Recursions and ODEs for the correlators in integrable systems

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Flat sections of Lie group

Take the product bundle over $\mathbb{CP}^1$ with the fiber G, $(\mathbb{CP}^1 \times G, \pi, \mathbb{CP}^1)$, (G is a reductive Lie group).
Consider the flat section s

$$\begin{aligned} s : \mathbb{CP}^1 &\to G \\ x &\mapsto \Psi(x) \end{aligned}$$

we associate the meromorphic connection :

$$\nabla \Psi = 0 \quad , \quad \nabla = d - \mathcal{D}(x) dx \tag{1}$$

where:

$$\mathcal{D}(x) \in \mathfrak{g} \otimes \mathbb{C}(x).$$

$\mathfrak{g}$ is the Lie algebra of the group G.

The flat section can be written in some faithful representation of G as :

$$\frac{d}{dx}\Psi(x) = \mathcal{D}(x)\Psi(x) \qquad \Psi(x) \in G. \tag{2}$$

Example (Airy differential system)

For $G = SL(2,\mathbb{C})$ ($\mathfrak{g} = \mathfrak{s}l(2,\mathbb{C})$), and

$$\mathcal{D}(x) = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}$$

The flat section Ψ is given in terms of two linearly independent solutions of the Airy differential equation:

$$f^{''} - xf = 0 \tag{3}$$

$$\Psi(x) = \begin{pmatrix} Ai(x) & Bi(x) \\ Ai'(x) & Bi'(x) \end{pmatrix} \in G = SL(2,\mathbb{C}).$$

Example (Gaussian Unitary Ensemble (GUE))

Let $N > 0$ be positive integer. For $G = GL(2, \mathbb{C})$ ($\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C})$), and

$$\mathcal{D}(x) = \begin{pmatrix} x & -1 \\ N & 0 \end{pmatrix}$$

The flat section is given by:

$$\Psi(x) = \begin{pmatrix} H_{N-1}(x) & \tilde{H}_{N-1}(x) \\ H_N(x) & \tilde{H}_N(x) \end{pmatrix}$$

where $H_N(x)$ is the N^{th} Hermite polynomial, and $\tilde{H}_N$ is the N^{th} Hermite function (which is not a polynomial of x). Both H_N and $\tilde{H}_N$ are solutions of the equation:

$$f''(x) - xf'(x) + Nf(x) = 0.$$

Adjoint Flat sections

Consider now a flat section M in the adjoint bundle with an adjoint connection:

$$\nabla_{\text{adj}} = d - [\mathcal{D}dx, .], \ \nabla_{\text{adj}} M = 0.$$

In other words:

$$\frac{d}{dx} M(x) = [\mathcal{D}(x), M(x)].$$

The connection ∇ acts in the adjoint bundle by the commutator. We have the following lemma:

Lemma

Let Ψ a flat section of the group bundle. Every adjoint flat section M must be of the form:

$$M(x) = \Psi(x) E \Psi(x)^{-1} = \mathrm{Ad}_{\Psi(x)} E$$

where $E \in \mathfrak{g}$ is constant.

Correlators

Take a faithful representation of the group G into $GL(r,\mathbb{C})$, we define the following functions:

Definition (Correlators)

Let $\Psi(x)$ a once for all chosen flat section of the group bundle. Let $M(x.E) = \Psi(x)E\Psi(x)^{-1}$ a flat section of the adjoint bundle, parametrized by $E \in \mathfrak{g}$. We define:

$$W_1(x.E) = \mathrm{Tr}\,\mathcal{D}(x)M(x.E)$$

and for $n \geq 2$

$$W_n(x_1.E_1,\dots,x_n.E_n) = \sum_{\sigma\in\mathfrak{S}_n^{1-\text{cycle}}} (-1)^\sigma \frac{\mathrm{Tr}\prod_{i=1}^n M(x_{\sigma^i(1)}.E_{\sigma^i(1)})}{\prod_{i=1}^n (x_i - x_{\sigma(i)})}$$

where $\mathfrak{S}_n^{1-\text{cycle}}$ is the subset of $\mathfrak{S}_n$ of permutations having just 1-cycle.

Correlators

with $X = x.E \in \mathbb{C}P^1 \times \mathfrak{g}$, is a pair of a point and a Lie algebra element, i.e. a Lie algebra weighted point.
These funcions were first introduced in [BE09] (for random matrix theory), and then used in [BDY16b; BDY16a](for integrable systems, KDV, intersection numbers). They are useful functions in many applications (random matrix theory, integrable systems...).

