INSTITUT DENIS POISSON — ORLÉANS • TOURS, Mathématiques & Physique Théorique

Conformal Anomalies: theory and applications

Conformal anomaly, nonlocal effective action and the metamorphosis of the running scale

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Plan of talk

Trace anomaly effective action debate (Deser-Duff-Schwimmer-Mottola)

Anomaly action from conformal gauge fixing (Riegert-Fradkin-Tseytlin and Fradkin-Vilkovisky gauges)

Covariant curvature expansion and trace anomaly action

Applications: renormalized stress tensor on conformally related spacetime, Casimir energy on static Einstein Universe, cosmological initial conditions

RG running of the cosmological and gravitational couplings and metamorphosis of their running scale to their higher-dimensional "partners"

Conundrum of higher derivative quantum gravity (modification of beta-functions in quadratic gravity?)

A.B., A. G. Mirzabekian and V. V. Zhytnikov, gr-qc/9510037

A.Mirzabekian, G.A.Vilkovisky, V.Zhytnikov, Phys.Lett. B369 (1996) 215-220, arXiv:hep-th/9510205

A.B.& W. Wachowski, Phys.Rev.D 108 (2023) 4, 045014, arXiv:2306.03780

Conformal anomaly debate

Trace (Weyl invariance) anomaly in curved spacetime

$$\langle T^\mu_\mu \rangle \equiv \frac{2}{g^{1/2}} g_{\mu\nu} \frac{\delta \Gamma_A}{\delta g_{\mu\nu}} = \frac{1}{16\pi^2} \left(\alpha C^2_{\mu\nu\alpha\beta} + \beta E + \gamma \Box R\right),$$

$$E = R^2_{\mu\nu\alpha\gamma} - 4R^2_{\mu\nu} + R^2$$

Riegert, Fradkin, Tseytlin (1984), Antoniadis, Mottola (1992), Antoniadis, Mazur, Mottola (1994)

$$\Gamma_A[g] = \frac{1}{64\pi^2} \int d^4x g^{1/2} \left[\alpha\, C^2_{\mu\nu\alpha\beta} + \frac{\beta}{2}\left(E - \frac{2}{3}\Box R\right)\right] \frac{1}{\Delta_4}\left(E - \frac{2}{3}\Box R\right)$$
$$- \frac{1}{32\pi^2}\left(\frac{\gamma}{6} + \frac{\beta}{9}\right) \int d^4x\, g^{1/2} R^2(g)$$

$$\int d^4x\, \varphi \frac{1}{\Delta_4}\psi = \int d^4x\, d^4y\, \varphi(x)\, \mathcal{G}(x,y)\, \psi(y)$$
$$\mathcal{G}(x,y) = \frac{1}{\Delta_4}\delta(x,y)$$

Paneitz operator $\quad \Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}(\nabla^\mu R)\nabla_\mu$

Objections: S.Deser, A.Schwimmer (hep-th/9302047), Γ_A vs (Weyl) log (box) (Weyl)
S.Deser (hep-th/9911129), double pole inconsistency

S.Deser, M.J.Duff and C.J.Isham

Ambiguity in the conformal split

$$\Gamma[g] = \Gamma_A[g] + \Gamma^{\rm conf}[g], \qquad \Gamma^{\rm conf}[g] = \text{conformally invariant}$$

Conformal gauge fixing

Orbit of the conformal group

$$g_{\mu\nu} = e^{2\sigma}\bar{g}_{\mu\nu}, \qquad \frac{\delta\Gamma[\,e^{2\sigma}\bar{g}\,]}{\delta\sigma} = g^{1/2}\langle\, T^{\mu}_{\mu}\,\rangle\Big|_{g=e^{2\sigma}\bar{g}}$$

$$\Gamma[\,g\,] - \Gamma[\,\bar{g}\,] = \frac{1}{16\pi^2}\int d^4x g^{1/2}\left\{\left[\alpha\, C^2_{\mu\nu\alpha\beta} + \beta\left(E - \frac{2}{3}\Box R\right)\right]\sigma - 2\beta\,\sigma\Delta_4\sigma\right\} - \frac{1}{32\pi^2}\left(\frac{\gamma}{6} + \frac{\beta}{9}\right)\int d^4x\left(g^{1/2}R^2(g) - \bar{g}^{1/2}R^2(\bar{g})\right) \equiv \Delta\Gamma[\,\bar{g},\sigma\,]$$

