Lecture 4. Further results.

An obstruction to a full solution of KLS. As we have seen above, the KLS conjecture would be implied by the following statement: in the isotropic log-concave case the expected operator norm of $\mathrm{cov}(X \mid X + \sqrt{s}G)$ remains of order 1 for all s. Unfortunately such an estimate cannot be true as we shall see now.

Let $X = (X_1, \dots, X_n)$ be a random vector whose coordinates are iid and such that $1 + X_i$ is an exponential variable of parameter 1. This is clearly an isotropic log-concave vector on $\mathbb{R}^n$.

Proposition 1. *We have $\mathbb{E}\|\mathrm{cov}(X \mid X + \sqrt{s}G)\| = O(1)$ for all $s \geq C \log n$. On the other hand, if $s \leq \log n$ then $\mathbb{E}\|cov(X \mid X + \sqrt{s}G)\| \geq cs$.*

Note that from the tensorization property of the Poincaré inequality, we have $C_P(X) = C_P(X_1)$, in particular the Poincaré constant of X does not depend on n. Recall also that we always have the bound $\mathbb{E}\|\mathrm{cov}(X \mid X + \sqrt{s}G)\| \leq s$ (acually this is true almost surely, not only in expectation). This examples shows that this bound can be essentially sharp on a time range $[0, s_0]$ with $s_0 \to \infty$, namely $s_0 = \log n$. In particular at time s_0 we have

$$\mathbb{E}\|\mathrm{cov}(X \mid X + \sqrt{s_0}G)\| \geq c \log n$$

so the expected norm of the conditional covariance is not bounded for all times. In view of this example, the best one could hope for is

$$\mathbb{E}\|cov(X \mid X + \sqrt{s}G)\| = O(1), \quad \forall s \geq C \log n \tag{1}$$

and for every log-concave isotropic X. Notice that there is still a gap between this and the bound that we obtained in our theorem (in which $\log n$ is replaced by $\log^2 n$). If true the estimate (1) would imply the bound

$$C_n = O((\log n)^{1/4}),$$

for the KLS constant. This seems to be the limit one could reach within this framework. Going below this mark would have to rely on different arguments.

The proof of the proposition only relies on some analysis in one dimension. Indeed since the coordinates of X and G are all independent it is clear that the conditional law of X given $X + \sqrt{s}G$ is just the n-fold product of the law of X_1 condition on $X_1 + \sqrt{s}G_1$. So the conditional covariance is diagonal with iid entries, The norm is just the maximum of this entries. So all we need to estimate is the expected maximum of n iid variables.

As a preliminary step, we need to compute the variance of a truncated Gaussian.

Lemma 2. *Let g be as standard Gaussian variable, then*

$$\mathrm{var}(g \mid g \geq x) \approx \frac{1}{1 + x_+^2}.$$

Proof. The quantity $\mathrm{var}(g \mid g \geq x)$ is clearly a positive continuous function of x, and it is also clear that it tends to 1 as x tends to $-\infty$. Now we look at its behaviour near $+\infty$. We use a lemma about log-concave measures in 1D, which was already mentioned by Bo'az. Namely, for a log-concave random variable on the line we have

$$\|f\|_\infty^2 \cdot \mathrm{var}(X) \approx 1,$$

where f is the density of X. Applying this to the conditional law of g given $g \geq x$ (which is indeed log-concave), we get

$$\operatorname{var}(g \mid g \geq x) \approx \left(\mathrm{e}^{x^2/2} \int_x^\infty \mathrm{e}^{-y^2/2} \, dy \right)^2 ,$$

for every positive x. Now it is well-known that the right-side is equivalent to x^{-2} as x tends to $+\infty$. □

