* = A$$

$$G = \begin{pmatrix} 20000 & 50 \\ 50 & 1 \end{pmatrix}$$

$$\kappa_1 = \kappa_2 = 1$$

Contour maps of $\log ||A - \omega \mathbf{1}||$

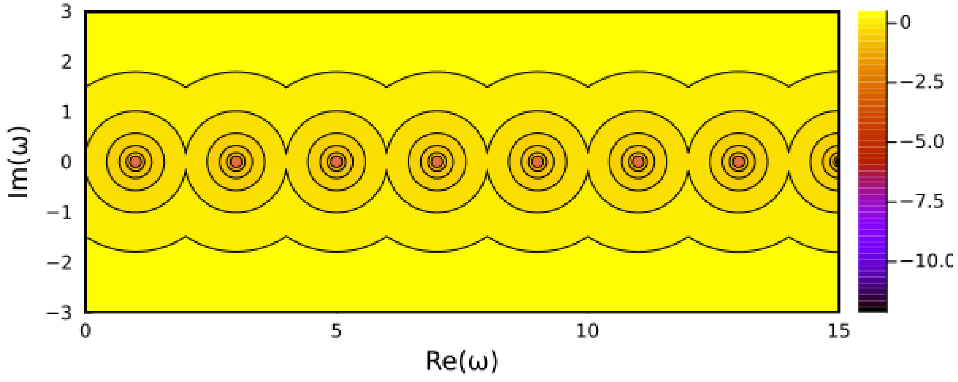




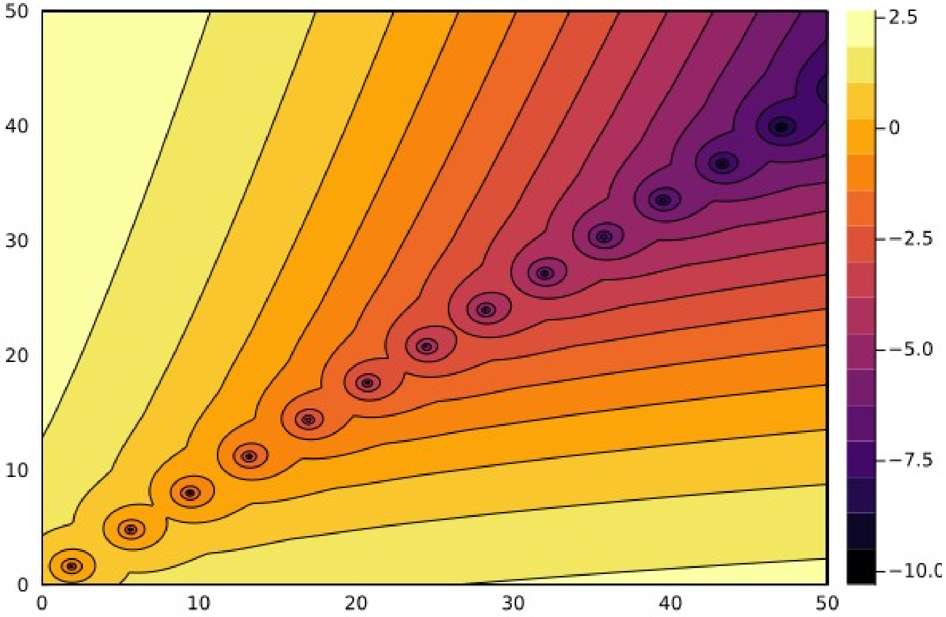
Pseudospectra

Harmonic Oscillator: $-\frac{d^2\phi}{dx^2} + c\,x^2\,\phi = \omega\phi$ $\qquad G_E = \int dx\bar{\phi}\phi$