Example (Correlators W_2, W_3)

$$W_2(X_1, X_2) = \frac{1}{(x_1 - x_2)^2} \operatorname{Tr} M(X_1) M(X_2)$$

$$W_3(X_1, X_2, X_3) = \frac{\operatorname{Tr} M(X_1)[M(X_2), M(X_3)]}{(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)}$$

We can now using these definitions exhibit the differential equations satisfied by these W_ns

Let's try to find the ODE for W_1, we have:

$$
\begin{aligned}
W_1(x) &= \operatorname{Tr}\mathcal{D}(x)M(X)\\
W_1'(X) &= \operatorname{Tr}\left(\mathcal{D}'(x)M(X) + \mathcal{D}(X)\left[\mathcal{D}(x), M(X)\right]\right) = \operatorname{Tr}\left(\mathcal{D}'(x)M(X)\right)\\
W_1''(X) &= \operatorname{Tr}\left(\mathcal{D}''(x) + \left[\mathcal{D}'(x), \mathcal{D}(x)\right]\right)M(X)
\end{aligned}
$$

$$\frac{d^k}{dx^k}W_1(X) = \operatorname{Tr}\mathcal{D}_k(x)M(X)$$

where:

$$\mathcal{D}_{k+1} = \mathcal{D}_k' + [\mathcal{D}_k, \mathcal{D}], \qquad \mathcal{D}_0 = \mathcal{D}$$

For each k, $\mathcal{D}_k(x) \in \mathfrak{g} \otimes \mathbb{C}(x)$. Since $\mathfrak{g}$ is finite dimensional, at most *dim* $\mathfrak{g}$ of them can be linearly independent over the field $\mathbb{C}(x)$. Therefore, there exist some rational coefficients $\alpha_k(x)$ such that

$$\sum_{k=0}^{\dim \mathfrak{g}} \alpha_k(x)\mathcal{D}_k(x) = 0.$$

Then we can state the following theorem:

Theorem

$W_1(x)$ satisfies a linear differential equation of order $\dim\mathfrak{g}$, *with polynomial coefficients $\alpha_k(x)$:*

$$\sum_{k=0}^{\dim\mathfrak{g}} \alpha_k(x) \frac{d^k}{dx^k} W_1(x.E) = 0. \tag{4}$$

The coefficients $\alpha_k(x)$ are determined by

$$\sum_{k=0}^{\dim\mathfrak{g}} \alpha_k(x)\mathcal{D}_k(x) = 0 \qquad \textit{where} \quad \mathcal{D}_0 = \mathcal{D} \quad , \quad \mathcal{D}_{k+1} = \mathcal{D}'_k + [\mathcal{D}_k, \mathcal{D}]. \tag{5}$$

Example (Airy)

$$\mathcal{D}_0(x) = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}, \quad \mathcal{D}_1(x) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{D}_2(x) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{D}_3(x) = \begin{pmatrix} 0 & -2 \\ 2x & 0 \end{pmatrix}$$

The linear relation is:

$$\frac{1}{2}\mathcal{D}_3 = 2x\mathcal{D}_1 - \mathcal{D}_0,$$

The ODE is:

$$\frac{1}{2}W_1''' = 2xW_1' - W_1.$$

The 3 linearly independent solutions to this equation are:

$$W_1(x.E)$$

with E any linear combination of the 3 Chevaley–basis vectors.

Example (GUE)

We have $\mathfrak{g} = \mathfrak{gl}(2)$ with $\dim \mathfrak{gl}(2) = 4$:

$$\mathcal{D}_0(x) = \begin{pmatrix} x & -1 \\ N & 0 \end{pmatrix}, \quad \mathcal{D}_1(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{D}_2(x) = \begin{pmatrix} 0 & -1 \\ -N & 0 \end{pmatrix}$$

$$\mathcal{D}_3(x) = \begin{pmatrix} -2N & x \\ -Nx & 2N \end{pmatrix}, \quad \mathcal{D}_4(x) = \begin{pmatrix} 0 & 4N - x^2 + 1 \\ 4N^2 - Nx^2 - N & 0 \end{pmatrix}.$$

The linear relation is

$$\mathcal{D}_4 = (x^2 - 4N)\mathcal{D}_2 + x\mathcal{D}_1 - \mathcal{D}.$$

The ODE is:

$$W_1'''' = (x^2 - 4N)\, W_1'' + xW_1' - W_1.$$

ODE for W_2

We have:

$$\frac{d^k}{dx_1^k} W_2(X_1, X_2) = \operatorname{Tr} \mathcal{D}_k(x_1, X_2) M(X_1)$$

where

$$\mathcal{D}_0(x_1, X_2) = \frac{M(X_2)}{(x_1 - x_2)^2}$$

$$\mathcal{D}_{k+1}(x_1, X_2) = \frac{d}{dx_1} \mathcal{D}_k(x_1, X_2) + [\mathcal{D}_k(x_1, X_2), \mathcal{D}(x_1)]$$

We can state the following theorem:

Theorem

$W_2(X_1, X_2)$ satisfies an x_1–differential equation of order $\dim \mathfrak{g}$, *with coefficients $\alpha_k(x_1; X_2)$ polynomials of x_1:*

$$\sum_{k=0}^{\dim \mathfrak{g}} \alpha_k(x_1; X_2) \frac{d^k}{dx_1^k} W_2(X_1, X_2) = 0.$$