Proof of Proposition 1. The conditional law of X_1 given $X_1 + \sqrt{s} G_1$ is just a truncated Gaussian. After some elementary computation we get

$$\operatorname{var}(X_1 \mid X_1 + \sqrt{s} G_1) = s \cdot v(\sqrt{s} - \frac{1}{\sqrt{s}} Y_1 - G_1) \tag{2}$$

where $Y_1 = X_1 + 1$ and v is the function given by

$$v(x) = \operatorname{var}(G_1 \mid G_1 \geq x).$$

Note that Y_1 is an exponential variable independent of G_1, hence

$$\mathbb{P}(Y_1 \geq s, G_1 \geq 0) = \frac{1}{2} \mathrm{e}^{-s}.$$

Since the function v is bounded away from 0 on $\mathbb{R}_-$, if $Y_1 \geq s$ and $G_1 \geq 0$ then

$$\operatorname{var}(X_1 \mid X_1 + \sqrt{s} G_1) \geq cs.$$

As a result

$$\mathbb{P}(\operatorname{var}(X_1 \mid X_1 + \sqrt{s} G_1) \geq cs) \geq \frac{1}{2} \mathrm{e}^{-s}.$$

By independence we get

$$\mathbb{P}(\| \operatorname{var}(X \mid X + \sqrt{s} G \|) \geq cs) \geq 1 - (1 - \frac{1}{2} \mathrm{e}^{-s})^n$$

If $s \leq \log n$, the right-hand side is at least $1 - \mathrm{e}^{-1/2}$. This implies $\mathbb{E} \| \operatorname{var}(X \mid X + \sqrt{s} G \|) \| \geq c' s$ by Markov inequality.

For the other inequality, since $v(x) \lesssim x^{-2}$ for large x, equation (2) and the union bound imply in particular that if C is a sufficiently large constant

$$\mathbb{P}(\| \operatorname{cov}(X_1 \mid X_1 + \sqrt{s} G_1) \| \geq C) \leq \mathbb{P}(Y_1 \geq \frac{s}{4}) + \mathbb{P}(G_1 \geq \frac{\sqrt{s}}{4}) \leq 2 \mathrm{e}^{-cs}.$$

Hence, by the union bound again,

$$\mathbb{P}(\| \operatorname{cov}(X \mid X + \sqrt{s} G) \| \geq C) \leq 2n \mathrm{e}^{-cs}.$$

Since $\| \operatorname{cov}(X \mid X + \sqrt{s} G) \| \leq s$ almost surely this implies

$$\mathbb{E} \| \operatorname{cov}(X \mid X + \sqrt{s} G) \| \leq 2ns \cdot \mathrm{e}^{-cs} + C.$$

This becomes $O(1)$ as soon as s exceeds a sufficiently large multiple of $\log n$. □

Concentration of measure.

Definition 3. Let (X, d, μ) be a metric measured space. The concentration function of μ is defined by

$$\alpha_\mu \colon r \mapsto \sup \{1 - \mu(S_r) \colon \mu(S) = 1/2\}$$

where S_r is the r-neighborhood of the set S.

In some very specific models such as the uniform measure on the sphere or the Gaussian measure the exact value of the concentration function is known. In general it is hopeless to compute it exactly and we are happy with an upper bound for α_μ. The most interesting types of upper bound for us are the case of Gaussian concentration and of exponential concentration.

Definition 4. We say that μ satisfies Gaussian concentration if there exist constants C_0, C_1 such that

$$\alpha_\mu(r) \leq C_0 \cdot \exp\left(-\frac{r^2}{C_G}\right), \quad \forall r \geq 0.$$

We say that μ satisfies exponential concentration if there exist constants C_0, C_1 such that

$$\alpha_\mu(r) \leq C_0 \cdot \exp\left(-\frac{r}{C_{\exp}}\right), \quad \forall r \geq 0.$$

Remark. The prefactor C_0 is usually of order 1. Once it has has been fixed, the best constant inside the exponent is called the Gaussian or exponential concentration constant.

Once again, the situation is well understood for uniformly log-concave measures. They satisfy a Gaussian concentration property with constants not depending on the dimension.