c=1



c=1+3i



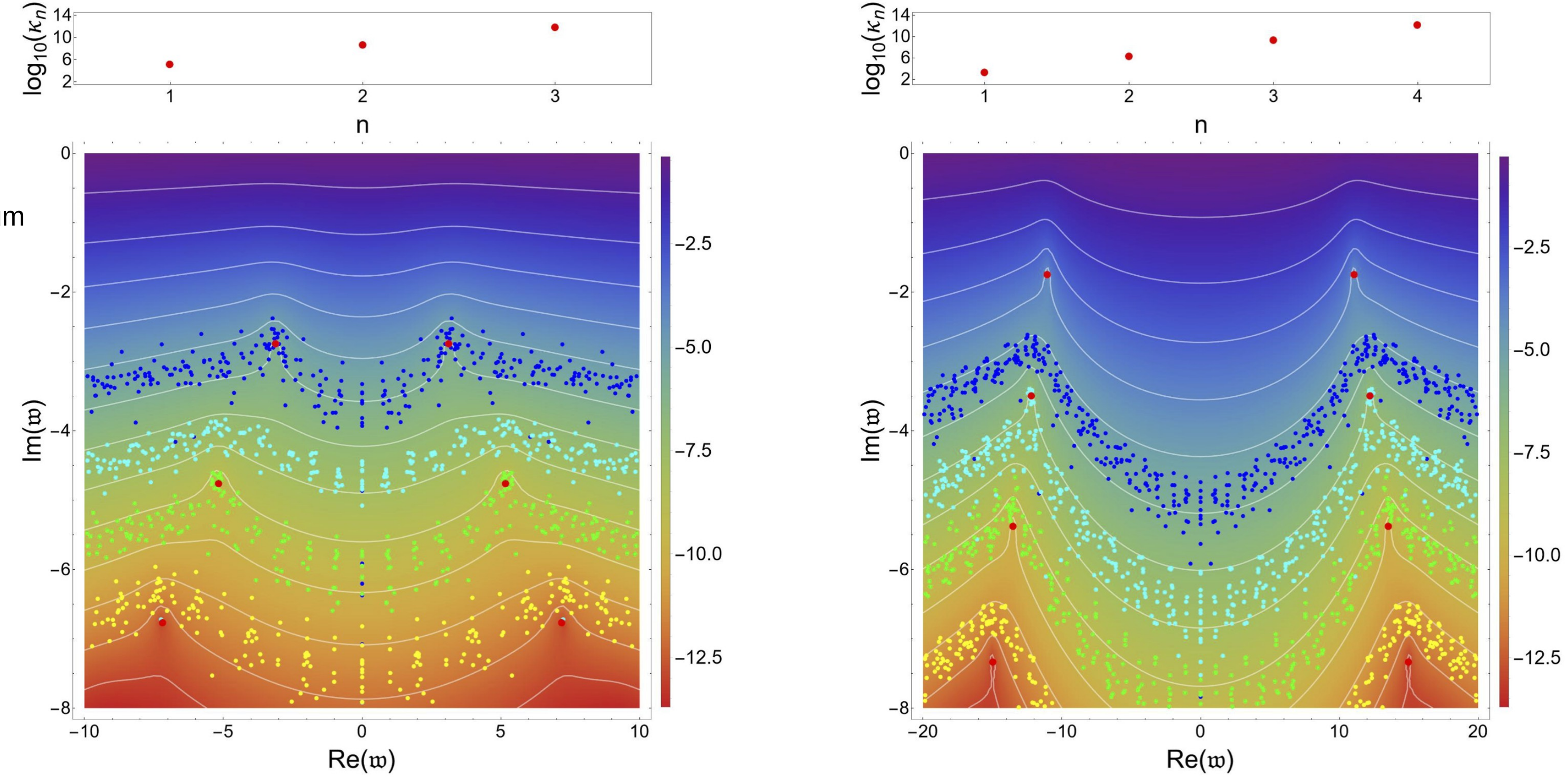
Pseudospectra

Pseudospectra of massless scalar in AdS_5:

“Selective” Pseudospectrum
Local random
Potential perturbations

$||V_{\mathrm{rand}}||$

- 10^{-1}
- 10^{-3}
- 10^{-5}
- 10^{-7}



(a) $m^2 l^2 = 0$, $\mathfrak{q} = 0$.