Example for Airy

$$
\begin{aligned}
\alpha_0(x_1,x_2) = & -6x_1^4 \operatorname{Ai}^4(x_2) + 4x_1^3 x_2 \operatorname{Ai}^4(x_2) + 16x_1^3 \operatorname{Ai}^2(x_2) \operatorname{Ai}'^2(x_2) \\
& + 2x_1^2 x_2^2 \operatorname{Ai}^4(x_2) - 16x_1^2 x_2 \operatorname{Ai}^2(x_2) \operatorname{Ai}'^2(x_2) - 2x_1^2 \operatorname{Ai}^3(x_2) \operatorname{Ai}'(x_2) \\
& - 10x_1^2 \operatorname{Ai}'^4(x_2) - 4x_1 x_2 \operatorname{Ai}^3(x_2) \operatorname{Ai}'(x_2) + 12x_1 x_2 \operatorname{Ai}'^4(x_2) \\
& + 8x_1 \operatorname{Ai}(x_2) \operatorname{Ai}'^3(x_2) + 6x_2^2 \operatorname{Ai}^3(x_2) \operatorname{Ai}'(x_2) - 2x_2^2 \operatorname{Ai}'^4(x_2) \\
& - 6x_2 \operatorname{Ai}^4(x_2) - 8x_2 \operatorname{Ai}(x_2) \operatorname{Ai}'^3(x_2) + 6 \operatorname{Ai}^2(x_2) \operatorname{Ai}'^2(x_2) \\
\alpha_1(x_1,x_2) = & -4x_1^5 \operatorname{Ai}^4(x_2) + 8x_1^4 x_2 \operatorname{Ai}^4(x_2) + 8x_1^4 \operatorname{Ai}^2(x_2) \operatorname{Ai}'^2(x_2) \\
& - 4x_1^3 x_2^2 \operatorname{Ai}^4(x_2) - 16x_1^3 x_2 \operatorname{Ai}^2(x_2) \operatorname{Ai}'^2(x_2) - 4x_1^3 \operatorname{Ai}^3(x_2) \operatorname{Ai}'(x_2) \\
& - 4x_1^3 \operatorname{Ai}'^4(x_2) + 8x_1^2 x_2^2 \operatorname{Ai}^2(x_2) \operatorname{Ai}'^2(x_2) + 8x_1^2 x_2 \operatorname{Ai}^3(x_2) \operatorname{Ai}'(x_2) \\
& + 8x_1^2 x_2 \operatorname{Ai}'^4(x_2) - 3x_1^2 \operatorname{Ai}^4(x_2) + 4x_1^2 \operatorname{Ai}(x_2) \operatorname{Ai}'^3(x_2) \\
& - 4x_1 x_2^2 \operatorname{Ai}^3(x_2) \operatorname{Ai}'(x_2) - 4x_1 x_2^2 \operatorname{Ai}'^4(x_2) + 6x_1 x_2 \operatorname{Ai}^4(x_2) \\
& - 8x_1 x_2 \operatorname{Ai}(x_2) \operatorname{Ai}'^3(x_2) + 3x_2^2 \operatorname{Ai}^4(x_2) + 4x_2^2 \operatorname{Ai}(x_2) \operatorname{Ai}'^3(x_2) \\
& - 12x_2 \operatorname{Ai}^2(x_2) \operatorname{Ai}'^2(x_2) + 6\operatorname{Ai}'^4(x_2)
\end{aligned}
$$

ODE for W_n $n \geq 2$

There is a tensor $Q_{0,n} \in (\mathfrak{g}^*)^{\otimes n}$, such that:

$$W_n(X_1, \dots, X_n) = Q_{0,n}(M(X_1), M(X_2), \dots, M(X_n))$$

whose coefficients are rational functions of $x_1, ..., x_n$. For $\mathfrak{g}l(r, \mathbb{C})$, the expression of $Q_{0,n}$ can be written as:

$$Q_{0,n} = \sum_{\sigma \in \mathfrak{S}_n^{1-cycle}} (-1)^\sigma \sum_{l_1, \dots, l_n} \frac{e^1_{l_1 l_{\sigma(1)}} \otimes \cdots \otimes e^n_{l_n l_{\sigma(n)}}}{\prod_i (x_i - x_{\sigma(i)})} \tag{7}$$

with $e_{i,j} \in \mathfrak{gl}(r, \mathbb{C})^*$ and satisfy $e_{i,j}(M) = M_{i,j}$ for any $M \in \mathfrak{gl}(r, \mathbb{C})$. Morevoer, we get,

$$\frac{d^k}{dx_1^k} W_n(X_1, \dots, X_n) = Q_{k,n}(M(X_1), M(X_2), \dots, M(X_n))$$

where

$$Q_{k+1,n} = \frac{d}{dx_1} Q_{k,n} + [Q_{k,n}, \mathcal{D}(x_1)]_1.$$

The space $(\mathfrak{g}^*)^{\otimes n}$ has dimension $(\dim \mathfrak{g})^n$, and so at most $(\dim \mathfrak{g})^n$ such tensors can be linearly independent. This implies

