Euclidean Rhythms

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Student Seminar

27 novembre 2023

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Figure – "Garota De Ipanema" by Tom Jobim

Figure – "Libertango" by Astor Piazzolla

Introduction

Let us first listen to two music pieces. Examples of music genres in the world !

Figure – "Garota De Ipanema" by Tom Jobim

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The Distance Geometry of Music

Erik D. Demaine* Francisco Gomez-Martin† Henk Meijer‡ David Rappaport‡
Perouz Taslakian§ Godfried T. Toussaint§¶ Terry Winograd‖ David R. Wood**

Figure – https ://arxiv.org/pdf/0705.4085.pdf

How to write rhythms ?

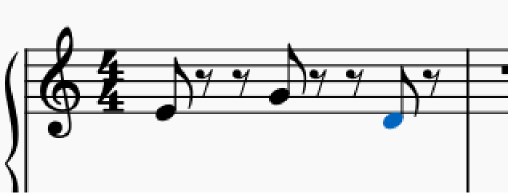


Figure – An example of rhythm

A measure contains 8 ♪ and in this example, we play 3 ♪, we will call them *onsets*. The other symbol represents a *silence*. The *timespan* of a rhythm is the length of the rhythm and depends of the time unit we use. In this example, the time unit is the ♪. It will be called *pulse*.

Definition (The Time Unit Box System (TUBS))

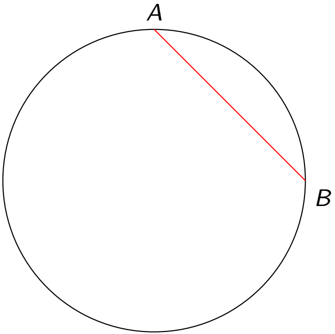
We represent the onsets by $\times$ and the silences by $\cdot$. For example, the previous rhythm can be represented as follows : $[\times \cdot \cdot \times \cdot \cdot \times \cdot]$. It is a rhythm of timespan 8 with 3 onsets.

Definition (The subset notation)

Let R a rhythm of timespan n. We number the pulses $\{0, 1, ..., n-1\}$. The rhythm will be represented by the subset of the onsets. In the previous example, we can represent R by $\{0, 3, 6\}_8$.

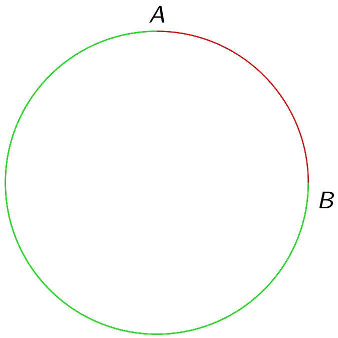
Definition

Let $\mathcal{C}$ a circle and $x, y \in \mathcal{C}$. The *chordal distance* between x and y and denoted $\bar{d}(x, y)$ is the length of the line segment $\overline{xy}$.



Definition

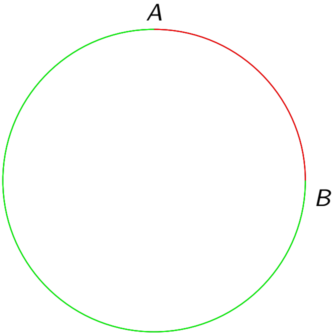
Let $\mathcal{C}$ a circle and $x, y \in \mathcal{C}$. The *clockwise distance* between x and y and denoted $\mathrm{d}_c(x, y)$ is the length of clockwise arc (x, y).



The clockwise distance (in red) of (A, B) is different from the clockwise distance (in green) of (B, A).

Definition

Let $\mathcal{C}$ a circle and $x, y \in \mathcal{C}$. The *geodesic distance* between x and y and denoted $\mathrm{d}_g(x, y)$ is the minimum of the clockwise distances of (x, y) and (y, x) :
$d_g(x, y) = \min\{d_c(x, y), d_c(y, x)\}$.



