# Relaxation of first-class constraints and the quantization of gauge theories: from "matter without matter" to the reappearance of time in quantum gravity

Alexander Kamenshchik

University of Bologna and INFN, Bologna

June 5-7, 2024

Based on

Roberto Casadio, Leonardo Chataignier,
Alexander Yu. Kamenshchik, Francisco G. Pedro,
Alessandro Tronconi and Giovanni Venturi,

Relaxation of first-class constraints and the quantization of gauge theories: from “matter without matter” to the reappearance of time in quantum gravity,

arXiv:2402.12437 [gr-qc]

# Content

# Introduction

- Canonical gauge theories are the Hamiltonian description of theories that possess local symmetries.
- These theories have systems of first class constraints.
- The well-known examples of such theories are electrodynamics, Yang-Mills, gravity, strings and superstrings.
- The invariance of General Relativity with respect to spacetime diffeomorphisms implies that the Hamiltonian represents a linear combination of constraints and vanishes.

$$H = 0.$$

- That means that non all initial values of fields and momenta are allowed.
- If one uses the Dirac prescription for the quantization of the system with constraints, then one obtains

$$\hat{H}|\Psi\rangle = 0,$$

known as Wheeler-DeWitt equation.

- This is time-independent Schrödinger-type equation.
- It looks like a disappearance of time or problem of time in quantum gravity.
- The different established approaches to the re-introduction of time are based on the identification of some variables (degrees of freedom) with time parameter.
- In this way one can obtained the standard time-dependent Schrödinger equation for quantum states living on a properly constructed Hilbert space.

- In a recent preprint
  A.K. Burns, D.E. Kaplan, T. Melia and S. Rajendran, Time Evolution in Quantum Cosmology, arXiv: 2204.03043 [gr-qc]
  a recipe for a solution of the time problem was suggested: to do not respect the super-Hamiltonian constraint.
- In this case the Hamiltonian becomes non-vanishing and the time reappears.
- However, it is not a method of quantization of gravity. It is a different theory.
- If one relaxes some constraints, one obtains instead some additional degrees of freedom.

- Different examples of such a creation of new degrees of freedom due to relaxation of constraints are known.
- We have tried to develop a formalism describing such models in a unified language.
- This formalism is rather cumbersome.
- I shall present some particular examples.

# Fock and Stueckelberg approach

V.A. Fock in
V. Fock, *Sobstvennoe vremya v klassichskoj i kvantovoj mekhanike* (in Russian), Izv. AN SSSR, Ser. Fizika, No 4-5, 551 (1937); republished in the book *Raboty po kvantovoj teorii polya (Works on Quantum Field Theory)*, publishing house of Leningrad University (1957); also published as *Die Eigenzeit in der klassischen und in der Quantennechanik*, Phys. Zeit. d. Sowjetunion **12**, 404 (1937),

and more explicitly,
E. Stueckelberg in
E. C. G. Stueckelberg, *Remarque à propos de la création de paires de particules en théorie de relativité*, Helv. Phys. Acta **14**, 588 (1941);
*La signification du temps propre en mécanique ondulatoire*, Helv. Phys. Acta **14**, 322 (1941).
*La mécanique du point matériel en théorie de relativité et en théorie des quants*, Helv. Phys. Acta **15**, 23 (1942),

proposed a relaxation of the relativistic particle mass-shell constraint:

$$p_0^2 - \vec{p}^2 = m^2,$$

(which can be seen as a particle's version of the Wheeler-DeWitt equation, the role of diffeomorphism invariance is played by time reparametrization invariance).

- That means that the particle's motion is not restricted to a timelike worldline.
- Its mass is determined by the initial values of its four-momentum.
- The trajectories need no longer be timelike.
- This phenomenon Stueckelberg used to describe pair creation and annihilation by identifying antiparticles with particles going backward in Minkowski time.
- The idea of Fock and Stueckelberg may be seen as a passage from the Jacobi action principle, which yields the trajectories in a reparametrization-invariant way to the Hamilton action principle, where there is a preferred time parameter.
- It is equivalent to promoting the mass to a New degree of freedom, which is conjugate to particle's proper time, also elevated to a new degree of freedom.

$$S_{\mathrm{mech}} = \int_{\tau_0}^{\tau_1} d\tau (p_a \dot{q}^a - H(p_a, q^a)).$$

Gauging the time with an independent auxiliary field $\lambda(\tau)$, one obtains

$$\tilde{S}_{\mathrm{mech}} = \int_{\tau_0}^{\tau_1} d\tau (p_a \dot{q}^a - \lambda H(p_a, q^a)).$$

The arbitrariness of $\tau$ is inherited from the arbitrariness of $\lambda(\tau)$. One can define time internally by fixing $\tau$ to be a function of the particle phase variables. This, together with the Hamiltonian constraint $H \approx 0$, means that two phase variables are not independent. Thus, by gauging the time translations, we exclude independent degrees of freedom.