Theorem

There exist some coefficients $\tilde{\alpha}_{k,n}(x_1; x_2, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$, such that

$$\sum_{k=0}^{\dim \mathfrak{g}^n} \tilde{\alpha}_{k,n}(x_1, \ldots, x_n) \frac{d^k}{dx_1^k} W_n(X_1, X_2, \ldots, X_n) = 0.$$

The $(\dim \mathfrak{g})^n$ linearly independent solutions are obtained by choosing $(E_1, E_2, \ldots, E_n) \in \mathfrak{g}^{\otimes n}$.

ODE for W_2

Let's apply the previous formalism to $W_2(X_1, X_2)$, for $\mathfrak{g} = \mathfrak{gl}(r, \mathbb{C})$, $M(X_1)$ and $M(X_2)$ are both $r \times r$ matrices, we can write:

$$W_2(X_1, X_2) = \sum_{i,j,i',j'=1}^{r} M_{i',j'}(X_2) Q_{0;i',j',i,j}(x_1, x_2) M_{i,j}(X_1)$$

with $Q_0(x_1, x_2)$ the $r^2 \times r^2$ matrix

$$Q_{0;i',j',i,j}(x_1, x_2) = \frac{\delta_{i,j'} \delta_{i',j}}{(x_1 - x_2)^2}.$$

Then we have:

$$\frac{d^k}{dx_1^k} W_2(X_1, X_2) = Q_k(M(X_1), M(X_2)).$$

Q_k is a rational quadratic form, belongs to $\mathfrak{g}^* \otimes \mathfrak{g}^*$

We have the recursion:

$$Q_{k+1} = \frac{d}{dx_1} Q_k + [Q_k, \mathcal{D}(x_1)]_1$$

where the commutator acts only on the 1st factor of $\mathfrak{g}^* \otimes \mathfrak{g}^*$ The space $\mathfrak{g}^* \otimes \mathfrak{g}^*$ has dimension $(\dim \mathfrak{g})^2$ and thus there must exist some polynomial coefficients $\tilde{\alpha}_k(x_1, x_2) \in \mathbb{C}[x_1, x_2]$, such that

$$\sum_{k=0}^{(\dim \mathfrak{g})^2} \tilde{\alpha}_k(x_1, x_2) Q_k(x_1, x_2) = 0.$$

This implies,

Theorem

W_2 satisfies a polynomial ODE of order $(\dim \mathfrak{g})^2$. There exist some coefficients $\tilde{\alpha}_k(x_1, x_2) \in \mathbb{C}[x_1, x_2]$, such that

$$\sum_{k=0}^{(\dim \mathfrak{g})^2} \tilde{\alpha}_k(x_1, x_2) \frac{d^k}{dx_1^k} W_2(X_1, X_2) = 0.$$

Recursion relations

Let's consider a compatible sequence of integrable systems indexed by an integer N, i.e. we have a section $\Psi^{(N)}(x)$ of a Lie group G principal bundle over $\mathbb{C}P^1$ with rational connections:

$$\frac{d}{dx}\Psi^{(N)}(x) = \mathcal{D}^{(N)}\Psi^{(N)}(x),$$

That satisfies a recursion

$$\Psi^{(N+1)}(x) = \mathcal{R}^{(N)}\Psi^{(N)}(x),$$

where each $\mathcal{R}^{(N)}(x)$ is rational in x.

Proposition (Compatibility relation /discrete Lax equation)

The ODE and recursion are compatible:

$$\mathcal{D}^{(N+1)}(x)\mathcal{R}^{(N)}(x) - \mathcal{R}^{(N)}(x)\mathcal{D}^{(N)}(x) = \frac{d}{dx}\mathcal{R}^{(N)}(x)$$

We also define adjoint flat sections

$$M^{(N)}(x.E) = \Psi^{(N)}(x)E\Psi^{(N)}(x)^{-1},$$

The adjoint flat section satisfies also a recursion and ODE:

Proposition (Recursion and ODE for the adjoint section)

$$M^{(N+1)}(x) = \mathcal{R}^{(N)}(x)M^{(N)}(x)\mathcal{R}^{(N)}(x)^{-1}.$$

$$\frac{d}{dx}M^{(N)}(x) = \left[\mathcal{D}^{(N)}(x), M^{(N)}(x)\right].$$

Recursion for $W_1^{(N)}$

$W_1^{(N)}$ satisfies a recursion relation:

Theorem

$W_1^{(N)}(x.E)$ satisfies a linear recursion relation of order $\dim \mathfrak{g}$, *with polynomial coefficients $\alpha_k^{(N)}(x)$, independent of E:*

$$\sum_{k=0}^{\dim \mathfrak{g}} \alpha_k^{(N)}(x) W_1^{(N+k)}(x) = 0.$$

where

$$W_1^{(N+k)}(x.E) = \operatorname{Tr} D_k^{(N)}(x)\ M^{(N)}(x.E)$$

with

$$D_0^{(N)}(x) = \mathcal{D}^{(N)}(x)$$

$$D_k^{(N)}(x) = \prod_{i=0}^{k-1} \mathcal{R}^{(N+i)}(x)^{-1}\ \ \mathcal{D}^{(N+k)}(x)\ \ \prod_{i=1}^{k} \mathcal{R}^{(N+k-i)}(x)$$

Recursion for $W_n^{(N)}$, $n \geq 2$

Theorem (Recursion equation for $W_n^{(N)}$)

$W_n^{(N)}(X_1, ..., X_n)$ satisfies a linear recursion relation of order $(\dim\mathfrak{g})^n$, with coefficients $\alpha_k^{(N)}(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$:

$$\sum_{k=0}^{(\dim\mathfrak{g})^n} \alpha_{k,n}^{(N)}(x_1, \ldots, x_n) W_n^{(N+k)}(X_1, \ldots, X_n) = 0.$$

where $\alpha_{k,n}^{(N)}$ are determined by:

$$\text{Initial term}: \tilde{Q}_{0,n}^{(N)}(x_1, ..., x_n) = Q_{0,n}(x_1, ..., x_n)$$

$$\text{Recursion}: \tilde{Q}_{k+1,n}^{(N)}(x_1, ..., x_n) = \left((\mathrm{Ad}_{R^{N+k}}(x_1) \otimes ... \otimes \mathrm{Ad}_{R^{N+k}}(x_1)) \, \tilde{Q}_{k,n}^{(N)} \right)(x_1, ..., x_n)$$

$$\text{Linear relation}: \sum_{k=0}^{(\dim\mathfrak{g})^n} \alpha_{k,n}^{(N)}(x_1, \ldots, x_n) \tilde{Q}_{k,n}^{(N)}(x_1, \ldots, x_n) = 0.$$

Applications

One-Matrix model

The partition function Z is defined as:

$$Z = \frac{1}{N!} \int_{\gamma^N} \Delta(\Lambda)^2 e^{-\operatorname{Tr} V(\Lambda)} d\lambda_1 ... d\lambda_n$$

where

$$\Lambda = \operatorname{diag}(\lambda_1, ..., \lambda_N) \in \gamma^N$$

is the diagonal matrix Λ of eigenvalues,

$$d\mu(\Lambda) = \frac{1}{N! Z} \Delta(\Lambda)^2 e^{-\operatorname{Tr} V(\Lambda)} d\Lambda$$

is the eigenvalues induced measure, V is called the potential, $d\Lambda$ is the product measure on γ^N, and the Vandermonde determinant $\Delta(\Lambda)$ is defined as

$$\Delta(\Lambda) = \prod_{i<j} (\lambda_i - \lambda_j)$$

We choose integration domain such that the partition function is well defined i.e. the integral is absolutely convergent.

Example (Gaussian Unitary Ensemble (GUE))

The Gaussian Unitary Ensemble is the case $\gamma = \mathbb{R}$ and $V(x) = \frac{1}{2}x^2$. It is, thus, a probability law on $\mathbb{R}^N$:

$$d\mu(\Lambda) = \frac{1}{Z^{(N)}}\Delta(\Lambda)^2 e^{-\frac{1}{2}\operatorname{Tr}\Lambda^2} d\Lambda$$

$Z^{(N)}$ is known to be proportional to the Barnes function

$$Z^{(N)} = (2\pi)^{\frac{N}{2}} \prod_{k=0}^{N-1} k!$$

Wave function and orthogonal polynomials

We define the wave function as the expectation value of the characteristic polynomial:

$$p^{(N)}(x) = \int_{\gamma^N} d\mu(\Lambda)\det(x - \Lambda).$$

It is a polynomial of x of degree N. These polynomials form a family of orthogonal polynomials for the measure $e^{-V(x)}dx$ on γ, namely

$$\int_\gamma dx\, p^{(N)}(x)\, p^{(N')}(x)\, e^{-V(x)} = h^{(N)}\delta_{N,N'}.$$

These wave functions can be normalized such that:

$$\psi^{(N)}(x) = \frac{e^{-\frac{1}{2}V(x)}}{\sqrt{h^{(N)}}}\; p^{(N)}(x)$$

We also define the dual wave function:

$$\phi^{(N)}(x) = \frac{e^{\frac{1}{2}V(x)}}{\sqrt{h^{(N)}}} \int_\gamma dx' e^{-V(x')} \frac{p^{(N)}(x')}{x - x'}$$

the matrix wave function can be written as:

$$\Psi^{(N)}(x) = \begin{pmatrix} \psi^{(N-1)}(x) & \phi^{(N-1)}(x) \\ \psi^{(N)}(x) & \phi^{(N)}(x) \end{pmatrix}$$