Wess-Zumino action

Choice of gauge selecting the representative of the equivalence class

$$\chi[\,\bar{g}\,] \equiv \chi[\,g\,e^{-2\sigma}\,] = 0 \quad\Rightarrow\quad \sigma = \Sigma_\chi[\,g\,] \quad\Rightarrow\quad \bar{g}_{\mu\nu} \equiv g_{\mu\nu}\,e^{-2\Sigma_\chi[\,g\,]} = \textbf{conformal invariant}$$

$$\Downarrow$$

$$\Gamma[\,\bar{g}\,] + \frac{1}{32\pi^2}\left(\frac{\gamma}{6} + \frac{\beta}{9}\right)\int d^4x\,\bar{g}^{1/2}R^2(\bar{g}) = \text{conformal invariant}$$

Decomposition of the total action in conformal and anomalous parts

$$\Gamma[g] = \Gamma_\chi[g] + \Gamma[\bar g]\Big|_{\bar g_{\mu\nu}[g]}, \qquad \bar g_{\mu\nu}[g] \equiv g_{\mu\nu}\, e^{-2\Sigma_\chi[g]}$$

''conformization''

Parameterization of anomalous actions by conformal gauges χ:

$$\Gamma_\chi[g] = \frac{1}{16\pi^2}\int d^4x g^{1/2}\Big\{\Big[\alpha\, C^2_{\mu\nu\alpha\beta} + \beta\,\Big(E - \frac{2}{3}\Box R\Big)\Big]\,\Sigma_\chi - 2\beta\,\Sigma_\chi \Delta_4 \Sigma_\chi\Big\}$$
$$- \frac{1}{32\pi^2}\Big(\frac{\gamma}{6} + \frac{\beta}{9}\Big)\int d^4x\, g^{1/2} R^2$$

Exactly ``solvable" gauges:

analogue of the Faddeev-Popov operator

Riegert-Fradkin-Tseytlin gauge

$$\chi_{\mathrm{RFT}}[\,\bar{g}\,] = \bar{E} - \frac{2}{3}\bar{\Box}\bar{R} = 0$$

$$g^{1/2}\Big(E - \frac{2}{3}\Box R\Big) = \bar{g}^{1/2}\Big(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R}\Big) + 4\,\bar{g}^{1/2}\bar{\Delta}_4\sigma, \quad g_{\mu\nu} = e^{2\sigma}\bar{g}_{\mu\nu}$$

$$\bar{g}^{1/2}\bar{\Delta}_4 = g^{1/2}\Delta_4$$

$$\Sigma_{\mathrm{RFT}} = \frac{1}{4\Delta_4}\Big(E - \frac{2}{3}\Box R\Big) = -\frac{1}{6\Box}R + O[\,R^2\,]$$

Fradkin-Vilkovisky gauge

$$\chi_{\mathrm{FV}}[\,\bar{g}\,] = \bar{R} = 0$$

$$\bar{R} \sim \Big(\Box - \frac{1}{6}R\Big)\,e^{-2\sigma} = 0 \Rightarrow \Sigma_{\mathrm{FV}} = -\ln\Big(1 + \frac{1}{6}\frac{1}{\Box - R/6}R\Big) = -\frac{1}{6\,\Box}R + O[\,R^2\,]$$

$$\Sigma_{\mathrm{RFT}} = \Sigma_{\mathrm{FV}} + O[\,R^2\,]$$

Comparison of anomaly actions in different gauges

$$\Gamma_{\chi 1}-\Gamma_{\chi 2}=\frac{1}{16\pi^2}\int d^4x g^{1/2}\Big(\Sigma_{\chi 1}-\Sigma_{\chi 2}\Big)\Big[\alpha\, C^2_{\mu\nu\alpha\beta}+\beta\Big(E-\frac{2}{3}\Box R\Big)-2\beta\Delta_4\Big(\Sigma_{\chi 1}+\Sigma_{\chi 2}\Big)\Big]$$

