A link between the central limit theorem and numerical schemes for the transport equation

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Goal of numerical analysis: For a given PDE, how can we approach its solutions?

Who cares about numerical analysis on the transport equation?

$$\begin{aligned} \partial_t u + c\partial_x u = 0, &\quad t \in]0, +\infty[, x \in \mathbb{R} \\ u(t=0,x) = u_0(x), &\quad x \in \mathbb{R}. \end{aligned}$$

Approximations of solutions of systems of conservation laws

We consider the systems of conservation laws

$$\partial_t u + \partial_x f(u) = 0, \quad t \in \mathbb{R}_+, x \in \mathbb{R},$$
$$u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R},$$

where the function $f : \mathbb{R} \to \mathbb{R}$ is smooth.

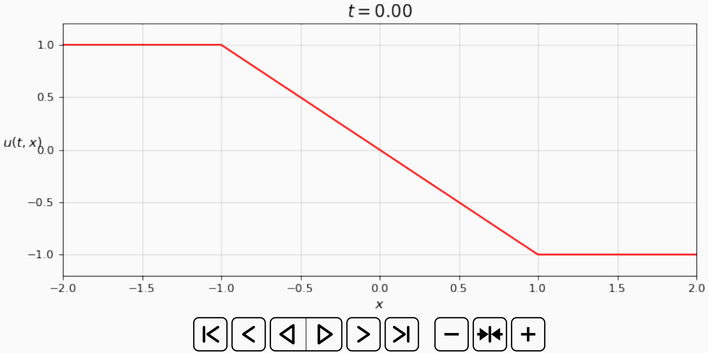
This type of PDE approaches several physical phenomena, for instance in fluid mechanics.

Particularities:

- The solutions of this type of PDE tends to have discontinuities. (shock waves)
- The solutions can be fairly difficult to find. Numerical analysis for those PDEs is a central theme.

Here is a solution for the Burgers equation:

$$\partial_t u + u \partial_x u = 0.$$



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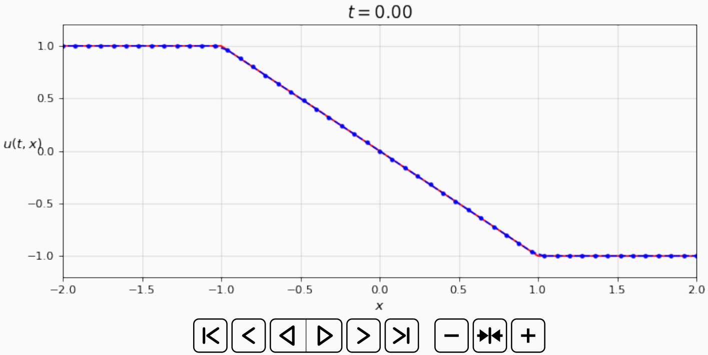
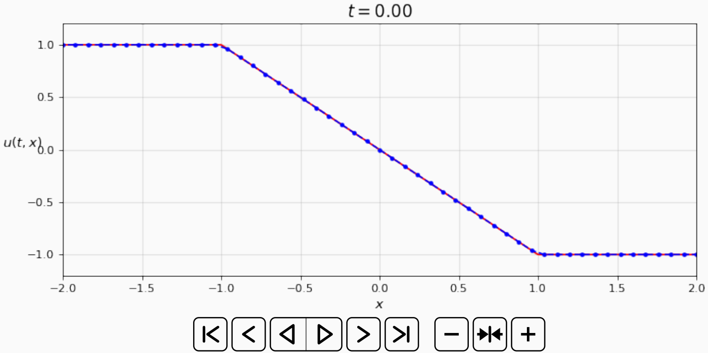


Here is a solution for the Burgers equation:

$$\partial_t u + u\partial_x u = \varepsilon\partial_{xx} u.$$



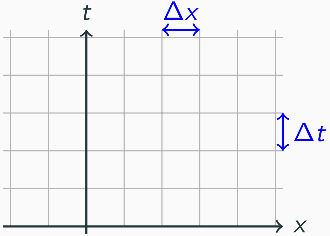
Finite difference schemes for the transport equation

We consider a velocity $c > 0$. We are looking to approach the solutions of the transport equation:

$$\partial_t u + c\partial_x u = 0, \quad t \in]0, +\infty[, x \in \mathbb{R}$$
$$u(t = 0, x) = u_0(x), \quad x \in \mathbb{R}.$$

We are going to apply a finite difference scheme.

We consider a small time step $\Delta t > 0$ and a small space step $\Delta x > 0$. We define $t^n := n\Delta t$ and $x_j := j\Delta x$ for $n \in \mathbb{N}$ and $j \in \mathbb{Z}$.