Example (GUE)

$$p^{(N)}(x) = H_N(x) = \textit{Hermite polynomial}$$

$$\int_{\mathbb{R}} H_N(x) H_M(x) e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi} N! \delta_{N,M}$$

$$h^{(N)} = \sqrt{2\pi} N! \qquad , \qquad \gamma^{(N)} = \sqrt{\frac{h^{(N)}}{h^{(N-1)}}} = \sqrt{N}$$

Resolvents and Cumulants

Definition (Resolvents)

Let

$$\hat{W}_1^{(N)} = \int_{\gamma^N} d\mu(\Lambda)\, \mathrm{Tr}\, \frac{1}{x-\Lambda}$$

and in general

$$\hat{W}_n^{(N)}(x_1, \ldots, x_n) = \frac{\delta_{n,2}}{(x_1-x_2)^2} + \int_{\gamma^N} d\mu(\Lambda) \prod_{i=1}^{n} \mathrm{Tr}\, \frac{1}{x_i-\Lambda}$$

Definition (Cumulants/ Connected correlators)

We define the cumulants $W_n^{(N)}$ by inverting the formula:

$$\hat{W}_n^{(N)}(x_1, \ldots, x_n) = \sum_{\mu \vdash \{x_1,\ldots,x_n\}} \prod_{i=1}^{\ell(\mu)} W_{|\mu_i|}^{(N)}(\mu_i) \qquad (8)$$

i.e. the sums over all partitions of the n variables, of products of cumulants of each part.

The system of equations (8) that define the cumulants is a triangular system.

Example

For example $W_1^{(N)} = \hat{W}_1^{(N)}$ and the $n = 2$ cumulant is the covariance of resolvents:

$$W_2^{(N)}(x_1, x_2) = \hat{W}_2^{(N)}(x_1, x_2) - W_1^{(N)}(x_1) W_1^{(N)}(x_2).$$

For $n = 3$ the cumulant is given by:

$$\begin{aligned} W_3^{(N)}(x_1, x_2, x_3) = {} & \hat{W}_3^{(N)}(x_1, x_2, x_3) - W_1^{(N)}(x_1) \hat{W}_2^{(N)}(x_2, x_3) \\ & - W_1^{(N)}(x_2) \hat{W}_2^{(N)}(x_1, x_3) - W_1^{(N)}(x_3) \hat{W}_2^{(N)}(x_1, x_2) \\ & + 2 W_1^{(N)}(x_1) W_1^{(N)}(x_2) W_1^{(N)}(x_3). \end{aligned}$$

There is a relation between cumulants and correlators:

Proposition (Correlators and Cumulants)

Let

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad , \quad M^{(N)}(x) = \Psi^{(N)}(x) E \Psi^{(N)}(x)^{-1}.$$

We have

$$W_1^{(N)}(x) = \mathrm{Tr}\Psi^{(N)\prime}(x) E \Psi^{(N)}(x)^{-1}$$

$$W_2^{(N)}(x_1, x_2) = \frac{1}{(x_1 - x_2)^2} \mathrm{Tr} M^{(N)}(x_1) M^{(N)}(x_2)$$

and in general we have

$$W_n^{(N)}(x_1, \ldots, x_n) = W_n(x_1.E, \ldots, x_n.E)$$

The right hand side is the W_n for the correlators.

Recursion and ODE for the wave function

The orthogonal polynomials satisfy a 3 terms recursion relation:

$$xp^{(N)}(x) = p^{(N+1)}(x) + S^{(N)}p^{(N)}(x) + \frac{h^{(N)}}{h^{(N-1)}}p^{(N-1)}(x)$$

Which gives:

$$x\psi^{(N)}(x) = \gamma^{(N+1)}\psi^{(N+1)}(x) + S^{(N)}\psi^{(N)}(x) + \gamma^{(N)}\psi^{(N-1)}(x).$$

It implies that the matrix wave function satisfies a recursion relation:

$$\Psi^{(N+1)}(x) = \mathcal{R}^{(N)}(x)\Psi^{(N)}(x) \tag{9}$$

where

$$\mathcal{R}^{(N)}(x) = \begin{pmatrix} 0 & 1 \\ \frac{-\gamma^{(N)}}{\gamma^{(N+1)}} & \frac{x-S^{(N)}}{\gamma^{(N+1)}} \end{pmatrix},$$

is called the shift operator.

