Modelling of system with a large number of agents: Birds and Hierarchy of models

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Student seminar

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Modelling of a large number N of identical agents of mass m.
Starting point : modelling through Newton's equations,
$\forall i = 1, \ldots, N:$

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- $2N$ equations $\gg 1 \longrightarrow$ Unreasonable computational cost
- First example : $F^{ext} = q_i\, E$, E electric field computed through Poisson's equation $\longrightarrow$ Not the focus of today's talk !
- **Today: Drone/bird modelling**

Outline of the talk

- Examples of common effects considered in Drone swarms/bird flocks modelling
- Case study of two collective models for birds : Cucker-Smale and three-zone model
- How to reduce the huge computational cost

Panorama of different kind of effects

- **Friction** with the environment : $F_i^{fric} = -\mu(v_i)\, v_i$. One can choose for instance a simple linear drag force, *i.e.* $\mu(v) := \beta$ with $= \beta > 0$, or $\mu(v) := \beta |v|^2$ (Rayleigh-Helmholtz).
- **Self-propulsion** : $F_i^{prop} = \alpha\, v_i$, with $\alpha > 0$ describing a constant acceleration of the particles.
- Normally self-propulsion and friction are **modelled together** via a force term $F^{fp}(v_i) = -(\beta\, |v_i|^2 - \alpha)\, v_i$, leading to an asymptotic velocity of magnitude $\sqrt{\alpha/\beta}$.

Obstacles and targets (drones)

- **Obstacle avoidance** repulsive artificial forces, prevent it from colliding with the occurring obstacles, *i.e.*

$$F_i^{obs} = -\nabla_{x_i}\left[\varphi_{obs}(|x_i - x_{obs}(t)|\right] = \varphi'_{obs}(|x_i - x_{obs}|)\,\frac{(x_i - x_{obs})}{|x_i - x_j|}, \qquad \varphi_{obs}(r) := \frac{1}{r^\alpha}\,, \;\; \alpha > 0\,;$$

- **Destination point (target)** attraction force, helps the drone to reach the goal, *i.e.*

$$F_i^{tar} = -\nabla_{x_i}\left[\varphi_{tar}(|x_i - x_{tar}(t)|\right] = \varphi'_{tar}(|x_i - x_{tar}|)\,\frac{(x_i - x_{tar})}{|x_i - x_j|}, \qquad \varphi_{tar}(r) := r^\alpha\,, \;\; \alpha > 0\,;$$

Example

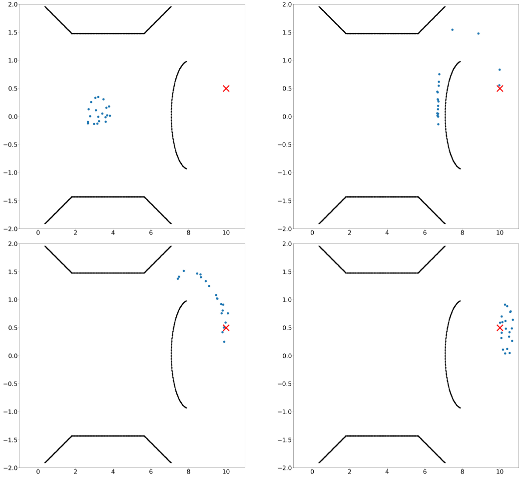

Figure: $N = 20$ drones converging towards a target (red point), avoiding obstacles.

Noise, environmental disturbances

- **Environmental disturbances**, like for example unpredictable fluctuations in the wind $\longrightarrow$ random force field in the model F_i^{fluc}, meaning noise terms representing the incessant impact of the environment on the drones;
- **Inner noise** inaccuracy of the sensors that measure the positions and velocities of the drones $\longrightarrow$ stochastic force fields.

Case study: modelling the flocking of birds

Figure: A flock of birds (Royalty free stock photo).

The Cucker-Smale model

$$\begin{cases} x_i'(t) &= v_i(t)\,, \\ v_i'(t) &= \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \psi(|x_i - x_j|)\,(v_j - v_i)\,, \end{cases} \qquad \forall i = 1, \dots, N\,.$$

$$\psi \in C^1(\mathbb{R}_*^+)\,, \quad \psi(r) > 0 \quad \text{and} \quad \psi'(r) \leq 0 \qquad \forall r > 0\,.$$

- Main feature: alignment of the agents with strength $\psi(|x_i - x_j|)$.
- Behaviour of the system depends solely on ψ !