Proposition 5. *Let μ be a t-uniformly log-concave measure. Then for every measurable set S and every $r \geq 0$ we have*

$$\mu(S)(1 - \mu(S_r)) \leq \exp(-c \cdot tr^2), \quad \forall r \geq 0,$$

where c is a universal constant. In particular the Gaussian concentration constant of μ is $O(t^{-1/2})$.

Proof. Recall the classical Prékopa-Leindler inequality, which asserts that if f, g, h satisfy the inequality

$$\sqrt{f(x)g(y)} \leq h(\frac{x+y}{2})$$

for every $x, y \in \mathbb{R}^n$ then

$$\sqrt{\int_{\mathbb{R}^n} f(x)\, dx \int_{\mathbb{R}^n} g(x)\, dy} \leq \int_{\mathbb{R}^n} h(x)\, dx.$$

If μ is t-uniformly log-concave then its potential V satisfies

$$V(\frac{x+y}{2}) \leq \frac{V(x) + V(y)}{2} - \frac{t}{8}|x - y|^2.$$

Given a set S and $\theta > 0$, one can then see that the hypothesis of Prékopa-Leinder applies to the functions $f(x) = \mathbf{1}_S(x)\mathrm{e}^{-V(x)}$, $g(y) = \mathrm{e}^{\theta d(y,S) - V(y)}$ and $h = \mathrm{e}^{-V + c\theta^2/t}$ for some suitable constant c (details left as an exercise). From the conclusion of Prékopa, we get

$$\mu(S) \cdot \int_{\mathbb{R}^n} \mathrm{e}^{\theta d(x,S)}\, d\mu \leq \mathrm{e}^{c\theta^2/t}.$$

The conclusion then follows from Chernov inequality. $\square$

One can be a bit more precise. As we already mentioned in the Gaussian case we know the exact value of the concentration function α_{γ_n}. Indeed, an integrated version of the isoperimetric inequality of Sudakov-Tsirelson / Borell asserts that $\gamma_n(S_r)$ is maximized when S is a halfspace. In particular

$$\alpha_{\gamma_n}(r) = 1 - \Phi(r), \quad \forall r \geq 0.$$

where Φ is the distribution function of the standard Gaussian variable. Moreover, a deep result of Bakry and Ledoux [2] asserts that if μ is t-uniformly log-concave then the concentration function of μ is bounded from above by that of the Gaussian variable of variance $1/t$. This implies in particular that

$$\alpha_\mu(r) \leq 1 - \Phi(\sqrt{t} \cdot r), \quad \forall r \geq 0.$$

Since $1 - \Phi(r) \leq \frac{1}{2}\mathrm{e}^{-r^2/2}$ for $r \geq 0$, this implies Gaussian concentration. However this only improves upon Proposition 5 at the level of the value of the universal constant c, which is irrelevant for our purposes.

Again we are interested here in concentration properties of log-concave measures on $\mathbb{R}^n$ and on the dependence on the dimension of the concentration function. Gaussian concentration cannot be true in general (think of μ being the exponential measure) but there is no obstruction to having exponential concentration with a dimension free constant for isotropic log-concave measures, and this is in fact equivalent to the KLS conjecture. Indeed, it is well-known that the Poincaré inequality yields exponential concentration, and more precisely that for any probability measure μ on $\mathbb{R}^n$ satisfying the Poincaré inequality we have

$$\alpha_\mu(r) \leq C \cdot \exp\left(-\frac{r}{L \cdot \sqrt{C_P(\mu)}} \right), \quad \forall r \geq 0,$$

where C and L are universal constants. See for instance [1, section 4.4.2] for the details. Once again, in the log-concave case this implication can be reversed. Indeed, by E. Milman's theorem we know that

$$C_P(\mu) \approx \sup\{\mathrm{var}_\mu(f) \colon \|f\|_{\mathrm{Lip}} = 1\}.$$