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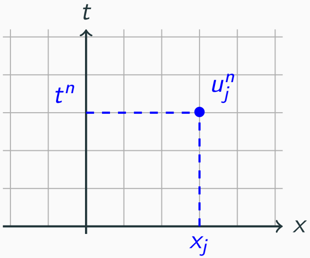

For a regular function u, we have:

$$\partial_t u(t,x) \approx \frac{u(t+\Delta t, x) - u(t,x)}{\Delta t}$$
$$\partial_x u(t,x) \approx \frac{u(t,x) - u(t, x-\Delta x)}{\Delta x}$$

Thus, for u a regular solution of the transport equation, we have for all $t \in [0, +\infty[$ and $x \in \mathbb{R}$:

$$\frac{u(t+\Delta t, x) - u(t,x)}{\Delta t} + c\frac{u(t,x) - u(t, x-\Delta x)}{\Delta x} \approx 0.$$

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Thus, for u a regular solution of the transport equation, we have for all $t \in [0,+\infty[$ and $x \in \mathbb{R}$:

$$u(t+\Delta t,x) \approx \left(1 - c\frac{\Delta t}{\Delta x}\right) u(t,x) + c\frac{\Delta t}{\Delta x} u(t,x-\Delta x).$$

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This leads us to study the solution $(u^n)_{n\in\mathbb{N}}$ of

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad u_j^{n+1} = \left(1 - c\frac{\Delta t}{\Delta x}\right) u_j^n + c\frac{\Delta t}{\Delta x} u_{j-1}^n$$

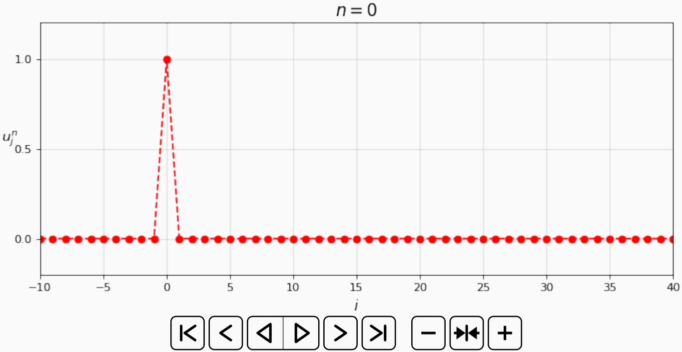
with $u^0 \in \mathbb{R}^Z$.

From now on we have:

$$\lambda := \frac{\Delta t}{\Delta x}.$$

We consider the numerical scheme for $c\lambda = 0.25$:

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad u_j^{n+1} = (1 - c\lambda)\, u_j^n + c\lambda u_{j-1}^n$$



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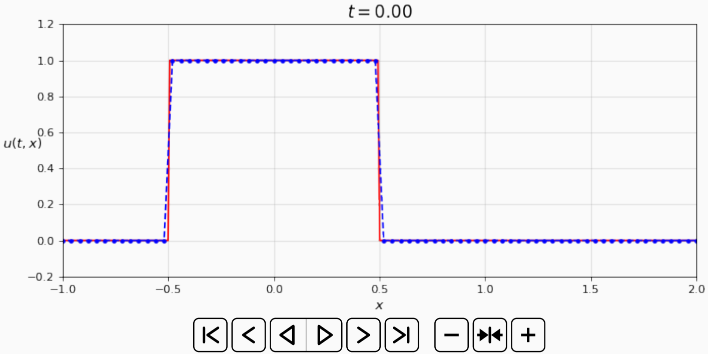

"The numerical scheme introduces numerical viscosity"

Hypothesis: We have fixed the ratio between Δt and Δx.

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"The numerical scheme introduces numerical viscosity"

Hypothesis: We have fixed the ratio between Δt and Δx.

We consider u a regular solution of the transport equation:

$$\begin{aligned} u(t+\Delta t, x) - (1-c\lambda)\, u(t,x) - c\lambda u(t, x-\Delta x) \\ = -\Delta x^2 \beta \partial_{xx} u + o\left(\Delta x^2\right) \end{aligned}$$

where β is some constant in $]0, +\infty[$.

Rewriting the finite difference schemes in a better way

We consider the Modified Lax Friedrichs scheme (D is some constant)

$$u_j^{n+1} = \left(\lambda D + \frac{c\lambda}{2}\right) u_{j-1}^n + (1 - 2D\lambda) u_j^n + \left(\lambda D - \frac{c\lambda}{2}\right) u_{j+1}^n$$

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$$u_j^{n+1} = \underbrace{\left(\lambda D + \frac{c\lambda}{2}\right)}_{= \mathbf{a}_1} u_{j-1}^n + \underbrace{(1 - 2D\lambda)}_{= \mathbf{a}_0} u_j^n + \underbrace{\left(\lambda D - \frac{c\lambda}{2}\right)}_{= \mathbf{a}_{-1}} u_{j+1}^n$$

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Thus,

$$\forall n \in \mathbb{N}, \quad u^{n+1} = \mathbf{a} * u^n,$$

where $\mathbf{a} = (\cdots, 0, \mathbf{a}_{-1}, \mathbf{a}_0, \mathbf{a}_1, 0, \cdots) \in \mathbb{C}^{\mathbb{Z}}$.

