What's the deal about Topology?

A.k.a. how to turn concrete problems into (hopefully) easier abstract ones.

Anthony Saint-Criq

Nov. 23rd, 2023

A few weird questions

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How to hang a frame to a wall in a sneaky way?

How to prove that a rope and metal ring puzzle is impossible to solve?

The Loony Loop
This loony loop puzzle was stumbled upon by the great American puzzler, Stewart T. Coffin. The aim of this puzzle is to free the tied cord from the figure-eight metal loop, without breaking or untying the cord. But — beware — the simplicity of the wire loop and inter-twined cord may be deceptive.

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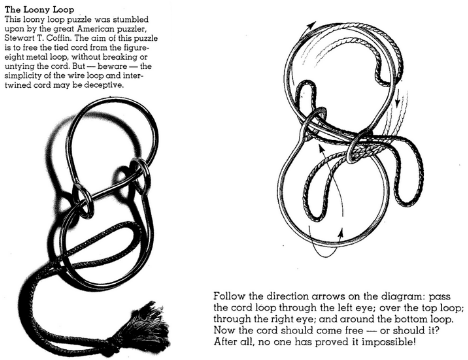

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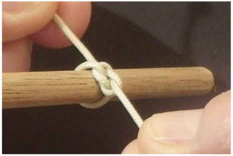
How to be sure that the granny knot and the figure-eight knot are actually different?



Distinguishing spaces: the fundamental group

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Two topological spaces X and Y are **homeomorphic** if there exists a bijection $f : X \to Y$ such that both f and f^{-1} are continuous.

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Topology is (a lot of times) the study of topological spaces *up to homeomorphism* (up to continuous deformations). The motto: imagine that everything is made out of clay.

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Fact

The doughnut $\mathscr{D}$ and the pretzel $\mathscr{P}$ are *not* homeomorphic topological spaces.

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How does one go proving this? The answer: find invariants!

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Funnier fact

The doughnut and the coffee mug are homeomorphic. The pretzel and the hand spinner are too.

How does one go proving this? The answer: find invariants!

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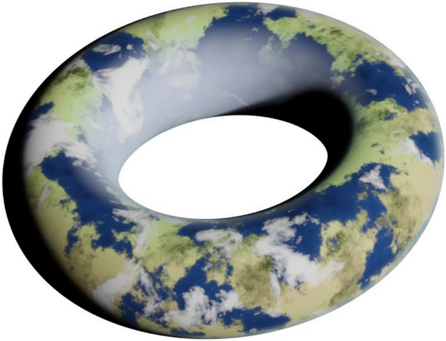
Let us play a game.



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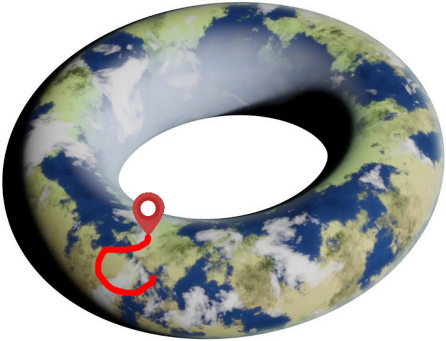

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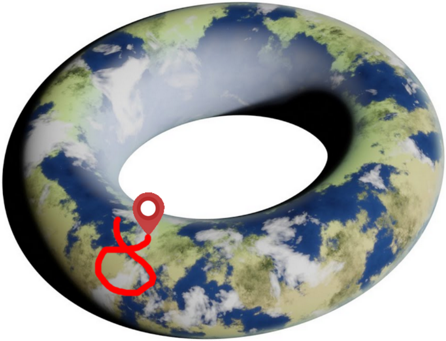

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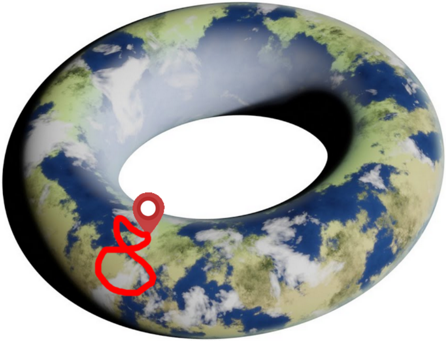

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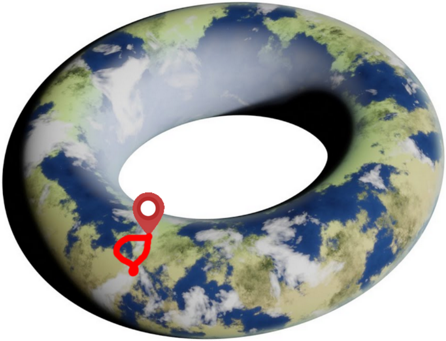