Moreover by definition of α_μ if f is a 1-Lipschitz function then

$$\mu(f - m_f \geq r) \leq \alpha_\mu(r).$$

where m_f is a median for f. By Fubini this tail bound implies

$$\mathrm{var}_\mu(f) \lesssim \int_0^\infty r \alpha_\mu(r)\, dr,$$

hence the inequality

$$C_P(\mu) \lesssim \cdot \int_0^\infty r \cdot \alpha_\mu(r)\, dr, \tag{3}$$

from which it follows that $C_P(\mu)$ is at most the exponential concentration constant squared, up to a universal factor. Therefore, the KLS conjecture amounts to the following bound

$$\alpha_\mu(r) \leq C\mathrm{e}^{-cr}$$

for the concentration function of an isotropic log-concave measure, whereas the the current best bound is equivalent to

$$\alpha_\mu(r) \leq C \cdot \exp\left(-c \cdot \frac{r}{\sqrt{\log n}} \right). \tag{4}$$

One of the points of this lecture is to show that one can go a bit beyond this estimate.

Theorem 6. *If μ is log-concave and isotropic then its concentration function satisfies*

$$\alpha_\mu(r) \leq C \exp\left(-c \cdot \min\left(r, \frac{r^2}{\log^2 n}\right)\right), \quad \forall r \geq 0. \tag{5}$$

Proof of Theorem 6. Fix a set S of measure $1/2$ and write

$$1 - \mu(S_r) = \mathbb{E}(1 - \mu_t(S_r)) \leq \mathbb{E}(1 - \mu_t(S_r))\mathbb{1}_{\{\mu_t(S) \geq 1/4\}} + \mathbb{P}(\mu_t(S) \leq 1/4),$$

where (μ_t) is the stochastic localization of μ. Since μ_t is t-uniformly log-concave, the first term is at most $4\mathrm{e}^{-ctr^2}$, by Proposition 5. To handle the second term recall that the martingale $M_t := \mu_t(S)$ satisfies

$$dM_t = \left(\int_S (x - a_t)\, d\mu_t\right) \cdot dW_t.$$

Applying Cauchy-Schwarz we obtain

$$\left|\int_S (x - a_t)\, d\mu_t\right|^2 \leq \mu_t(S)\|A_t\| \leq \|A_t\|.$$

Hence the inequality

$$\langle M\rangle_t \leq \int_0^t \|A_s\|\, ds.$$

In particular if $\|A_s\| \leq 2$ on $[0, t]$ then $\langle M\rangle_t \leq 2t$. Therefore

$$\mathbb{P}(M_t \leq \frac{1}{4}) \leq \mathbb{P}(M_t \leq \frac{1}{4} \,\&\, \langle M\rangle_t \leq 2t) + \mathbb{P}(\exists s \leq t\colon \|A_s\| \geq 2).$$

We have seen in the previous lecture that the second term is at most $\exp(-(Ct)^{-1})$, provided $t \leq (C\log^2 n)^{-1}$. On the other hand since $M_0 = \mu(S) = 1/2$, Freedman's inequality (also from the previous lecture) insures that

$$\mathbb{P}(M_t \leq \frac{1}{4} \,\&\, \langle M\rangle_t \leq 2t) \leq \exp\left(-\frac{1}{C_1 t}\right).$$

Putting everything together we get

$$\mu(\overline{S_r}) \leq 4\exp(-c \cdot tr^2) + 2\exp(-(C_2 t)^{-1})$$

for every $t \leq (C \cdot \log n)^{-2}$. Choosing $t = \min(r^{-1}, (C \cdot \log n)^{-2})$ yields the result. □