We can restore the eliminated degrees of freedom with the Fock-Stueckelberg mechanism. Using the gauge condition $\chi = \lambda - \Lambda$, the Fock-Stueckelberg action reads

$$S_{\mathrm{FS}} = \int_{\tau_0}^{\tau_1} d\tau (p_a \dot{q}^a - \lambda H(p_a, q^a) - w(\lambda - \Lambda)),$$

and it leads to the constraints $H + w \approx 0, \lambda \approx \Lambda$. The Hamiltonian constraint is relaxed, the symmetry under local time reparametrization is lost and eliminated degrees of freedom are restored.

## Relativistic particle

$$H(q,p) = \frac{1}{2} g^{ab}(q) p_a p_b + m(V(q) - E).$$

The configuration space action is

$$S_{\rm particle} = \int_{\tau_0}^{\tau_1} d\tau \left( \frac{g_{ab}\dot{q}^a \dot{q}^b}{2\lambda} - \lambda m(V - E) \right).$$

The variation with respect to $\lambda$ gives

$$-\frac{g_{ab}\dot{q}^a \dot{q}^b}{2\lambda^2} - m(V - E) = 0.$$

Then

$$S_{\rm Jacobi} = -\int_{\tau_0}^{\tau_1} d\tau \sqrt{2m(E - V) g_{ab}\dot{q}^a \dot{q}^b}.$$

Using the Fock-Stueckelberg mechanism we are led from this theory to

$$S_{\mathrm{Hamilton}} = \int_{\tau_0}^{\tau_1} d\tau \left( \frac{m}{2} g_{ab} \dot{q}^a \dot{q}^b - V + E \right),$$

where we have taken the gauge $\lambda = \frac{1}{m}$.

The new Hamiltonian is identical to the old one, but it is not constrained to vanish.

It is conserved.

To transition back from the Hamilton action principle to the Jacobi principle it is enough to restrict oneself to initial values for which Hamiltonian vanishes.

# Dirac non-linear electrodynamics

P.A.M. Dirac, A new classical theory of electrons, Proc. Roy. Soc. London A 209, 291 (1951).

“In the theory of the electromagnetic field without charges, the potentials are not fixed by the field, but are subject to gauge transformations. The theory thus involves more dynamical variables than are physically needed. It is possible by destroying the gauge transformations to make the superfluous variables acquire a physical significance and describe electric charges.”

$$\mathcal{L} = -\frac{1}{4} F^{ab} F_{ab} \ ,$$
$$F_{ab} = \partial_a A_b - \partial_b A_a .$$

Let us consider the gauge condition

$$\chi = \frac{1}{2}(A^a A_a + k^2) \approx 0 \ ,$$

with $k$ being a constant. It fixes the Lagrange multiplier $A_0$:

$$A_0 = \pm\sqrt{A_1^2 + A_2^2 + A_3^2 + k^2}.$$

The Dirac Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F^{ab} F_{ab} - \frac{w}{2}(A^a A_a + k^2).$$

The field equations

$$\partial_a F^{ab} = w A^b =: -J^b \ ,$$

with $\partial_a J^a = 0$ due to the antisymmetry of $F_{ab}$.

The appearance of the four-current $J^a$ signals the presence of new degrees of freedom with respect to the parent theory. We obtain a modification of the vacuum Gauss law:

$$\mathrm{div}\vec{E} = -wA^0 =: J^0 =: \rho \ .$$

The Dirac treatment can be generalized by considering a more general gauge condition.

# Generalized unimodular gravity

The unimodular gravity was first considered by Einstein in
A. Einstein, Spielen Gravitationsfelder im Aufbau der materiellen Elementarteilchen eine wesentliche Rolle? Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1919**, 349 (1919).
It was further developed in
W. G. Unruh, A unimodular theory of canonical quantum gravity, Phys. Rev. D **40**, 1048 (1989) and
M. Henneaux and C. Teitelboim, The cosmological constants and general covariance, Phys. Lett. B **222**, 195 (1989).
The action in this theory is invariant with respect to volume-preserving diffeomorhisms, which constitute a subgroup of the group of the spacetime diffeomorphisms.
The cosmological constant arises as an integration constant.