$$\delta_\sigma\Sigma_{\chi_{1,2}}=\sigma, \qquad \delta_\sigma\equiv 2\int d^4x\,\sigma(x)g_{\mu\nu}(x)\frac{\delta}{\delta g_{\mu\nu}(x)}$$

$$\delta_\sigma\Big[g^{1/2}\Big(E-\frac{2}{3}\Box R\Big)\Big]=4g^{1/2}\Delta_4\sigma$$

$$\Downarrow$$

$$\delta_\sigma\Big(\Gamma_{\chi 1}-\Gamma_{\chi 2}\Big)=0$$

$$\Gamma_{\chi_{\rm RFT}}-\Gamma_{\chi_{\rm FV}}=\frac{1}{16\pi^2}\int d^4x g^{1/2}\Big(\Sigma_{\rm RFT}-\Sigma_{\rm FV}\Big)\Big[\alpha\, C^2_{\mu\nu\alpha\beta}+2\beta\Delta_4\Big(\Sigma_{\rm RFT}-\Sigma_{\rm FV}\Big)\Big]$$

$$\Sigma_{\rm RFT}-\Sigma_{\rm FV}=\frac{1}{2}\frac{1}{\Delta_4}\Big(E-\frac{2}{3}\Box R\Big)+\ln\Big(1+\frac{1}{6}\frac{1}{\Box-R/6}R\Big)=\text{conformal invariant}$$

Covariant curvature expansion $\equiv$ Feynman diagram expansion

$$\Gamma = \frac{1}{2}\,\mathrm{Tr}\,\ln F(\nabla), \qquad F(\nabla) = \Box + P - \frac{1}{6}R,$$

$$\Gamma = \overbrace{\Gamma_0 + \Gamma_1}^{\text{local power div}} + \Gamma_2 + \Gamma_3 + \mathrm{O}[\Re^4]$$

$$\Gamma_n \sim \Re^n, \qquad \Re = (R^\mu{}_{\nu\alpha\beta}, \mathcal{R}_{\mu\nu}, P), \qquad [\nabla_\mu, \nabla_\nu] = \mathcal{R}_{\mu\nu}$$

Quadratic order

S.Deser, M.J.Duff and C.J.Isham, Nucl. Phys. B111 (1976) 45

$$\Gamma_2^{\text{dim reg}} = -\frac{\Gamma(2-\omega)\Gamma(\omega+1)\Gamma(\omega-1)}{2(4\pi)^\omega\Gamma(2\omega+2)}\int dx\, g^{1/2}(x)\,\mathrm{tr}\Big\{R_{\mu\nu}(-\Box)^{\omega-2}R^{\mu\nu}\hat{1}$$
$$-\frac{1}{18}(4-\omega)(\omega+1)R(-\Box)^{\omega-2}R\hat{1} - \frac{2}{3}(2-\omega)(2\omega+1)\,\hat{P}(-\Box)^{\omega-2}R$$
$$+2(4\omega^2-1)\,\hat{P}(-\Box)^{\omega-2}\hat{P} + (2\omega+1)\hat{\mathcal{R}}_{\mu\nu}(-\Box)^{\omega-2}\hat{\mathcal{R}}^{\mu\nu}\Big\}, \qquad \omega = \frac{d}{2} \to 2$$

A.O.Barvinsky and G.A.Vilkovisky, Nucl.Phys. B 333 (1990) 471-511

Origin of logarithmic form factors

$$\Gamma(2-\omega)(-\Box)^{\omega-2} \;\Rightarrow\; \gamma(\Box) = \ln\left(-\frac{\Box}{\mu^2}\right)$$