Example (GUE)

Hermite polynomials satisfy

$$p^{(N+1)}(x) = xp^{(N)}(x) - Np^{(N-1)}(x),$$

hence, matrix wave function satisfies a recursion relation, with

$$\mathcal{R}^{(N)}(x) = \begin{pmatrix} 0 & 1 \\ -\sqrt{\frac{N}{N+1}} & \frac{x}{\sqrt{N+1}} \end{pmatrix}$$

$$h^{(N)} = \sqrt{2\pi}N!, \qquad \gamma^{(N)} = \sqrt{\frac{h^{(N)}}{h^{(N-1)}}} = \sqrt{N}, \qquad S^{(N)} = 0$$

Now we have a recursion relation for the wave function matrix. In order to apply our general method for the correlators to one-matrix model, we need an ODE for $\Psi(x)$. The following theorem makes this possible.

ODE for the wave function

Proposition (ODE for Ψ)

If $V'(x)$ is a rational function, then the orthogonal polynomials satisfy an ODE with rational coefficients, which in matrix form can be written

$$\frac{d}{dx}\Psi^{(N)}(x) = \mathcal{D}^{(N)}(x)\Psi^{(N)}(x) \qquad (10)$$

where

$$\mathcal{D}^{(N)}(x) = \begin{pmatrix} \frac{V'(x)}{2} & 0 \\ 0 & -\frac{V'(x)}{2} \end{pmatrix} + \begin{pmatrix} w_{N-1,N-1}(x) & w_{N-1,N}(x) \\ w_{N,N-1}(x) & w_{N,N}(x) \end{pmatrix} \begin{pmatrix} 0 & -\gamma^{(N)} \\ \gamma^{(N)} & 0 \end{pmatrix}$$

where

$$w_{N,N'}(x) = \frac{1}{\sqrt{h^{(N)}h^{(N')}}} \int_{\gamma} e^{-V(x')} dx' \; p^{(N)}(x') \frac{V'(x) - V'(x')}{x - x'} p^{(N')}(x')$$

We have

$$\mathrm{Tr}\, \mathcal{D}^{(N)}(x) = 0$$

Example (GUE)

Hermite polynomials satisfy

$$\frac{d}{dx}\begin{pmatrix} p^{(N-1)}(x) \\ p^{(N)}(x) \end{pmatrix} = \begin{pmatrix} x & -1 \\ N & 0 \end{pmatrix}\begin{pmatrix} p^{(N-1)}(x) \\ p^{(N)}(x) \end{pmatrix},$$

which implies

$$\frac{d}{dx}\Psi^{(N)}(x) = \mathcal{D}^{(N)}(x)\Psi^{(N)}(x),$$

where

$$\mathcal{D}^{(N)}(x) = \begin{pmatrix} \frac{1}{2}x & -\gamma^{(N)} \\ \gamma^{(N)} & -\frac{1}{2}x \end{pmatrix}$$

Recursion and ODE for $W_1^{(N)}(x)$

Theorem (Recursion)

At most 3 of the matrices $\mathcal{D}^{(N,0)}(x), \ldots, \mathcal{D}^{(N,3)}(x)$ can be linearly independent, therefore there exist some coefficients $C^{(N,k)}(x)$, polynomials of x, such that

$$\sum_{k=0}^{3} C^{(N,k)}(x) W_1^{(N+k)}(x) = 0$$

Theorem (ODE)

At most 3 of the matrices $\hat{\mathcal{D}}^{(N,0)}(x), \ldots, \hat{\mathcal{D}}^{(N,3)}(x)$ can be linearly independent, therefore there exist some coefficients $\hat{C}^{(N,k)}(x)$, polynomials of x, such that

$$\sum_{k=0}^{3} \hat{C}^{(N,k)}(x) \frac{d^k}{dx^k} \; W_1^{(N)}(x) = 0$$

$$\hat{\mathcal{D}}^{(N,0)}(x) = \mathcal{D}^{(N)}(x) \, , \; \hat{\mathcal{D}}^{(N,k+1)}(x) = \frac{d}{dx} \hat{\mathcal{D}}^{(N,k)}(x) + [\hat{\mathcal{D}}^{(N,k)}(x), \mathcal{D}^{(N)}(x)]$$

Examples

Example (Recursion and ODE of $W_1^{(N)}$ in GUE)

The recursion is given by

$$(N+2)W_1^{(N+3)}(x) - (x^2 - N)W_1^{(N+2)}(x) + (x^2 - N - 3)W_1^{(N+1)}(x) - (N+1)W_1^{(N)}(x) = 0.$$

The ODE of order 3 is

$$\left(x + (4N - x^2)\frac{d}{dx} + \frac{d^3}{dx^3}\right)W_1^{(N)}(x) = 0$$

Minimal models

Minimal models arise as limits of matrix models. They can also be defined independently of matrix models, from integrable hierarchies, KdV, KP, Toda, see [BBT03]. They are classified by two integers (p,q), in short, the (p,q) minimal model can be formulated in terms of a differential system of order q, with polynomial coefficients whose degree is bounded by p.