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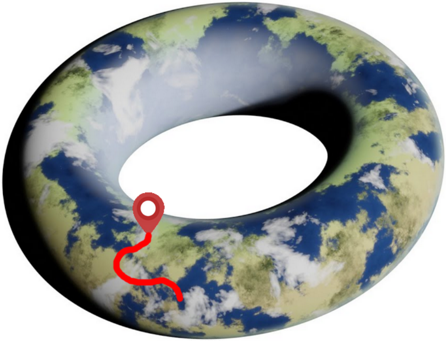

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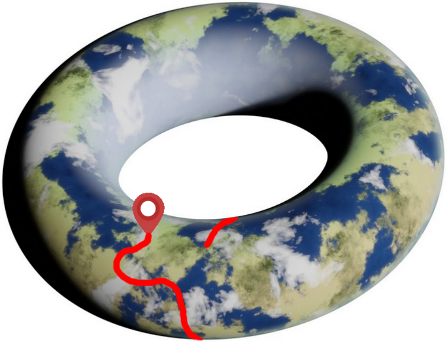

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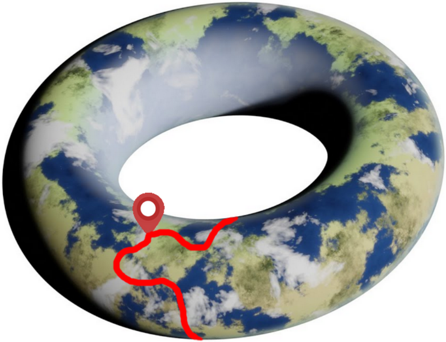

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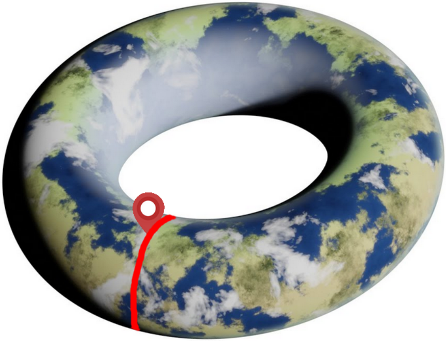

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A **loop** in a space X is a continuous path $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1)$. A **homotopy** of a loop γ is a family $(\gamma_s)_{s\in[0,1]}$ of loops in X such that:

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2. If Y and X are homeomorphic, then $\pi_1(X)$ and $\pi_1(Y)$ are isomorphic.

Distinguishing spaces: the fundamental group

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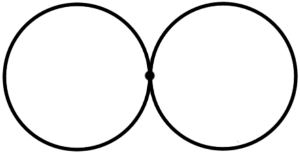

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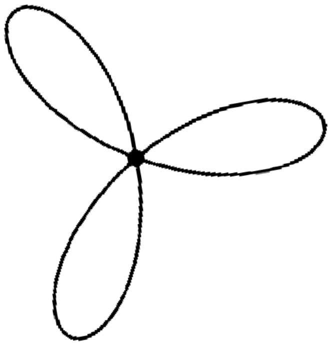

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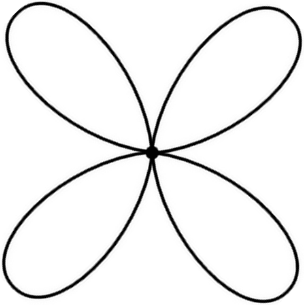

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Imagine that they are made of cheese, and that our friend Mickey lives inside.

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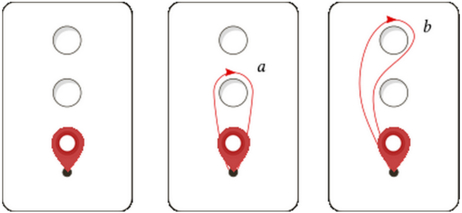
How to hang a frame to a wall in a sneaky way? The plane minus two points $\mathbb{R}^2 \setminus \{*, *\}$ has $\pi_1 = \pi_1(\infty) = F_2$:



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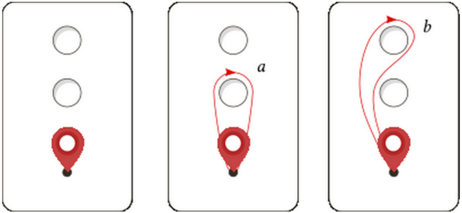

Here, $a \cdot b \neq b \cdot a$, so $aba^{-1}b^{-1} \neq 0$.