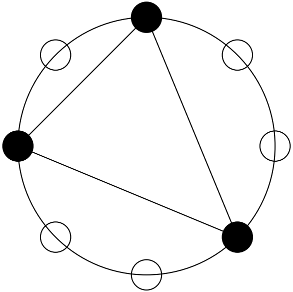


Figure – Clock Diagram Representation of a rhythm

Definition (The clockwise distance sequence representation)

Let R a rhythm of timespan. We place its pulses on a circle of circumference n. The *clockwise distance sequence* of R is the sequence of the clockwise distances between two consecutive onsets. In the example, the clockwise distance sequence representation is $(3, 3, 2)$. We notice that the sum of the elements of the sequence is always equal to n.

Even Rhythms

Which rhythm is the more "even" ?

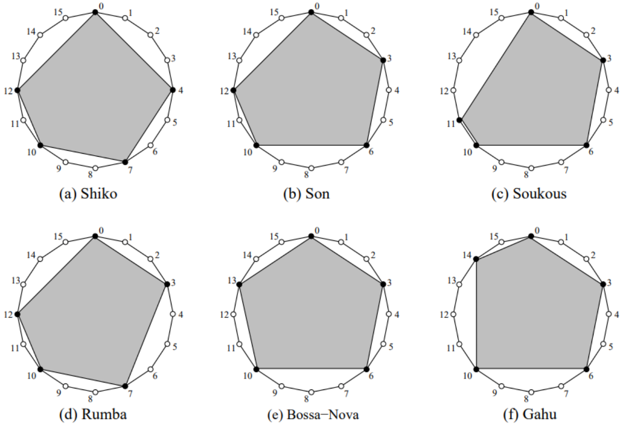


Figure – Source : "The Distance Geometry of Music "
https ://arxiv.org/pdf/0705.4085.pdf

How to measure the "eveness" of a rhythm ?

Definition (Geodesic Eveness)

Let R a rhythm of timespan n, et $r_1, \ldots, r_k$ its onsets. We define $E_g(R)$ the geodesic eveness of R as follows :

$$E_g(R) = \sum_{i=1}^{k} \sum_{j=1}^{k} d_g(r_i, r_k).$$

The subset representation of the *Shiko* rhythm is $(4, 3, 3, 2, 4)$. Its geodesic eveness is 48.

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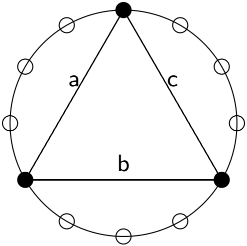
The subset representation of the *Shiko* rhythm is $(4, 3, 3, 2, 4)$. Its geodesic eveness is 48. Actually the geodesic eveness of all these rhythms is 48 ! We have to find another method to measure the eveness of a rhythm.

Definition (The chordal eveness)

Let R a rhythm of timespan n, et $r_1, ..., r_k$ its onsets. We define $\bar{E}(R)$ the chordal eveness of R as follows :

$$\bar{E}(R) = \sum_{i=1}^{k} \sum_{j=1}^{k} \bar{d}(r_i, r_k).$$



With this definition of the eveness, we can discriminate the rhythms. The Bossa-Nova rhythm is the most even, followed by the Son, Rumba, Shiko, Gahu, and Soukous.

Proposition

Let $k < n$ two integers. We consider the set of rhythms of timespan n with k onsets. There exists a unique rhythm (up to a rotation) maximizing the eveness. This rhythm is called euclidean and is noted $E[k, n]$.

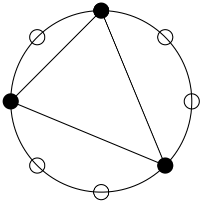


Figure – Tresillo $E[3, 8]$

This rhythm is one of the most used rhythms in the world. It is in particular very popular in Cuba or called *Habanera* rhythms in other countries.

Why Habanera ?