Definition (Gelfand-Dikii polynomials)

We define by recursion the following differential polynomials $R_n(U) \in \mathbb{Q}[U, \dot{U}, \ddot{U}, \dddot{U}, \dots]$

$$R_0(U) = 2$$

$$\dot{R}_{n+1} = -2U\dot{R}_n - \dot{U}R_n + \frac{1}{4}\dddot{R}_n$$

and we choose the primitive that is homogeneous (in U and $\ddot{}$).
The first few are

$$R_0 = 2 \quad , \quad R_1 = -2U \quad , \quad R_2 = 3U^2 - \frac{1}{2}\ddot{U} \quad , \quad R_3 = -5U^3 + \frac{5}{2}U\ddot{U} + \frac{5}{4}\dot{U}^2 - \frac{1}{8}\ddddot{U}$$

Minimal models

Definition (Lax pair)

Let

$$\mathcal{R}(x,t) \coloneqq \begin{pmatrix} 0 & 1 \\ x + 2U(t) & 0 \end{pmatrix}$$

Define,

$$B_n(x,U) \coloneqq \frac{1}{2}\sum_{j=0}^{n} x^{n-j} R_j(U)$$

Define

$$\mathcal{D}_n(x,t) \coloneqq \begin{pmatrix} -\frac{1}{2}\dot{B}_n & B_n \\ (x+2U)B_n - \frac{1}{2}\ddot{B}_n & \frac{1}{2}\dot{B}_n \end{pmatrix}$$

Notice that $\mathcal{R}(x,t)$ and $\mathcal{D}_n(x,t)$ are traceless, so that they belong to the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$.

ODE with respect to time for W_1

Proposition (Gelfand-Dikii)

We have

$$\dot{\mathcal{D}}_n(x,t) + [\mathcal{D}_n(x,t), \mathcal{R}(x,t)] = -\dot{R}_{n+1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

the following proposition is a consequence of this equation, which enables us to determine the two differential equations (with respect to time and x) of the wave function Ψ,

Proposition (Compatibility and wave function)

Let $m \geq 0$ an integer. Let $\vec{t} = (t_0, t_1, t_2, \dots, t_m)$ a set of parameters (called higher times). Let

$$\mathcal{D}(x, t, \vec{t}) = \sum_{k=0}^{m} t_k \mathcal{D}_k(x, t).$$

$\mathcal{D}(x, t; \vec{t})$ satisfies the Lax equation

$$\frac{d}{dt}\mathcal{D}(x, t; \vec{t}) + [\mathcal{D}(x, t; \vec{t}), \mathcal{R}(x, t)] = \frac{d}{dx}\mathcal{R}(x, t)$$

if and only if U is solution to the equation

$$\sum_{k=0}^{m} t_k R_{k+1}(U) = -t. \tag{11}$$

In this case, there exists a matrix $\Psi(x, t; \vec{t})$ such that

$$\frac{d}{dx}\Psi = \mathcal{D}\Psi \quad , \quad \frac{d}{dt}\Psi = \mathcal{R}\Psi$$

Example ($m = 0$: Airy)

take $\vec{t} = (1)$, eq (11) reduces to:

$$-2U = -t$$

We have

$$\mathcal{R}(x,t) = \begin{pmatrix} 0 & 1 \\ x+t & 0 \end{pmatrix} = \mathcal{D}(x,t)$$

The matrix $\Psi(x,t) = \Psi(x+t)$ satisfies the Airy equation, and is worth:

$$\Psi(x,t) = \Psi(x+t) = \begin{pmatrix} Ai(x+t) & Bi(x+t) \\ Ai'(x+t) & Bi'(x+t) \end{pmatrix}$$

Proposition (ODE for W_1)

W_1 satisfies the following ODE of order 3:

$$W_1 = \sum_{k=0}^{m} t_k \left((2(x+2U)B_k - \frac{1}{2}\ddot{B}_k)\dot{W}_1 + \frac{1}{2}\dot{B}_k \ddot{W}_1 - \frac{1}{2} B_k \dddot{W}_1 \right)$$

Examples

Example (Airy $m = 0$, ODE W_1)

$$W_1 - 2(x+t)\dot{W}_1 + \frac{1}{2}\dddot{W}_1 = 0$$

Example (Painlevé 1, $m = 1$, ODE W_1)

$$W_1 - (2x^2 + 2Ux - U^2 + t)\dot{W}_1 + \frac{1}{2}\dot{U}\ddot{W}_1 + \frac{1}{2}(x-U)\dddot{W}_1 = 0$$

Conclusion

As conclusion we have shown how to exploit the ODE satisfied by the wave function (flat section of a Lie group bundle) or equivalently the adjoint ODE satisfied by M (flat section of the adjoint Lie algebra bundle) to derive, in a systematic way, linear ODE and recursions with polynomial coefficients for every correlator W_n. This approach can be further generalized to non trivial bundles, it is a general method that can be applied to every Lax system, in particular systems that depend on other parameters (time dependent wave function), these ODEs can be shown useful in resurgence theory for example.

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Merci pour votre attention

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