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- Let us introduce furthermore the notation

$$X(t) := (x_i(t) - x_c(t))_{i=1}^{N}\,, \quad V(t) := (v_i(t) - v_c(t))_{i=1}^{N}\,.$$

Proposition (**Flocking with bounded ψ**)

- *Assume* $\psi = \psi_b := \frac{1}{(1+r^2)^{\beta/2}}$.
- *Non collisional I.C:* $x_i^0 \neq x_j^0$ *for all* $1 \leq i \neq j \leq N$

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(i) if $\beta \in [0,1]$, *one has an* **unconditional flocking** , *meaning there exists* $d_M > 0$ *such that*

$$||X(t)||_2 \leq d_M\,, \quad ||V(t)||_2 \leq ||V^0||_2\, e^{-\psi_b(d_M)t}\,, \ \forall t \in \mathbb{R}^+\,.$$

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(ii) if $\beta \in (1, \infty)$, *we are in the* **conditional flocking** *case, namely assuming* $||V^0||_2 < \int_{||X^0||_2}^{\infty} \psi_b(r)\, dr$, *there exists* $d_M > 0$ *such that*

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We need to refine the modelling.

The three-zone model

$$\begin{cases} x_i'(t) = v_i(t)\,, \\ v_i'(t) = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \psi(|x_i - x_j|)\,(v_j - v_i) \\ \qquad - \dfrac{1}{N}\displaystyle\sum_{j=1, j\neq i}^{N} \underbrace{\nabla_{x_i}\left[\varphi(|x_i - x_j|)\right]}_{\varphi'(|x_i - x_j|)\,\frac{(x_i - x_j)}{|x_i - x_j|}}\,, \end{cases} \quad \forall i = 1, \ldots, N\,,$$

- Attraction between particles i, j such that $\varphi'(|x_i - x_j|) > 0$.
- Repulsion between particles i, j such that $\varphi'(|x_i - x_j|) < 0$.

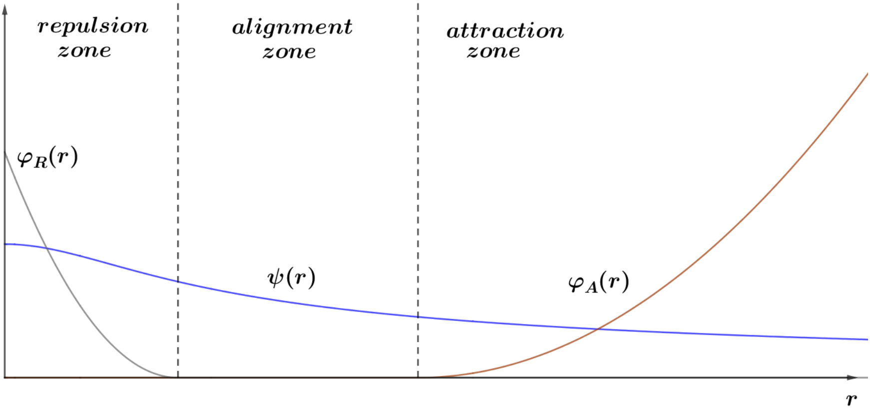

Figure: Example of attraction, alignment and repulsion potentials (bounded in $r = 0$) for the 3zone model.

Flocking for the three-zone model

Proposition (Flocking for the 3-zone model)

Assume $\varphi \in C^1(\mathbb{R}_+^*)$ *is such that* $\lim_{r\to 0,\infty} \varphi(r) = +\infty$.

Then for any non-collisional I.C $(x_i^0, v_i^0)_{i=1}^N$ *then there exist two constants* $r_m, r_M > 0$ *(dependent on* N *but not on* t*), such that for all* $i, j = 1, \cdots, N$

$$0 < r_m \leq |x_i(t) - x_j(t)| \leq r_M \quad \forall t \geq 0\,, \qquad V(t) \to_{t\to\infty} 0\,.$$

What about the equilibrium configurations ?

$$\begin{cases} x_i'(t) = v_i(t)\,, \\ v_i'(t) = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \psi(|x_i - x_j|)\,(v_j - v_i) \\ \qquad -\dfrac{1}{N}\displaystyle\sum_{j=1, j\neq i}^{N} \nabla_{x_i}\left[\varphi(|x_i - x_j|)\right]\,, \end{cases} \quad \forall i = 1, \dots, N\,,$$