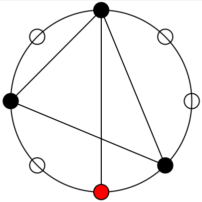


Figure – Tresillo $E[3,8] + E[2,8]$

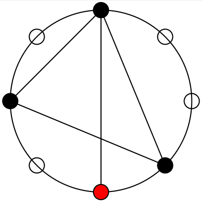


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Figure – "Veinte Años" by Maria Teresa Vera

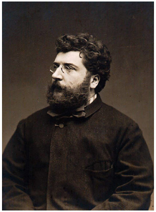
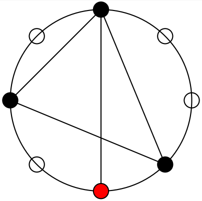


Figure – "Habanera (Carmen)" by Georges Bizet



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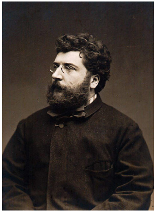

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Definition

Let $k < n$.

- We say that an euclidean rhythm $E[k, n]$ is trivial if and only if $k = 1$.
- We say that an euclidean rhythm $E[k, n]$ is perodic if and only if there exists d such that d divides k and n. In this case, $E[k, n] = dE[\frac{k}{d}, \frac{n}{d}]$.

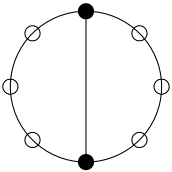

Figure – $E[2, 8]$

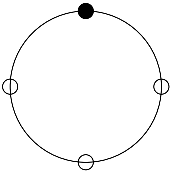

Figure – $E[1, 4]$

The rhythm $E[2, 8]$ is contructed with two copies of the trivial rhythm $E[1, 4]$.

Definition

Let R a rhythm of timespan n with k onsets. The complementary of R, denoted by R^c, is the rhythm where all the onsets of R^c are silences in R and all the silences of R^c are onsets in R.

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Proposition

Let $k < n$. The complementary of the euclidean rhythm $E[k, n]$ is a rotation of the the euclidean rhythm $E[n - k, n]$.

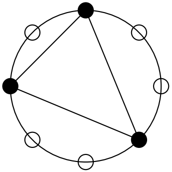

Figure – $E[3, 8]$, Tresillo

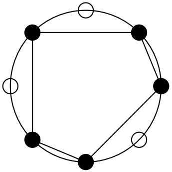

Figure – Rotation of $E[5, 8]$, Cinquillo

We saw earlier that the tresillo is a popular rhythm in Cuba, but the cinquillo, its complementary, is also one of the typical Cuban rhythms.

The article "The Distance Geometry of Music" gives an exhaustive list of euclidean rhythms in traditional world music. Here a small part of the list.

Mary [Far92], as well as the drum rhythmic pattern of the Moroccan *Al Kudum* [Hag03].

$E(5,13) = [\times \cdot \cdot \times \cdot \times \cdot \cdot \times \cdot \times \cdot \cdot] = (32323)$ is a Macedonian rhythm which is also played by starting it on the fourth onset as follows: $[\times \cdot \times \cdot \cdot \times \cdot \cdot \cdot \times \cdot \times \cdot \cdot]$ [Aro04].

$E(5,16) = [\times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \cdot] = (33334)$ is the *Bossa-Nova* rhythm necklace of Brazil. The actual Bossa-Nova rhythm usually starts on the third onset as follows: $[\times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \cdot \times \cdot \cdot \times \cdot \cdot]$ [Tou02]. However, other starting places are also documented in world music practices, such as $[\times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \times \cdot \cdot \cdot \times \cdot \cdot]$ [Beh73].

$E(6,7) = [\times \times \times \times \times \times \cdot] = (111112)$ is the *Póntakos* rhythm of Greece when started on the sixth (last) onset [Hag03].

$E(6,13) = [\times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \times \cdot \cdot] = (222223)$ is the rhythm of the Macedonian dance *Mama Cone pita* [Sin74]. Started on the third onset, it is the rhythm of the Macedonian dance *Postupano Oro* [Sin74], as well as the *Krivo Plovdivsko Horo* of Bulgaria [Ric04].