One can see the origin of the unimodular gravity in other terms.
In the Arnowitt-Deser-Misner formalism:

$$g_{\mu\nu} = \begin{pmatrix} N_a N^a - N^2 & N_b \\ N_b & h_{ab} \end{pmatrix} .$$

The lapse function $N$ and the shift functions $N^a$ play the role of Lagrange multipliers.
If one fixes the value of the lapse function as

$$N = \frac{1}{\sqrt{h}}, \;\; h = \det h_{ab},$$

one obtains the unimodular gravity.
Indeed,

$$N = \frac{1}{\sqrt{h}} \Rightarrow \det(|g|) = N\sqrt{h} = 1$$

accept only the volume-preserving diffeomorphisms.

Choosing a general function

$$N = \tilde{N}(h),$$

we obtain the generalized unimodular gravity.

A. O. Barvinsky and A. Yu. Kamenshchik, Darkness without dark matter and energy - generalized unimodular gravity, Phys. Lett. B **774**, 59 (2017).

We add to the action the term

$$\lambda(N - \tilde{N}(h))$$

to fix the lapse function. The variation of this term gives a contribution into the effective energy-momentum tensor, corresponding to the appearance of dark matter without dark matter:

$$T^{\mu\nu}_{\rm eff} = \frac{2}{\sqrt{|g|}} \frac{\delta}{\delta g_{\mu\nu}} \int d^4x \, (-\lambda(N - \tilde{N}(h)).$$

It gives us a perfect fluid with the four-velocity

$$v^\mu = \left(\frac{1}{\tilde{N}}, -\frac{N^a}{\tilde{N}}\right),$$

energy density

$$\varepsilon = \frac{\lambda}{\sqrt{h}}$$

and pressure

$$p = 2\frac{d \ln \tilde{N}}{dh}\varepsilon = w\varepsilon.$$

We should fix the acceptable forms of the new Lagrange multiplier $\lambda$.
It is possible to do it requiring the conservation of the energy-momentum tensor. As a result one can find such expressions:

$$\lambda = \frac{\lambda_0}{\tilde{N}\sqrt{h}},$$
$$\varepsilon = \frac{w_0}{\tilde{N}\sqrt{h}},$$
$$p = 2\frac{d\ln\tilde{N}}{dh}\frac{\lambda_0}{\tilde{N}\sqrt{h}}.$$

One can see that at

$$\tilde{N} = \frac{1}{\sqrt{h}}$$

we reproduce the expressions for the unimodular gravity, while at

$$\tilde{N} = 1,$$

we obtain dust without dust.

General analysis of the new degrees of freedom is rather a complicated problem (A.O. Barvinsky, N. Kolganov, A. Kurov and D. Nesterov, Dynamics of the generalized unimodular gravity theory, Phys. Rev. D 100 (2019) 2, 023542).

# Generalized unimodular gravity and Friedmann universe

$$ds^2 = N^2(\tau)d\tau^2 - a^2(\tau)dl^2,$$

where $a$ is the scale factor and $dl^2$ is the interval of the three-dimensional Euclidean space.

The gravitational part of the action is proportional to

$$\int d\tau \frac{\dot{a}^2 a}{N},$$

where "dot" means the differentiation with respect to the time parameter $\tau$.

The variation with respect to the lapse function $N$ gives the term

$$-\frac{\dot{a}^2 a}{N^2}.$$

In the presence of a perfect fluid, variating the corresponding part of the action one arrives to the first Friedmann equation, which is nothing but the 00 component of the Einstein equations or the super-Hamiltonian constraint

$$\frac{\dot{a}^2}{N^2 a^2} = \frac{1}{a^2}\left(\frac{da}{dt}\right)^2 = \varepsilon,$$

where $t$ is the cosmic time parameter. If the fluid satisfies the equation of state