A finite difference approximation of our initial transport equation can be written as

$$\forall n \in \mathbb{N}, \quad u^n = \mathbf{a}^n * u^0,$$
$$u^0 \in \mathbb{C}^{\mathbb{Z}},$$

where $\mathbf{a} \in \mathbb{C}^{\mathbb{Z}}$ is finitely supported and $\mathbf{a}^n = \mathbf{a} * \cdots * \mathbf{a}$.

We must also have :

$$\sum_{k\in\mathbb{Z}} \mathbf{a}_k = 1 \qquad \text{and} \qquad \sum_{k\in\mathbb{Z}} k\mathbf{a}_k = c\lambda \quad .$$

constants are solutions

the solutions of the scheme travel at the correct speed

Goal:

- We want to study $\mathbf{a}^n$ for large values of n to better understand the large time behavior of the finite difference scheme. Maybe this would explain the diffusive behavior we observe.
- Could we prove some estimates on $\mathbf{a}^n$ in $\ell^p(\mathbb{Z})$?

A detour through Probability - The Local Limit Theorem

We consider a random walk

$$S_n := I + X_1 + \cdots + X_n$$

where X_n are i.i.d. random variables (same law as some random variable X with values in $\mathbb{Z}$) and I is also an independent random variable with values in $\mathbb{Z}$. We use the notation

$$\forall j \in \mathbb{Z}, \quad u_j^0 := P(I = j) \quad \text{and} \quad \mathbf{a}_j := P(X = j).$$

We then have

$$\forall j \in \mathbb{Z}, \quad P(S_0 = j) = u_j^0.$$

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We then have

$$\forall j \in \mathbb{Z}, \quad P(S_1 = j) = \sum_{k \in \mathbb{Z}} P((I = k) \cap (X_1 = j - k)) = (\mathbf{a} * u^0)_j.$$

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We then have

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad P(S_n = j) = (\mathbf{a}^n * u^0)_j.$$

Central Limit Theorem :

$$\sqrt{n}\left(\frac{S_n}{n} - \mathbb{E}(X)\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, V(X)).$$

Local Limit Theorem : Under suitable conditions on the sequence **a**

$$\mathbf{a}_j^n - \frac{1}{\sqrt{2\pi V(X) n}} \exp\left(-\frac{|j - n\mathbb{E}(X)|^2}{2nV(X)}\right) \underset{n\to+\infty}{=} o\left(\frac{1}{\sqrt{n}}\right) \quad \text{uniformly on } \mathbb{Z}.$$

A detour through Probability - The Local Limit Theorem

$$\mathbf{a}_j^n - \frac{1}{\sqrt{2\pi V(X)n}} \exp\left(-\frac{|j - n\mathbb{E}(X)|^2}{2nV(X)}\right) - \frac{1}{n} q\left(\frac{j - n\mathbb{E}(X)}{\sqrt{V(X)n}}\right) \underset{n\to+\infty}{=} o\left(\frac{1}{n}\right),$$

where

$$\forall x \in \mathbb{R}, \quad q(x) := C(X)(x^3 - 3x)e^{-\frac{x^2}{2}}$$

with $C(X)$ a constant depending on the random variable X.

Question solved ?

A detour through Probability - The Local Limit Theorem

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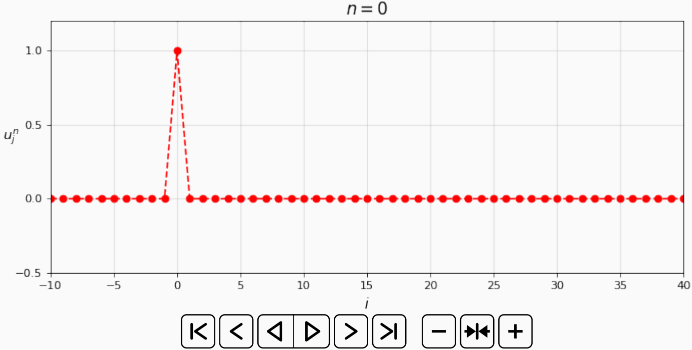
Question solved ? **No !**

Two remaining issues:

- The Local limit theorem does not allow to obtain estimates on $\mathbf{a}^n$ in ℓ^p.
- The Local limit theorem only applies to sequences $\mathbf{a}$ with positive coefficients. The case with real (or even complex) coefficients can create strange new behavior!