Origin of RFT action in quadratic order of curvature expansion

Conformal scalar field: $P = 0,\ \hat{\mathcal{R}}_{\mu\nu} = 0,\ \gamma = -\frac{1}{180},\ \beta = \frac{1}{360},\ \alpha = -\frac{1}{120}$

$$\Gamma_2^{\rm renormalized} = \frac{1}{2(4\pi)^2}\int dx\, g^{1/2}(x)\left\{\frac{1}{60}\Big[R_{\mu\nu}\gamma(-\Box)R^{\mu\nu} - \frac{1}{3}R\gamma(-\Box)R\Big] + \frac{1}{1080}R^2\right\}$$

$$= \frac{1}{2(4\pi)^2}\int dx\, g^{1/2}(x)\left\{\frac{1}{120}C_{\mu\nu\alpha\beta}\gamma(-\Box)C^{\mu\nu\alpha\beta} + \frac{1}{1080}R^2\right\} + O[\Re^3]$$

Weyl invariant up to $O[\Re^3]$

⇕

$$\Gamma_{\rm RFT}\Big|_{\gamma=-\frac{1}{180},\,\beta=\frac{1}{360},\,\alpha=-\frac{1}{120}} = \frac{1}{2(4\pi)^2}\int dx\, g^{1/2}(x)\,\frac{1}{1080}R^2 + O[\Re^3]$$

Cubic order

A.Barrvinsky, Yu.Gusev, G.Vilkovisky and V.Zhytnikov, University of Manitoba report (Winnipeg, 1993) 192 p., arXiv:0911.1168

$$\Gamma_3 = \frac{1}{2(4\pi)^2}\int dx\, g^{1/2}(x) \sum_{M=1}^{29} \Gamma_M(-\Box_1,-\Box_2,-\Box_3)\, I_M(x_1,x_2,x_3)\Big|_{x_1=x_2=x_3=x},$$

Curvature invariants

$$I_M(x_1,x_2,x_3) \sim \nabla ...\nabla \Re(x_1)\Re(x_2)\Re(x_3)$$

Structure of formfactors

$$\Gamma_M = A_M\,\Gamma + \sum_{1\leq i<k}^{3} C_M^{ik}\frac{\ln(\Box_i/\Box_k)}{(\Box_i-\Box_k)} + B_M$$

Fundamental one-loop form factor

$$\Gamma = \int\limits_{\alpha\geq 0} d^3\alpha\, \frac{\delta(1-\alpha_1-\alpha_2-\alpha_3)}{\alpha_1\alpha_2(-\Box_3)+\alpha_1\alpha_3(-\Box_2)+\alpha_2\alpha_3(-\Box_1)}$$

$$(\,A_M,\ C_M^{ik}\,) = \frac{P_M(\Box_1,\Box_2,\Box_3)}{D^6\,\Box_1\,\Box_2\,\Box_3}$$

$$D = \Box_1{}^2+\Box_2{}^2+\Box_3{}^2-2\Box_1\Box_2-2\Box_1\Box_3-2\Box_2\Box_3$$

Essentially ``quantum"

Tree structure form factors

$$B_M = \frac{\tilde{P}_M(\Box_1,\Box_2,\Box_3)}{\Box_1\,\Box_2\,\Box_3}$$

Cubic part of the Fradkin-Vilkovisky anomaly action expanded in curvature

Conformal resummation: Fradkin-Vilkovisky anomaly action

A.Mirzabekian, G.A.Vilkovisky, V.Zhytnikov,
Phys.Lett. B369 (1996) 215-220,
arXiv:hep-th/9510205

Transition to Weyl – scalar curvature basis of invariants:

$$\nabla\ldots\nabla\Re\Re\Re \Rightarrow \underbrace{\nabla\ldots\nabla RRR, \nabla\ldots\nabla RRC, \nabla\ldots\nabla RCC}, \nabla\ldots\nabla CCC,$$

irreducible
Weyl tensor part

$$\varGamma_M \Rightarrow \varGamma_M = A_M \varGamma + \sum_{1\le i<k}^{3} C_M^{ik} \frac{\ln(\Box_i/\Box_k)}{(\Box_i-\Box_k)} + B_M$$

purely tree formfactors
-- expansion of anomaly action

$$C_{\mu\nu\alpha\beta}\gamma(-\Box)C^{\mu\nu\alpha\beta} \Rightarrow C_{\mu\nu\alpha\beta}\gamma(-\bar\Box)C^{\mu\nu\alpha\beta}$$