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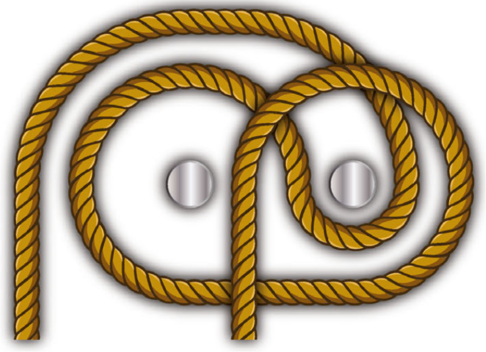

The loop $aba^{-1}b^{-1} \in \pi_1(\mathbb{R}^2 \setminus \{*, *\}) = F_2$.

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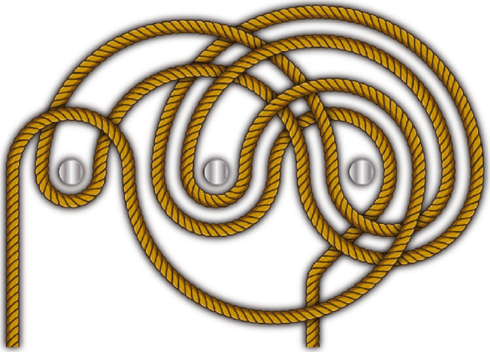

Take the loop $[a,[b,c]] \in \pi_1(\mathbb{R}^2 \setminus \{*,*,*\}) = \pi_1($$) = F_3$.

Answering the questions!

With n nails:

$$\pi_1(\mathbb{R}^2 \setminus \{z_1, \ldots, z_n\}) = F_n.$$

Take the loop:

$$[a_1, [a_2, [\ldots, [a_{n-1}, a_n]] \ldots]].$$

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Whatever; now, it's *just* algebra!

How to prove that a rope and metal ring puzzle is impossible to solve?

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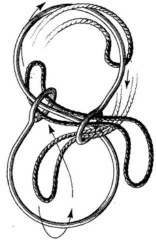
This loony loop puzzle was stumbled upon by the great American puzzler, Stewart T. Coffin. The aim of this puzzle is to free the tied cord from the figure-eight metal loop, without breaking or untying the cord. But — beware — the simplicity of the wire loop and inter-twined cord may be deceptive.



Follow the direction arrows on the diagram: pass the cord loop through the left eye; over the top loop; through the right eye; and around the bottom loop. Now the cord should come free — or should it? After all, no one has proved it impossible!

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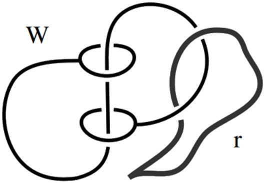

Take $X = \mathbb{R}^3 \setminus W$. Then r is a loop in X.

Answering the questions!

It is possible to find a ”presentation” of the fundamental group:

$$\pi_1(X) = \langle a, b, c \mid a = [c, [b^{-1}, a]] \rangle.$$

(For the topologists: either using Van Kampen, or by a Wirtinger-type argument.)

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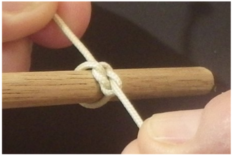

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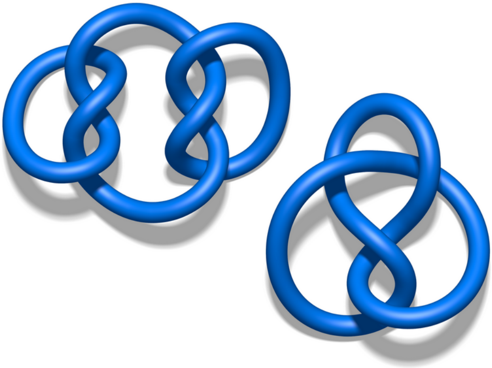

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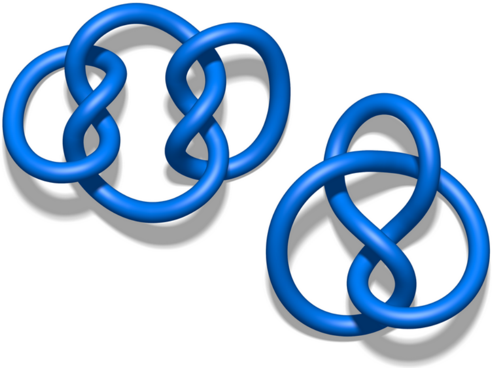

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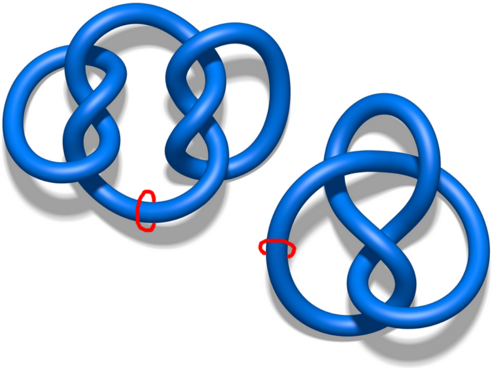