$E(7,8) = [\times \times \times \times \times \times \times \cdot] = (1111112)$, when started on the seventh (last) onset, is a typical rhythm played on the *Bendir* (frame drum), and used in the accompaniment of songs of the *Tuareg* people of Libya [Sta88].

$E(7,9) = [\times \cdot \times \times \times \cdot \times \times \times] = (2112111)$ is the *Bazaragana* rhythmic pattern of Greece [Hag03].

$E(7,10) = [\times \cdot \times \times \cdot \times \times \cdot \times \times] = (2121211)$ is the *Lenk fahhte* rhythmic pattern of Turkey [Hag03].

$E(7,12) = [\times \cdot \times \times \cdot \times \cdot \times \times \cdot \times \cdot] = (2122122)$ is a common West African bell pattern. For

Figure – Source : "The Distance Geometry of Music", https ://arxiv.org/pdf/0705.4085.pdf

Goal : Let $k < n$, how to maximize the eveness of the rhythms of timespan n with k onsets ?

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Answer : The Bjorklund algorithm.

In the paper "A metric for measuring the evenness of timing system rep-rate patterns" [E. Bjorklund 03'], he answers to the question in another context : timing systems in neutron accelerators. We divide a time window in n intervals. A timing system sends signals to enable a gate during any subset of the n intervals. The goal of Bjorklund is for k fixed to distribute the signals as evenly as possible among the n intervals. It is exactly how we define euclidean rhythms !

Illustration of the algorithm. How to compute $E[5, 11]$?

Step 1 [1][1][1][1][1][0][0][0][0][0][0] [We put all the 1 to the left and the 0 to the right.]

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Step 3 [100] [10] [10] [10] [10] [We concatene the remaining 0 to some [10].]

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- Step 1 [1][1][1][1][1][0][0][0][0][0][0] [We put all the 1 to the left and the 0 to the right.]
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- Step 3 [100] [10] [10] [10] [10] [We concatene the remaining 0 to some [10].]
- Step 4 [10010101010] [We concatene everything.]

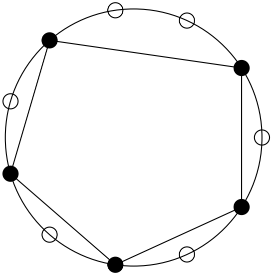

Figure – The rhythm $E[5, 11]$ can be found in Hindustani, Macedonian, Bulgarian, Serbian music.

Algorithm CLOUGH-DOUTHETT(n, k)

1. **return** $\{\lfloor \frac{in}{k} \rfloor : i \in [0, k-1]\}$

Figure – The Clough-Douttet algorithm

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Algorithm SNAP(n, k)

1. Let D be a set of k evenly spaced points on C_n such that $D \cap C_n = \emptyset$.
2. For each point $x \in D$, let x' be the first point in C_n clockwise from x.
3. **return** $\{x' : x \in D\}$

Figure – The SNAP algorithm

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Figure – The SNAP algorithm

Algorithm EUCLIDEAN(n, k)

1. **if** k evenly divides n **then return** $(\underbrace{\frac{n}{k}, \frac{n}{k}, \ldots, \frac{n}{k}}_{k})$
2. $a \leftarrow n \bmod k$
3. $(x_1, x_2, \ldots, x_a) \leftarrow$ EUCLIDEAN(k, a)
4. **return** $(\underbrace{\lfloor\frac{n}{k}\rfloor, \ldots, \lfloor\frac{n}{k}\rfloor}_{x_1-1}, \lceil\frac{n}{k}\rceil; \underbrace{\lfloor\frac{n}{k}\rfloor, \ldots, \lfloor\frac{n}{k}\rfloor}_{x_2-1}, \lceil\frac{n}{k}\rceil; \ldots; \underbrace{\lfloor\frac{n}{k}\rfloor, \ldots, \lfloor\frac{n}{k}\rfloor}_{x_a-1}, \lceil\frac{n}{k}\rceil)$

Figure – The Euclidean algorithm

Let us illustrate the last algorithm in order to compute $EUCLIDEAN(5, 13)$.