$$\bar\Box = \Box\Big|_{g_{\mu\nu}\to\exp(-\varSigma_{\rm FV})\,g_{\mu\nu}}$$

"conformization"

$$\varGamma[\,g\,] = \varGamma_{\rm FV}[\,g\,] + \varGamma[\,\bar g\,]\Big|_{\bar g_{\mu\nu}[\,g\,]}$$

$$\bar g_{\mu\nu}[\,g\,] \equiv g_{\mu\nu}\,e^{-2\varSigma_{\rm FV}[\,g\,]}$$

Applications

1) Renormalized stress tensors on conformally related spacetimes

$$\sqrt{g}\left\langle T^{\alpha}_{\beta}\right\rangle - \sqrt{\bar{g}}\left\langle \bar{T}^{\alpha}_{\beta}\right\rangle = 2\bar{g}_{\beta\gamma}\frac{\delta}{\delta\bar{g}_{\alpha\gamma}}\Delta\Gamma[\,\bar{g},\sigma\,]$$

Generalization of Brown-Cassidy formula to conformally non-flat spacetime:

$${}^{(1)}H^{\alpha}_{\beta} = -\frac{1}{2}\delta^{\alpha}_{\beta}R^2 + 2RR^{\alpha}_{\beta} + 2\delta^{\alpha}_{\beta}\Box R - 2\nabla^{\alpha}\nabla_{\beta}R,$$

$${}^{(3)}H^{\alpha\beta} = R^{\alpha\mu}R^{\beta}_{\mu} - \frac{2}{3}RR^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R^2_{\mu\nu} + \frac{1}{4}g^{\alpha\beta}R^2$$

$$\sqrt{g}\left\langle T^{\alpha}_{\beta}\right\rangle - \sqrt{\bar{g}}\left\langle \bar{T}^{\alpha}_{\beta}\right\rangle = \frac{\sqrt{g}}{8\pi^2}\left[\beta\,{}^{(3)}H^{\alpha}_{\beta} + \frac{\alpha}{18}\,{}^{(1)}H^{\alpha}_{\beta} + 2\beta R^{\mu\nu}C^{\alpha}{}_{\mu\beta\nu}\right]^{g}_{\bar{g}} - \frac{\sqrt{\bar{g}}}{4\pi^2}\,\alpha\left(\bar{R}^{\mu\nu} + 2\bar{\nabla}^{(\mu}\bar{\nabla}^{\nu)}\right)\left(\sigma\bar{C}^{\alpha}{}_{\mu\beta\nu}\right)$$

2) Transition to flat metric from a conformally flat one – covariant expression

$$C_{\alpha\beta\mu\nu} = 0 \;\Rightarrow\; \bar{R}^{\alpha}{}_{\beta\mu\nu} = 0, \quad \bar{g}_{\mu\nu} = e^{-2\Sigma_{\rm RFT}[g]} g_{\mu\nu}\Big|_{C_{\alpha\beta\mu\nu}=0}$$

$$\Sigma_{\rm RFT} = \frac{1}{4\Delta_4}\mathcal{E}_4$$

covariant expression

3) Casimir vacuum energy $E_{\rm vac}$ on static Einstein Universe

$$ds^2_{EU} = a_0^2\left(d\eta^2 + d\Omega^2_{(3)}\right) = e^{2\sigma}\left(d\rho^2 + \rho^2 d\Omega^2_{(3)}\right) \equiv e^{2\sigma} ds^2_{\rm flat}, \quad \sigma = -\eta = \ln\frac{a_0}{\rho},$$

$$\Gamma_{EU} - \Gamma_{\rm flat} = \frac{3}{4}\left(\beta - \frac{\gamma}{2}\right)\int d\eta \;\Rightarrow\; E_{\rm vac} = \frac{3}{4}\left(\beta - \frac{\gamma}{2}\right)$$

($\Gamma_{\rm flat} \to 0$)

in terms of the parameters of the trace anomaly

Applications in the theory of microcanonical density matrix of the Universe: eradication of the vacuum no-boundary state, quasi-thermal initial conditions, origin of the Universe is a sub-Planckian phenomenon, the need of conformal higher-spin fields