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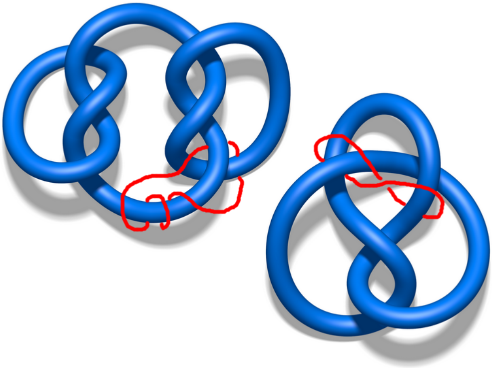

Take $X_1 = \mathbb{R}^3 \setminus K_1$ and $X_2 = \mathbb{R}^3 \setminus K_2$.

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There is an algorithm to compute $\pi_1(X_i)$, which gives:

$$\pi_1(X_1) = \langle x, y, z \mid xyx = yxy,\ xzx = zxz \rangle$$

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Showing that $K_1 \neq K_2$ has "just" become an algebra problem only.
This is still not obvious, but those groups are distinct!

To infinity, and beyond!

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A volley ball is "hollow", there *is* a hole *right in the middle!* And yet, $\pi_1(\mathscr{S}) = 0$...

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Definition

The second homotopy group $\pi_2(X)$ is the set of all homotopy classes of 2-loops into X.

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The second homotopy group $\pi_2(X)$ is the set of all homotopy classes of 2-loops into X.

Fact

This *is* a group. Moreover, it is always abelian.

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We have:

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An n-loop in X is a continuous map $f : \mathbb{S}^n \to X$. Homotopies and concatenation are well-defined.

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Definition & Fact

The n-th homotopy group $\pi_n(X)$ is the set of homotopies of all n-loops in X.

This forms a group, which is always abelian when $n \geqslant 2$.

To infinity, and beyond!

It is not very difficult to show that:

$$\forall n \geqslant 2,\ \pi_n(\mathbb{S}^1) = 0,\ \pi_1(\mathbb{S}^n) = \cdots = \pi_{n-1}(\mathbb{S}^n) = 0 \text{ and } \pi_n(\mathbb{S}^n) = \mathbb{Z}.$$

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We have:

$$\pi_4(\mathbb{S}^3) = \mathbb{Z}/2.$$

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We have:

$$\pi_7(\mathbb{S}^4) = \mathbb{Z} \times \mathbb{Z}/12.$$

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$$\forall n \geqslant 2,\ \pi_n(\mathbb{S}^1) = 0,\ \pi_1(\mathbb{S}^n) = \cdots = \pi_{n-1}(\mathbb{S}^n) = 0 \text{ and } \pi_n(\mathbb{S}^n) = \mathbb{Z}.$$

We have:

$$\pi_{14}(\mathbb{S}^3) = \mathbb{Z}/84 \times \mathbb{Z}/2 \times \mathbb{Z}/2.$$

Wait, *WHAT?!*

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	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}{\times}Z_2$	$Z_{84}{\times}Z_2^2$	Z_2^2
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}{\times}Z_2$	$Z_{84}{\times}Z_2^2$	Z_2^2
S^4	0	0	0	Z	Z_2	Z_2	$Z{\times}Z_{12}$	Z_2^2	Z_2^2	$Z_{24}{\times}Z_3$	Z_{15}	Z_2	Z_2^3	$Z_{120}{\times}Z_{12}{\times}Z_2$	$Z_{84}{\times}Z_2^5$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}	Z_2	Z_2^3	$Z_{72}{\times}Z_2$
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	Z	Z_2	Z_{60}	$Z_{24}{\times}Z_2$	Z_2^3
S^7	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{120}	Z_2^3
S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	$Z{\times}Z_{120}$

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The ramifications are incredible though: they allow to compute the homotopy groups of virtually any reasonable space.

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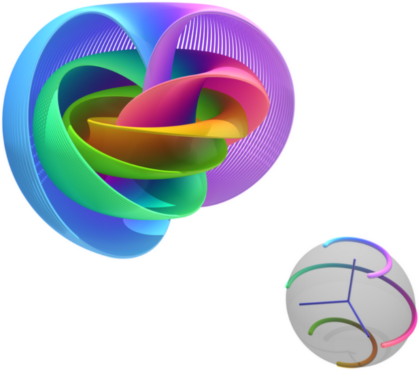
To conclude, a picture of the Hopf map $h : \mathbb{S}^3 \to \mathbb{S}^2$, whose homotopy class spans $\pi_3(\mathbb{S}^2) = \mathbb{Z}$:



Questions?