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Flocking solutions are such that

$$\sum_{j=1, j\neq i}^{N} \nabla_{x_i}\left[\varphi(|x_i - x_j|)\right] = 0\,.$$

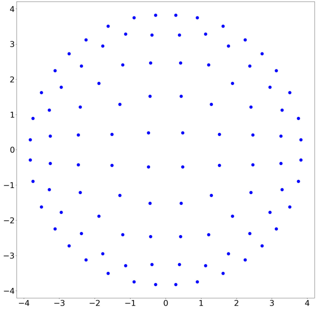

Figure: Annular formation for a potential $\varphi_{ann}(r) := (r-5)^2$.

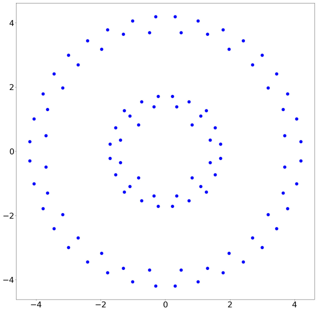

Figure: Ring formation for the potential $\varphi_{ring}(r)$.

$$\varphi_{ring}(r) := \begin{cases} \frac{1}{10}\,(r-5)^2\,, & 0 \leq r \leq 5 \\ \frac{1}{3}(r-5)^2\,, & 5 \leq r \leq 6 \\ 1 - \frac{2}{3\,(r-5)}\,, & r \geq 6 \end{cases}.$$

Microscopic description

$$\begin{cases} x_i'(t) = v_i(t)\,, \\ v_i'(t) = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \psi(|x_i - x_j|)\,(v_j - v_i) \\ \qquad -\dfrac{1}{N}\displaystyle\sum_{j=1, j\neq i}^{N} \nabla_{x_i}\left[\varphi(|x_i - x_j|)\right]\,, \end{cases} \qquad \forall i = 1, \ldots, N\,,$$

Very high dimensionality when number of birds/drones is very high.

- No one cares about the movement of each individual agent.

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Very high dimensionality when number of birds/drones is very high.

- No one cares about the movement of each individual agent.
  $\longrightarrow$ Idea, follow the statistical distribution of agents: this is called the **Mean-Field limit**.

Mean Field Limit

Idea to obtain such a statistical distribution of agents.

(i) Start from the solution of that model $(x_1(t), \ldots, x_N(t))$.

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(iii) Assuming that $f^N(t = 0) \to f^0$ as $N \to \infty$ we obtain (formally or rigorously, in some sense)

$$f^N(t, x, v) \to f(t, x, v)$$

Mesoscopic description

- $f(t, x, v)\, dx\, dv$ is the probability to find a drone/bird/agent at instant t, in a small volume $dx\, dv$ around the phase space point (x, v).

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  f satisfies (formally or rigorously, and in some sense)

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) - (\nabla_x \varphi * n) \cdot \nabla_v f + \nabla_v (F_a(f) f) = 0$$

Macroscopic description

Even the previous description is a computationally intensive approach: **Macroscopic approach**.

Define (some) moments of f:

$$n(t,x) = \int_{\mathbb{R}^3_v} f(t,x,v)dv\,, \tag{1}$$

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Integrating the mesoscopic description gives

$$\begin{aligned} \partial_t n \quad & + \nabla_x (nu) = 0 \\ \partial_t (n\,u) \quad & + \nabla_x (n\,(u \otimes u)) + \nabla_x \mathbb{P} + n(\nabla \varphi * n) \\ & \qquad = \int_{\mathbb{R}^3} \psi(x-y) n(t,x) n(t,x)(u(t,y) - u(t,x))dy \end{aligned}$$

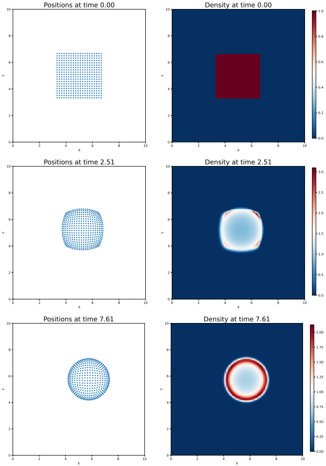

Figure: Drone swarm particle (left, $N = 400$) and fluid (right) simulations at $t = 0, 2.51, 7.61s$ respectively.

Thank you for your attention !