The problem of running Λ and G

Running coupling constant = nonlocal form factor in effective action

$$g \Rightarrow g(\mu), \qquad \mu\frac{d}{d\mu}g(\mu) = \beta\big(g(\mu)\big),$$

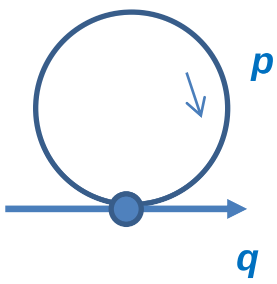
$$-\frac{1}{4g^2(\mu)}\int d^4x\, F_{\mu\nu}^2 \;\Rightarrow\; -\int d^4x\, F_{\mu\nu}(x)\frac{1}{4g^2\big(\sqrt{-\Box/\mu^2}\big)}F^{\mu\nu}(x)$$

Tadpole (total derivative) nature problem for running Λ and G

$$-\frac{\Lambda}{16\pi G}\int d^4x\, g^{1/2} \;\Rightarrow\; -\frac{1}{16\pi}\int d^4x\, g^{1/2}\frac{\Lambda(\Box)}{G(\Box)}1 = -\frac{1}{16\pi}\int d^4x\, g^{1/2}\frac{\Lambda(0)}{G(0)}$$

$$-\frac{1}{16\pi G}\int d^4x\, g^{1/2}R \;\Rightarrow\; -\frac{1}{16\pi}\int d^4x\, g^{1/2}\frac{1}{G(\Box)}R = -\frac{1}{16\pi}\int d^4x\, g^{1/2}\frac{1}{G(0)}R$$

not running



Renormalization group and metamorphosis of the running scale

The idea of RG:

$$S_{\{\Lambda\}}[\phi] = \sum_{\dim I, N} \Lambda_N \int d^d x\, I_N(\phi), \quad I_N(\phi) = \partial ... \partial \underbrace{\phi ... \phi}_{N}, \quad \dim \Lambda = d - \dim I$$

⇩ *quantization and renormalization*

$$\Gamma_{\{\Lambda(\mu)\}}[\phi] = S_{\{\Lambda(\mu)\}}[\phi] + \sum_I \int d^d x\, \mu^{d-\dim I} \Gamma_{\rm loop}\Big(\lambda(\mu)\,\Big|\, -\frac{\partial}{\mu}\Big)\, I(\phi)$$

$$= S_{\{\Lambda(\mu)\}}[\phi] + \sum_I \int dp_1 ... dp_N\, \delta\left(\sum p\right) \mu^{d-\dim I}\, \Gamma_{\rm loop}\Big(\lambda(\mu)\,\Big|\, -\frac{ip}{\mu}\Big)\, \tilde I(p_1, ... p_N)$$

in momentum reoresentation

Dimensionless nonlocal form factors

Dimensionless coupling $\lambda(\mu) = \mu^{-\dim \Lambda} \Lambda(\mu)$

RG equation $\mu \frac{d}{d\mu} \Gamma_{\{\Lambda(\mu)\}}[\phi] = 0 \;\Rightarrow\; \mu \frac{d}{d\mu} \lambda(\mu) = \beta\big(\{\lambda(\mu)\}\big)$

Choice of scale in momentum representation

$$\mu \to P \equiv \sqrt{p_1^2 + ... + p_N^2} \text{ or } \sqrt{(p_1+p_2)^2 + (p_3+p_4)^2 + ...}$$

Mandelstam variables

Replacement $\mu \to P$ *and UV limit* $\{p\} \to \infty, \quad \frac{\{p\}}{P} \to O(1) \quad \Rightarrow \quad \Gamma^{\text{loop}}_{\{\Lambda(\mu)\}} \ll 1$

Renormalized effective action in UV limit

$$\Gamma_{\text{renorm}}[\phi] \simeq S_{\{\Lambda(P)\}}[\phi], \quad P \to \infty$$