(b) $m^2 l^2 = 0$, $\mathfrak{q} = 10$.

Pseudospectra

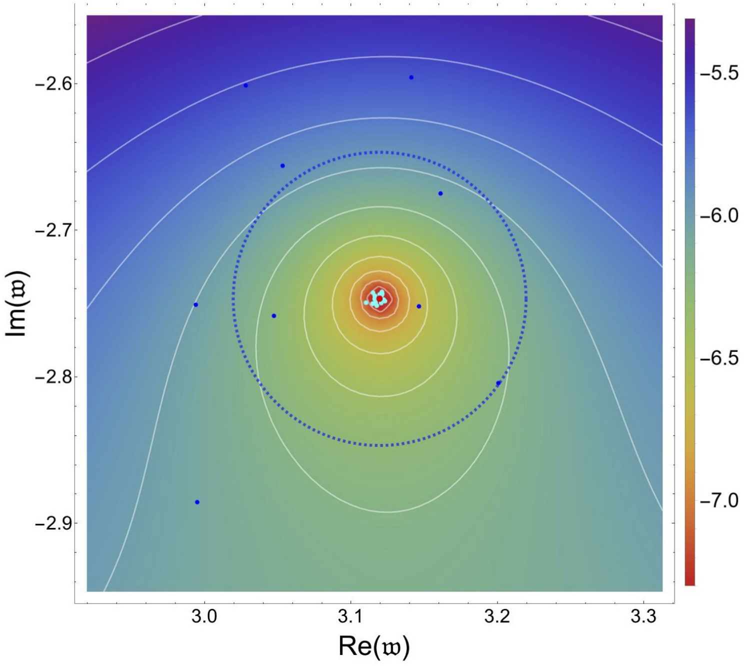
Zoom into first QNM:

Circle of stability

$$|\lambda(\epsilon)_i - \lambda_i| \leq \epsilon\kappa_i$$

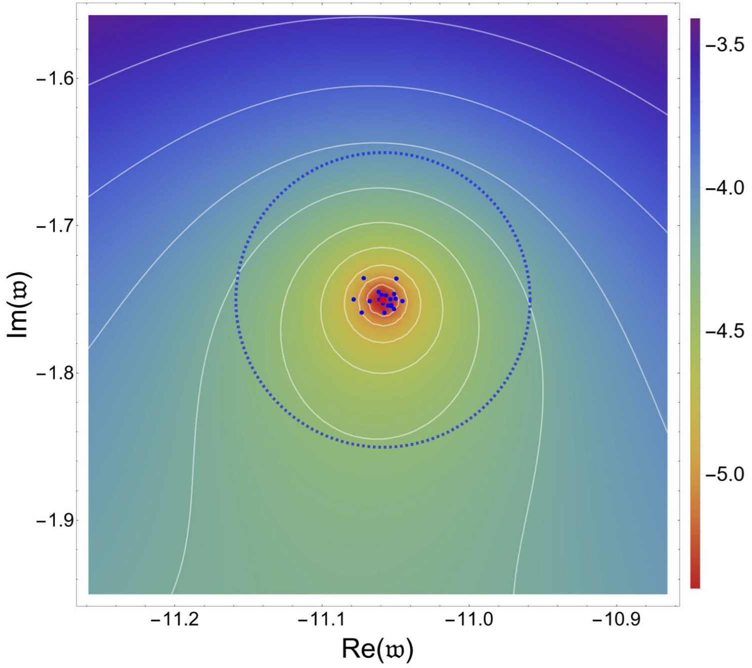
- 10^{-1}
- 10^{-3}

Restricted spectral instability



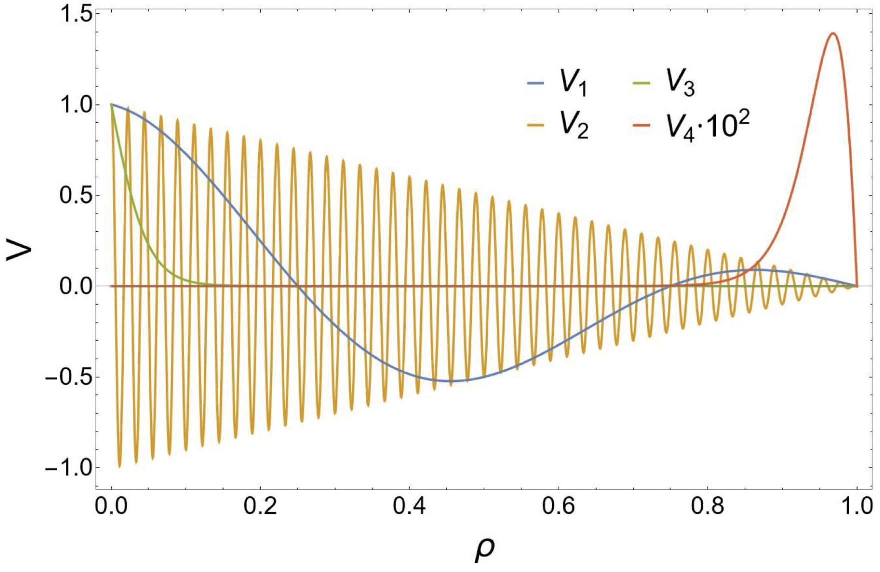
(a) $m^2l^2 = 0$, $\mathfrak{q} = 0$.

Restricted spectral stability



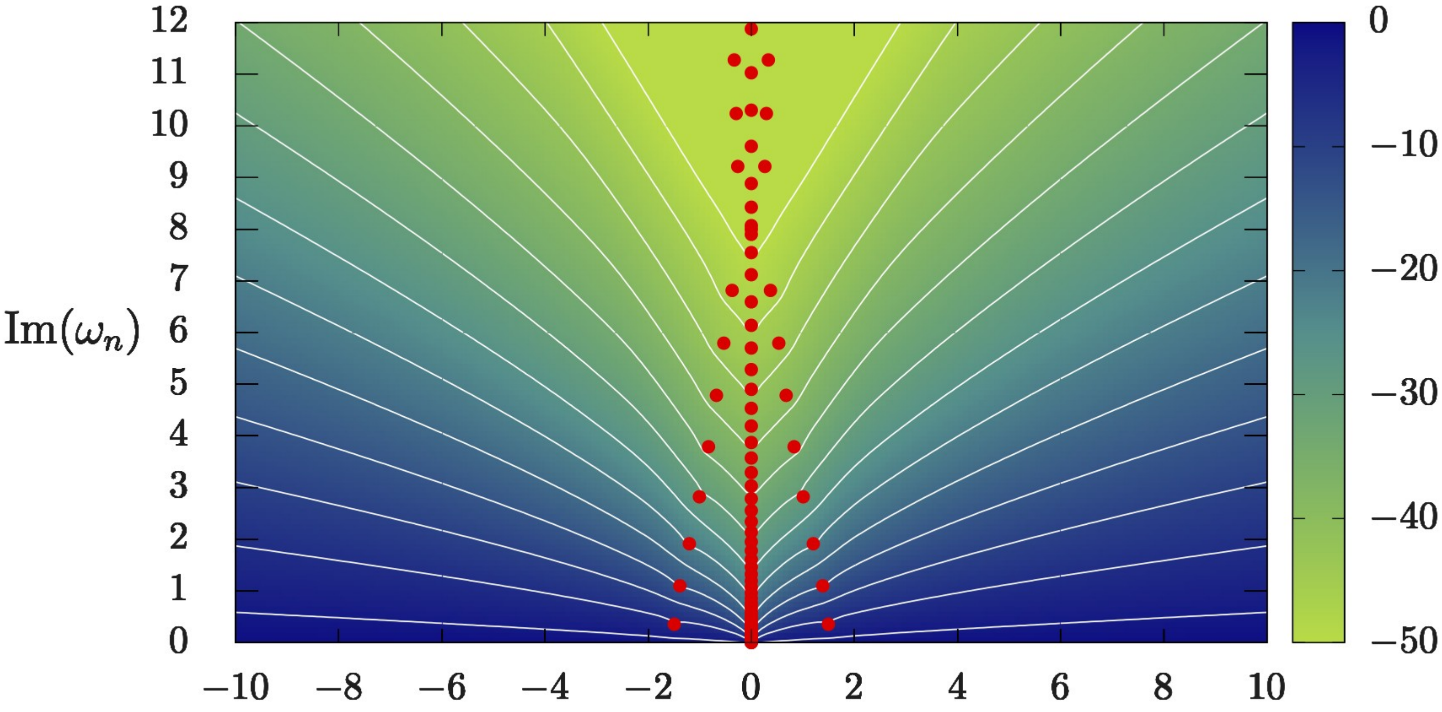
(b) $m^2l^2 = 0$, $\mathfrak{q} = 10$.