We consider the Lax -Wendroff scheme for $c\lambda = 0.25$:

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z}, \quad u_j^{n+1} = \frac{c\lambda + (c\lambda)^2}{2} u_{j-1}^n + (1 - (c\lambda)^2) u_j^n - \frac{c\lambda - (c\lambda)^2}{2} u_{j+1}^n.$$



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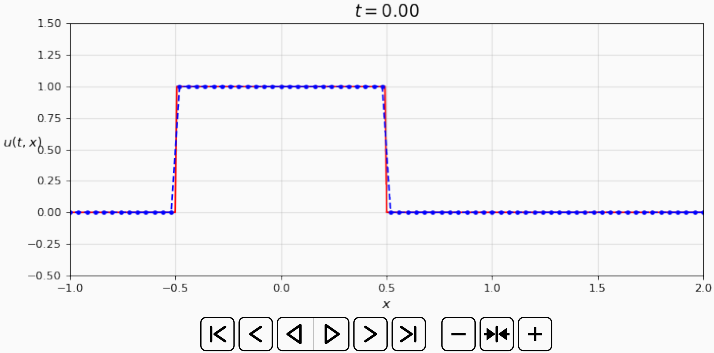


A spectral issue

We consider the convolution operator acting on $\ell^p(\mathbb{Z})$:

$$\forall u \in \ell^p(\mathbb{Z}), \quad Lu := \mathbf{a} * u.$$

The spectrum of the convolution operator L is:

$$\left\{ \sum_{k \in \mathbb{Z}} \mathbf{a}_k \kappa^k, \quad \kappa \in \mathbb{S}^1 \right\} = F(\mathbb{S}^1)$$

where F is the Fourier series associated to the sequence $\mathbf{a}$.

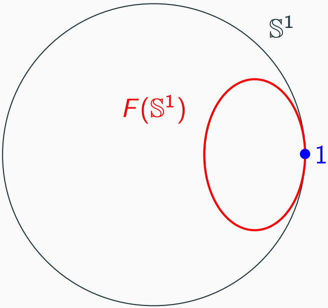
Hypotheses on the Fourier series

We assume that:

- $F(0) = \sum_{k \in \mathbb{Z}} \mathbf{a}_k = 1$
- $F'(0) = \sum_{k \in \mathbb{Z}} k\mathbf{a}_k = c\lambda$
- $\forall \kappa \in \mathbb{S}^1, \quad |F(\kappa)| < 1$

We notate $m \in \mathbb{N} \setminus \{0, 1\}$ the first integer such that there exists a constant $D \neq 0$ such that:

$$F(e^{it}) \underset{t \to 0}{=} \exp(ic\lambda t + Dt^m + o(t^m)).$$



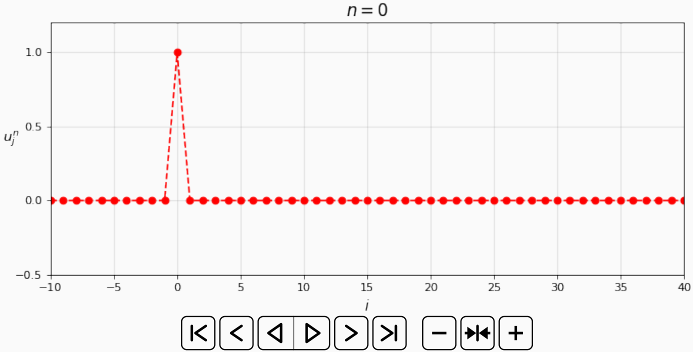
Generalized local limit theorem - Randles and Saloff-Coste 15'

The asymptotic behavior of $\mathbf{a}^n$ is described by:

$$\mathbf{a}_j^n - \frac{1}{n^{\frac{1}{m}}} H_m\left(\frac{j - nc\lambda}{n^{\frac{1}{m}}}\right) \underset{n\to+\infty}{=} o\left(\frac{1}{n^{\frac{1}{m}}}\right)$$

where the generalized Gaussian H_m is the fundamental solution of

$$\partial_t u + i^m D \partial_x^m u = 0.$$



Generalized local limit theorem - C. 22'

When m is even, there exists two positive constants C, c such that:

$$\forall n \in \mathbb{N}, \forall j \in \mathbb{Z},$$
$$\left| \mathbf{a}_j^n - \frac{1}{n^{\frac{1}{m}}} H_m \left(\frac{j - nc\lambda}{n^{\frac{1}{m}}} \right) \right| \leq \frac{C}{n^{\frac{2}{m}}} \exp \left(-c \left(\frac{|j - nc\lambda|}{n^{\frac{1}{m}}} \right)^{\frac{m}{m-1}} \right).$$

Thank you for your attention!