We should make some comments on this result. First of all, the rate provided by Theorem 6 is not smaller than that of (4) on the whole halfline. In particular combining the Theorem with (3) only leads to a $\log^2 n$ bound for the Poincaré constant of an isotropic log-concave measure (rather than $\log n$). The reason for this is that we did not use the improved Lichnerowicz estimate in the proof of this Theorem. That being said, the theorem yields in particular the rate e^{-cr}, which is predicted by the KLS conjecture, as soon as r is larger than $\log^2 n$ or so. As far as I know, this information cannot be inferred from bound $C_n = O(\log n)$ alone. Let me also mention that one can prove the following variant of (5), in which the concentration depends on the KLS constant C_n, namely

$$\alpha_\mu(r) \leq C \exp\left(c \cdot \min\left(r, \frac{r^2}{C_n \cdot \log n}\right)\right), \quad \forall r \geq 0. \tag{6}$$

This inequality is taken from Bizeul [3], as is most of the material of this lecture.

This concentration is reminiscent of Guédon-Milman [6] from 2011, where they proved that every isotropic log-concave measure μ satisfies

$$\mu\left(||x| - \sqrt{n}| \geq r\right) \leq C \cdot \exp\left(c \cdot \min\left(r, \frac{r^2}{n^{2/3}}\right)\right), \quad \forall r \geq 0.$$

This is weaker than (5) in two ways, first of all the constant is much worse ($n^{2/3}$ vs $\log^2 n$) and the deviation inequality is only for the Euclidean norm, and not for every 1-Lipschitz function, as in (5). This application of stochastic localization to concentration dates back to Lee and Vempala [7]. Their main result in that paper is the bound $C_n = O(n^{1/2})$ for the KLS constant, but they also obtain the inequality

$$\alpha_\mu(r) \leq C \cdot \exp\left(c \cdot \min\left(r, \frac{r^2}{n^{1/2}}\right)\right). \tag{7}$$

In contrast with (6), they do not loose a logarithm when they pass from the bound on the KLS constant to the deviation inequality. Their argument is very delicate and clever but it only works with a polynomial estimate for C_n and it does not allow to remove the logarithm from (6) now that we have a logarithmic estimate for C_n. Here is an example of an application of the theorem.

Corollary 7 (Paouris theorem [8])**.** *Suppose μ is log-concave and isotropic then*

$$\mu(|x| \geq r) \leq \exp(-cr), \quad \forall r \geq C\sqrt{n},$$

where as usual c, C are universal constants.

Remark. The inequality can also be expressed in terms of moments. It asserts that if X is log concave and isotropic on $\mathbb{R}^n$ then the moments of the Euclidean norm of X remain constant for quite a while, namely

$$(\mathbb{E}|X|^p)^{1/p} \approx (\mathbb{E}|X|^2)^{1/2}$$

for p as large as $\sqrt{n}$.

Proof. We apply the concentration estimate to the 1-Lipschitz function $f(x) = |x|$. We get in particular

$$\mu(|x| \geq m + r) \leq \mathrm{e}^{-cr},$$

provided that $r \geq C \cdot \log^2 n$, where m is a median for $|x|$. Since $m \leq 2\int |x|\, d\mu \leq 2\sqrt{n}$, the latest display is easily seen to imply the desired inequality. □

Our main point here is that from the inequality (4) one can only recover the Paouris inequality up to some logarithmic factor. That being said, the inequality (5) is an overkill for this application, and the argument would go through using (7) instead, and as a matter of fact this application to the Paouris inequality is taken from Lee and Vempala's paper [7].