Pseudospectra



Pseudospectra

In asymptotically flat space: [Jaramillo, Macedo, Al Sheik, PRX 11 (2021) 3, 031003 • e-Print: 2004.06434]



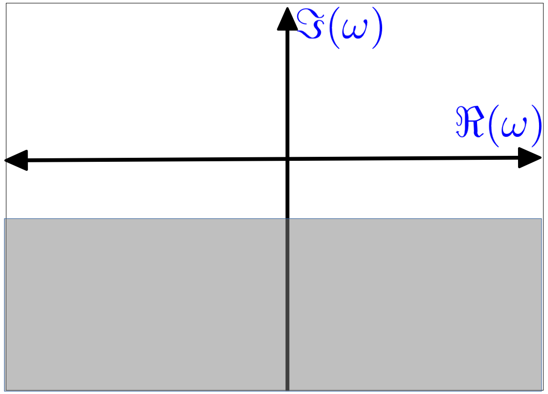
Review: [Destounis, Duque] e-Print: 2308.16227

Pseudospectra

Caveats: Ingoing modes are integrable in the energy norm:

$$\phi_{\rm in} \approx \rho^{i\omega} = \rho^{i\Omega}\rho^{\Gamma}$$

$$|\phi'_{\rm in}|^2 \approx \rho^{2\Gamma-1} \qquad \Gamma = -\Im(\omega) > \frac{1}{2}$$



- Hilbert space of square integrable functions with energy norm
  In this sense the ingoing solutions belong to the spectrum
- Pseudospectrum in IEF N → infinity limit converges to this limit

- Math: Sobolev norm, Hilbert space $H^{(k)}$
  Physics: higher derivative theories!

$$||\Phi||^2 = \int \sum_{m=0}^{k} |D^m\Phi|^2$$

- In regular coords resolvent seems to be divergent in general

[Warnick: CMP. 333 (2015) 2, 959-1035 • e-Print: 1306.5760 [gr-qc]]

- Finite N provides a natural cutoff → what is the optimal N?
  e.g stretched Horizon, Planck length away from horizon

[Boyanov, Cardoso, Destounis,Jaramillo, Macedo] e-Print: 2312.11998 [gr-qc]

- Universality? Holographic interpretation?

Summary

- Quasinormal Modes are central in gauge/gravity duality
- Subject to spectral instability
- Choice of norm is important → Energy norm
- Energy norm "$H^{(1)}$" Hilbert space
- Hydrodynamic modes seem safe ? $$\lim_{k\to 0} \omega_{\text{hydro}}(k) = 0$$
- Physical significance of higher modes seems less clear
- Optimal lattice N ?
- Regular boundary conditions: beyond two derivative theories?

Thank you!