Step 1 5 does not divide 13 and $13 = 2 * 5 + 3$, $a_1 = 3$.

Step 2 3 does not divide 5 and $5 = 3 * 1 + 2$, $a_2 = 2$.

Step 3 2 does not divide 3 and $3 = 2 * 1 + 1$, $a_3 = 1$.

Step 4 1 divide 2, we obtain that $EUCLIDEAN(1, 2) = (2)$.

Step 5 It follows that $EUCLIDEAN(2, 3) = (1, 2)$.

Step 6 It follows that $EUCLIDEAN(3, 5) = (2, 1, 2)$.

Step 7 We obtain that $EUCLIDEAN(5, 13) = (2, 3, 3, 2, 3)$.

The Euclidean Rhythm $E[5, 13]$ can be represented in its subset representation by $(2, 3, 2, 3, 3)$, $EUCLIDEAN(5, 13)$ is indeed a rotation of $E[5, 13]$.

The main theorem of [DGMRTTWW07] is the following one, ensuring the unicity of the euclidean rhythm for k and n fixed.

Theorem (Theorem 4.1 (DGMRTTWW07))

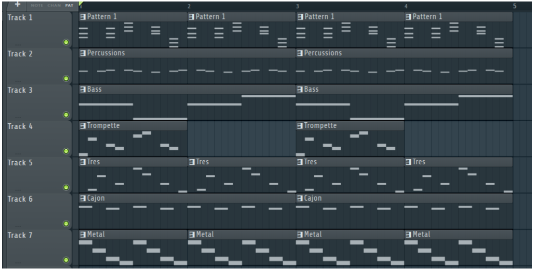
Let $2 \leq k \leq n$ two integers and $R = \{r_0, ..., r_{k-1}\}_n$ a rhythm of timespan n with k onsets. The following assumptions are equivalent :

1. *R has maximum eveness.*
2. *R is a rotation of the $CLOUGH - DOUTHETT(n, k)$ rhythm.*
3. *R is a rotation of the $SNAP(n, k)$ rhythm.*
4. *R is a rotation of the $EUCLIDEAN(n, k)$ rhythm.*

Moreover, up to a rotation there exists a unique rhythm R satisfying these conditions.

Let us try to compose some music with euclidean rhythms. Every instrument will play an euclidean rhythm.

1. The piano plays $E[10, 24]$.
2. The percussions play $E[15, 24]$.
3. The bass plays $E[4, 24]$.
4. The trumpet plays $E[8, 12]$.
5. The tres plays $E[7, 12]$.
6. The cajon plays $E[8, 24]$.



1. "The Distance Geometry of Music" by Erik D. Demaine, Francisco Gomez-Martin, Henk Meijer, David Rappaport, Perouz Taslakian, Godfried T. Toussaint, Terry Winograd, David R. Wood [2007] https ://arxiv.org/abs/0705.4085
2. "C'est quoi un RYTHME EUCLIDIEN ?" by the channel "Mathémusique" (on Youtube).

Thank you for your attention !

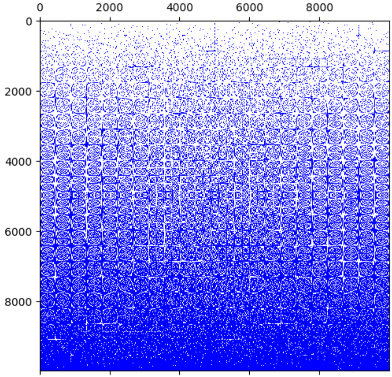


Figure – List of $E[k, n]$ for $n = 10000$ and $1 \leq k \leq n - 1$