Logarithmic Sobolev inequality and a variant of the KLS conjecture. A probability measure on $\mathbb{R}^n$ is said to satisfy the logarithmic Sobolev inequality if there exists a constant $C > 0$ such that

$$\mathrm{ent}_\mu(f) \leq \frac{C}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} \, d\mu$$

for every positive Lipschitz function f, where

$$\mathrm{ent}_\mu(f) = \int_{\mathbb{R}^n} f \log f \, d\mu - \left(\int_{\mathbb{R}^n} f \, d\mu \right) \log \left(\int_{\mathbb{R}^n} f \, d\mu \right),$$

denotes the entropy of f. The best constant C is called the log-Sobolev constant, denoted $C_{LS}(\mu)$ below. The factor $1/2$ is just a matter of convention. With this convention the log-Sobolev constant of the standard Gaussian 1. This is a stronger inequality than Poincaré. More precisely we have $C_P(\mu) \leq C_{LS}(\mu)$ for any μ. This is easily seen by applying log-Sobolev to $f = 1 + \varepsilon g$ and letting ε tend to 0. Once again, the uniformly log-concave case is well understood.

Theorem 8 (Bakry-Émery criterion)**.** *If μ is t-uniformly log-concave then $C_{LS}(\mu) \leq t^{-1}$. The inequality is sharp, equality is attained for the Gaussian measure of covariance $t^{-1} \cdot \mathrm{Id}$.*

There are many ways to prove this inequality, see for instance [5] for an overview. Again we are interested in the log-concave case. However, in contrast with the Poincaré inequality, not every log-concave measure satisfies log Sobolev. Indeed, it is well known that log-Sobolev implies Gaussian concentration, with explicit control of the constants, this is called Herbst argument. Again the inequality can be reversed in the log concave case, so that the log-Sobolev constant and the Gaussian concentration constant are actually of the same order in this case. This is again due to E. Milman. We say more about this in the uncovered material section of the first lecture. In any case, to insure log-Sobolev, one has to impose another condition on top of log-concavity, such as having bounded support. The following result is due to Lee-Vempala [7].

Theorem 9. *Suppose μ is log-concave, isotropic, and supported on a set of diameter D. Then $C_{LS}(\mu) \leq C \cdot D$.*

Let us remark that because of the equivalence between log-Sobolev and Gaussian concentration in the log-concave case, a log-concave measure supported on a set of diameter D trivially has $O(D^2)$ log-Sobolev constant. Since the diameter of the support of an isotropic measure is at least $\sqrt{n}$ the theorem improves greatly upon the trivial bound in the isotropic case. It should also be noted that for the uniform measure on the ℓ_1 ball rescaled to be isotropic, the diameter of the support and the log-Sobolev constant both are of order n.

Proof. This is actually an easy consequence of our concentration result Theorem 6. Indeed, the latter asserts that if μ is log concave and isotropic then

$$\alpha_\mu(r) \leq C \cdot \exp\left(-c \cdot \min(r, \frac{r^2}{\log^2 n}) \right), \quad \forall r \geq 0.$$

On the other hand if μ is supported on a set of diameter D then trivially $\alpha_\mu(r) = 0$ if $r > D$. On the interval $[0, D]$ we have $r \leq r^2/D$, and since $D \geq c_1 \sqrt{n} \geq c_1 \log^2 n$, we finally obtain

$$\alpha_\mu(r) \leq C \cdot \exp\left(c' \cdot \frac{r^2}{D} \right).$$

The Gaussian concentration constant is thus $O(\sqrt{D})$, which implies the desired inequality by E. Milman's result. □

Let us try to relax the bounded support assumption. We know that log-Sobolev implies Gaussian concentration. In particular linear functions should have sub-Gaussian tails, at a rate controlled by the log-Sobolev constant. Let us be a bit more precise.

Definition 10. Suppose f is a function having mean zero for μ. We denote by $\|f\|_{\psi_2(\mu)}$ the Orlicz norm of f associated to the Orlicz function $\mathrm{e}^{r^2}-1$, namely the best constant C in the inequality

$$\mu(|f| \geq r) \leq 2 \cdot \exp\left(-\frac{r^2}{C^2}\right).$$

The discussion above shows that for any probability measure and any direction θ we have

$$\|x \cdot \theta\|_{\psi_2(\mu)}^2 \lesssim C_{LS}(\mu).$$

It is natural to conjecture that this inequality could be reversed in the log-concave case. This amounts to saying that the log-Sobolev constant is the largest ψ_2-norm squared of a linear function, very much like the KLS conjecture predicts that Poincaré constant of a log-concave measure is up to a constant the largest L^2-norm squared of a linear function.