Renormalization group and metamorphosis of the running scale

$$S[g_{\mu\nu}]=\sum_{m,N}\Lambda_N^{(m)}\int d^dx\,\sqrt{g}\,\Re_N^{(d+m)} \qquad \Re_N^{(m)}=\underbrace{\nabla...\nabla}_{m-2N}\overbrace{\Re...\Re}^{N},\,, \quad \dim\Re_N^{(m)}=m$$

curvature invariants of dimension m

Covariant perturbation theory $\quad g_{\mu\nu}=\tilde{g}_{\mu\nu}+h_{\mu\nu},\quad \tilde{R}^{\mu}{}_{\nu\alpha\beta}=0$ G.A.Vilkovisky & A.B. (1986-1990)

$$I_M^{(m)}(h)\propto\underbrace{\tilde{\nabla}...\tilde{\nabla}}_{m}\overbrace{h(x)...h(x)}^{M}\equiv I_M^{(m)}(h_1,h_2,...h_M)\Big|_{x_1=x_2=...x_M=x}$$

RG scale μ $\qquad \Lambda=\mu^{d-\dim I}\lambda(\mu)$ dimensionless couplings $\{\lambda(\mu)\}$

Effective action

$$\Gamma[g_{\mu\nu}]=\sum_I\mu^{d-\dim I}\int d^dx\,\sqrt{\tilde{g}}\gamma_I\Big(\{\lambda(\mu)\},\frac{\tilde{\nabla}_1}{\mu},...\frac{\tilde{\nabla}_M}{\mu}\Big)I(h_1,...h_M)\Big|_{\{x\}=x}$$

nonlocal form factors

Employing RG:

$$\mu\frac{d}{d\mu}\Gamma[g_{\mu\nu}]=0\to\mu\frac{d}{d\mu}\lambda(\mu)=\beta\big(\{\lambda(\mu)\}\big)$$

Choice of scale

$\mu \to ?$

$$\tilde{D} \equiv \Big(-\sum_{N=1}^{\infty} \tilde{\Box}_N\Big)^{1/2}, \quad \tilde{\Box}_N \equiv \tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu = 0,$$

$$\tilde{D} I_N = \tilde{D}_N I_N, \quad \tilde{D}_N \equiv \Big(-\sum_{M=1}^{N} \tilde{\Box}_M\Big)^{1/2}, \quad \tilde{D}_0 = 0.$$

Replacement $\mu \to \tilde{D}$ *and UV limit* $\tilde{\nabla} \to \infty, \quad \dfrac{\tilde{\nabla}}{\tilde{D}_N} \to O(1)$

$$\mu^{4-\dim I}\,\gamma_I\Big(\lambda(\mu)\,\Big|\,\frac{\tilde{\nabla}_1}{\mu},\dots\frac{\tilde{\nabla}_N}{\mu}\Big)\Big|_{\mu\to\tilde{D}_N} \Longrightarrow (\tilde{D}_N)^{4-\dim I}\,\gamma_I\Big(\lambda(\tilde{D}_N)\,\Big|\,O(1)\Big) \equiv (\tilde{D}_N)^{4-\dim I}\lambda_I\big(\tilde{D}_N\big)$$

Covariantization:

$$h_{\mu\nu} = -\frac{2}{\Box}R_{\mu\nu} + O[\Re^2], \quad \tilde{\nabla}_\mu = \nabla_\mu + O[\Re],$$

$$I_N^{(m)}(h_1, h_2, \dots h_N) \to \frac{1}{\Box_1 \dots \Box_N}\underbrace{\nabla\dots\nabla}_{m}\Re_1\dots\Re_N + O[\Re^{N+1}]$$

$$\Downarrow$$

$$\varGamma[g_{\mu\nu}] \to \int d^4x\,\sqrt{g}\sum_{m,N\geq 1}^{\infty}\frac{(D_N)^{4-m}}{\Box_1\dots\Box_N}\lambda_N^{(m)}(D_N)\underbrace{\nabla\dots\nabla}_{m}\Re_1\dots\Re_N\Big|_{\{x\}=x}$$

RG form factor

dimensionful factors

Linear in curvature terms:

Does not contribute – total derivative: no running

$$D_1 = \sqrt{-\Box}$$

$$\int d^4x\,\sqrt{g}\sum_m^{\infty} \frac{(D_1)^{4-2m}}{\Box}\,\lambda_1^{(m)}(D_1)\underbrace{\nabla\ldots\nabla}_{2m}\Re(x) \sim \int d^4x\,\sqrt{g}\,\Box\,\sum_m^{\infty} c_m\,\lambda_1^{(m)}(\sqrt{-\Box})R(x)$$

Quadratic in curvature terms:

$$D_2 = \sqrt{-\Box_1 - \Box_2},$$

Integration by parts

$$\int d^4x\sqrt{g}\,F(\Box_1,\Box_2)\Re_1\,\Re_2 = \int d^4x\sqrt{g}\,\Re_1 F(\Box,\Box)\Re_2$$

$$\int d^4x\,\sqrt{g}\,\sum_m \lambda_2^{(m)}(D_2)\,\frac{(D_2)^{4-m}}{\Box_1\Box_2}\underbrace{\nabla\ldots\nabla}_{m}\Re_1\,\Re_2\Big|_{x_1=x_2}$$

all factors in $\frac{(D_2)^{4-m}}{\Box_1\Box_2}\underbrace{\nabla\ldots\nabla}_{m}$ ***completely cancel out due to integration by parts and Bianchi identities!***

$$= \int d^4x\sqrt{g}\Big(R_{\mu\nu}F_1(\Box)R^{\mu\nu} + RF_2(\Box)R\Big) + O[\Re^3]$$

dimensionless RG form factors

Running couplings of quadratic curvature terms:

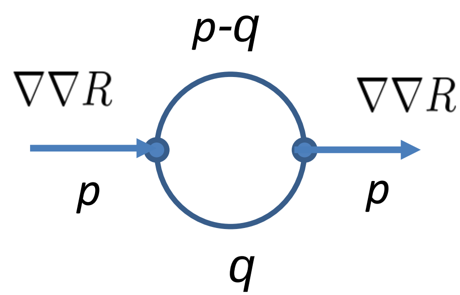
Conundrum of higher derivative quantum gravity

Quadratic gravity action and propagator

$$S \sim \int d^4x \sqrt{g}(-M^2 R + R^2) \sim \int d^4q \, h(q)(M^2q^2 + q^4)h(-q) + \ldots \Rightarrow \text{ propagator } \sim \frac{1}{M^2q^2 + q^4}$$

UV finite and IR divergent at M=0

$$\int \frac{d^4q}{(q^4 + M^2q^2)((p-q)^4 + M^2(p-q)^2)} \sim \frac{1}{p^4} \log \frac{p^2}{M^2}$$



⇩

$$\int d^4p \, \nabla\nabla R(p) \, \frac{1}{p^4} \log \frac{p^2}{M^2} \nabla\nabla R(-p) \sim \int d^4x \, R \, \log\left(\frac{-\Box}{M^2}\right) R$$

running constant of R^2- action

⇩

Modification of beta-functions by UV finite terms!

D.Buccio, J.Donoghue, G.Menezes, R.Percacci, arXiv:2403.02397

Conclusions

Nonlocal trace anomaly action from conformal gauge fixing --- RFT and FV anomaly actions

Recovery of anomaly action from covariant curvature expansion --- resolution of Deser-Duff-Schwimmer-Mottola debate?

Applications:

Renormalized stress tensor on conformally related spacetimes --- generalization of Brown-Cassidy formula;

Casimir energy on static Einstein Universe: CFT driven inflation --- microcanonical density matrix state of cosmology vs no-boundary wavefunction of the Universe (eradication of the vacuum no-boundary state, quasi-thermal initial conditions, origin of the Universe is a sub-Planckian phenomenon, the need of conformal higher-spin fields)

Metamorphosis of the running RG scale:
cosmological and Einstein terms to nonlocal form factors of their higher-order "partners" – long-distance modification of GR?

Enigma of higher-derivative gravity models *(modification of beta-functions in quadratic gravity?)*

Thank you!