$$p = w\varepsilon,$$

then the energy density behaves as

$$\varepsilon \sim \frac{1}{a^{3(1+w)}}.$$

Now, let us fix the gauge as

$$N = a^n$$

and consider an empty spacetime. Then the action is

$$\int d\tau \dot{a}^2 a^{1-n}.$$

The equation of motion is

$$\frac{d}{d\tau}(2\dot{a}a^{1-n}) + (n-1)\dot{a}^2 a^{-n} = 0.$$

Integrating this equation we obtain

$$\frac{\dot{a}^2}{a^2} = \frac{C}{a^{3-n}}.$$

The time parameter $\tau$ related to the lapse function $N$ is connected with the cosmic time $t$, corresponding to the special lapse function $N = 1$, by means of the relation

$$\frac{d}{d\tau} = \frac{d}{dt}\frac{dt}{d\tau} = N\frac{d}{dt}.$$

The effective Friedmann equation:

$$\frac{1}{a^2}\left(\frac{da}{dt}\right)^2 = \frac{C}{a^{3+n}}.$$

The choice of the cosmic time gauge fixing $N = 1$, or $n = 0$ gives us the effective Friedmann equation for the universe filled with some effective dust. The choice of the unimodular gauge $N = a^{-3}, n = -3$, gives us the universe filled with the cosmological constant.

Generally, the exponent $n$ is connected with the equation of state parameter as

$$n = 3w.$$

If $n = -1$ one has a string gas. If $n = 1$, one has a radiation, if $n = 3$, one has a stiff matter.

Note that for the case of the flat Friedmann universe

$$\gamma = a^6.$$

Thus, the exponent $r$ in

$$N = \gamma^r$$

is

$$r = \frac{n}{6}.$$

# Bianchi identites, Einstein equations and generalized unimodular gravity

Contracted Bianchi identity:

$$\nabla_i R^i_j - \frac{1}{2}\partial_j R = 0.$$

For $j = 0$:

$$\partial_0 \left( R^0_0 - \frac{1}{2} R \right) = \Gamma^\alpha_{0\beta} R^\beta_\alpha - \Gamma^\alpha_{0\alpha} R^0_0.$$

We have used the fact that the variation with respect to the shift functions gives the equality

$$R^0_\alpha = 0.$$

After having done this variation with respect to the shift functions, we can choose them equal to zero. Then

$$\Gamma^{\alpha}_{0\beta} = \frac{1}{2}\gamma^{\alpha\delta}\dot{\gamma}_{\delta\beta,0}.$$

Then

$$(\gamma^{p})_{,0} = p\gamma^{p}\gamma^{\alpha\beta}\gamma_{\alpha\beta,0} = 2p\gamma^{p}\Gamma^{\alpha}_{0\alpha}.$$

Hence,

$$\partial_0 \left[ \gamma^p \left( R_0^0 - \frac{1}{2} R \right) \right]$$
$$= \gamma^p \left[ \Gamma^\alpha_{0\beta} \left( R_\alpha^\beta - \frac{1}{2} \delta_\alpha^\beta R \right) + (2p-1) \Gamma^\alpha_{0\alpha} \left( R_0^0 - \frac{1}{2} R \right) \right].$$

If the lapse function $N$ is a fixed function of the determinant of the spatial metric $\gamma_{\alpha\beta}$, namely

$$N = \gamma^{p-\frac{1}{2}} = \gamma^{\frac{w}{2}},$$

then we obtain the following modified spatial-spatial Einstein equation:

$$R_\alpha^\beta - \frac{1}{2} \delta_\alpha^\beta R + w \delta_\alpha^\beta \left( R_0^0 - \frac{1}{2} R \right) = 0.$$

Using this equation and the Bianchi identities, we obtain

$$\partial_0 \left[ \gamma^{\frac{w+1}{2}} \left( R_0^0 - \frac{1}{2} R \right) \right] = 0.$$

Then

$$R_0^0 - \frac{1}{2} R = \frac{C(x^\alpha)}{\gamma^{\frac{w+1}{2}}},$$

where $C(x^\alpha)$ is an arbitrary function of the spatial coordinates.

This is 00 component of the effective Einstein equations.

Its right-hand is $T_0^0$ component of the energy-momentum tensor of an effective fluid.

$$T_{ij} = (\varepsilon + p) u_i u_j - g_{ij} p,$$

$$u^0 = \frac{1}{N}.$$

The energy density of the effective fluid

$$\varepsilon = \frac{C(x^\alpha)}{\gamma^{\frac{w+1}{2}}}.$$

Then

$$R_\alpha^\beta - \frac{1}{2}\delta_\alpha^\beta R = -w\delta_\alpha^\beta \varepsilon.$$

The right-hand side of this equation gives the pressure of the perfect fluid with the equation of state parameter $w$.
Should additional bounds on the function $C(x^\alpha)$ be imposed?