Definition 11 (Log-Sobolev version of KLS constant, [3])**.** Let D_n be the largest log-Sobolev constant of a log-concave measure for which every linear function has ψ_2 norm at most 1.

Conjecture 12 (Log-Sobolev KLS conjecture, Bizeul [3])**.**

$$D_n = O(1).$$

It follows from some result of Bobkov [4] from 2007 that $D_n = O(n)$. Using stochastic localization, one can show the following.

Theorem 13 (Bizeul (2023))**.**

$$D_n = O(n^{1/2}).$$

Proof. The idea is to combine Theorem 6 with a rather crude net argument. By E. Milman's theorem it is enough to prove that if μ log-concave is ψ_2 with norm 1 in all directions, then its concentration function satisfies

$$\alpha_\mu(r) \leq C\mathrm{e}^{-cr^2/\sqrt{n}}. \tag{8}$$

Note that the ψ_2 norm is larger than the L^2 norm, maybe up to a constant. So the covariance of μ has operator norm $O(1)$. The concentration function of μ thus satisfies

$$\alpha_\mu(r) \leq C\exp\left(-c\min(r, \frac{r^2}{\log^2 n})\right),$$

for every $r > 0$. Here there is a small gap which we can leave as an exercise: show that having an upper bound for the concentration function of isotropic log-concave μ of the form $\alpha_\mu \leq \alpha_*$ implies that if μ is log-concave but not necessarily isotropic then $\alpha_\mu(r) \leq \alpha_*(r/\sqrt{\|\mathrm{cov}(\mu)\|})$. We thus get an estimate that is smaller than our target concentration if $r > cr^2/\sqrt{n}$, namely

$r < C\sqrt{n}$. Therefore, it is enough to prove (8) when r is a sufficiently large multiple of $\sqrt{n}$. Moreover, by Markov inequality we have

$$\mu(|x| \geq 2\sqrt{n}) \leq \frac{1}{4n} \int |x|^2 \, d\mu \leq \frac{1}{4}.$$

So if S is a set of measure $1/2$ then S intersects the ball of radius $2\sqrt{n}$. If $r \geq 2\sqrt{n}$ this implies easily that $S_{2r} \supset \{|x| \leq r\}$, hence

$$\alpha_\mu(2r) \leq \mu(|x| > r).$$

So it is enough to prove that $\mu(|x| > r) \leq \mathrm{e}^{-cr^2/\sqrt{n}}$ for $r \geq C\sqrt{n}$. Now recall the ψ_2 hypothesis: For every direction θ and every r, we have

$$\mu\{|x \cdot \theta| > r\} \leq 2\mathrm{e}^{-r^2}. \tag{9}$$

It is well-known that there exists $1/2$-net of the unit sphere of cardinality 5^n at most. Let N be such a set. Since any element x in the sphere is at distance $1/2$ at most from a point of N we have

$$|x| \leq 2 \max_{\theta \in N} \{x \cdot \theta\},$$

for every $x \in \mathbb{R}^n$. Applying (9) to every θ in the net and the union bound we get

$$\mu(|x| > r) \leq 2 \cdot 5^n \mathrm{e}^{-r^2/4}.$$

If $r > C\sqrt{n}$ for a sufficiently large constant C, we deduce from this inequality

$$\mu(|x| > r) \leq \mathrm{e}^{-r^2/8}$$

which is even better than what we needed. □

This proof seems to have lots of slack. In particular it certainly does not look like the ψ_2 hypothesis was fully exploited. However, as far as the log-Sobolev version of the KLS conjecture is concerned this is the best result around, as of today.

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