Starting from the energy-momentum tensor conservation law

$$\nabla_i T^i_j = 0,$$

we arrive to the following equation:

$$wu^i u_j(\nabla_i \varepsilon) + (1+w)\varepsilon u^i(\nabla_i u_j) - w(\nabla_j \varepsilon) = 0.$$

Let us consider the case of the dust:

$$N = 1,\ w = 0,\ u_0 = u^0 = 1.$$

Then the equation above is reduced to

$$u^i \nabla_i u_j = 0,$$

which is always true and does not give bounds on the function $C(x^\alpha)$.

When $w \neq 0$, we arrive to the equation

$$(1+w)\varepsilon u^0(\nabla_0 u_\alpha) - w(\nabla_\alpha \varepsilon) = 0.$$

It is satisfied if and only if

$$C(x^\alpha) = const.$$

Thus, in the case of the non-dust effective matter with $w \neq 0$, the effective energy density is

$$\varepsilon = \frac{C_0}{\gamma^{\frac{w+1}{2}}}$$

.

# Comments concerning Dirac non-linear electrodynamics

In the Lagrangian

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

we substitute the zero component of the potential $A_0$, which is a Lagrange multiplier, by some given function

$$A_0 = A_0(\vec{A}^2).$$

Thus, we have a Lagrangian, which depends only on three components of the potential $\vec{A}$.

The Euler-Lagrange equations for this Lagrangian:

$$L = -\frac{1}{4}F_{ij}F^{ij} - \frac{1}{2}(\partial_0 A_i - 2A'_0 A^j \partial_i A_j)^2,$$

$$\frac{\partial L}{\partial(\partial_k A_i)} = -F^{ki} + 2A'_0 A^i F^{0k},$$

$$\frac{\partial L}{\partial(\partial_0 A_i)} = -F^{0i},$$

$$\frac{\partial}{\partial x^k}\frac{\partial L}{\partial(\partial_k A_i)} = -\partial_k F^{ki} + 2A'_0 A^i \partial_k F^{0k} + 2A'_0 A^i_{,k},$$

$$\frac{\partial}{\partial x^0}\frac{\partial L}{\partial(\partial_0 A_i)} = -\partial_0 F^{0i},$$

$$\frac{\partial L}{\partial A^i} = 2A'_o \partial_k A_i F^{0k}.$$

Finally,

$$\partial_0 F^{0i} + \partial_k F^{ki} = 2A_0' A^i \partial_k F^{0k}.$$

In the standard Maxwell electrodynamics we have three dynamical equations (including second time derivatives), obtained by the variation of the Lagrangian with respect to the spatial components $A_i$ of the potential and one constraint, obtained by the variation with respect to the time component $A_0$.

In Dirac electrodynamics we have three dynamical equations and no constraints.

Instead we have effective sources.

# Cosmological Consequences of Unconstrained Gravity and Electromagnetism

L. Del Grosso, D.E. Kaplan, T.Melia, V. Poulin, S. Rajendran and T. L. Smith,
Cosmological Consequences of Unconstrained Gravity and Electromagnetism, arXiv: 2405.06374 [hep-ph].

What happens if the gravitational constraints and the Gauss law are relaxed simultaneously?

The relaxation of the Gauss law implies the appearance of the so called Shadow charge which works as a source of electromagnetic fields.
However, it does not feel the electromagnetic field.
Thus, it is different from the standard electric charge.
It is well known that the equation of motion for charged particles Lorentz force is independent with respect to Maxwell equations.
Thus, the existence of two kinds of charges do not contradict to the common wisdom.
The interplay between possible existence of dark matter and shadow charge opens interesting phenomenological opportunities.

# Concluding remarks

- Generally, one can say that in a theory with gauge symmetries or in a theory with first-class constraints, the relaxation of some of these constraints implies the appearance of new physical degrees of freedom.
- The relaxation of some of constraints gives birth to a new classical theory different from the "parent" theory.
- It has nothing common with the problem of time in quantum gravity.
- The study of the new physical degrees of freedom arising in this new theory is an interesting mathematical problem.
- Is it possible to find the observational difference between the predictions of alternative theories?