Semiclassical asymptotics for Bergman projections: from smooth to analytic

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Joint work with A. Deleporte and J. Sjöstrand and with M. Stone

Introduction. The Bargmann space

Let Φ be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \Phi(x)}{\partial x_j \partial \overline{x}_k} \zeta_j \overline{\zeta}_k > 0, \quad x \in \mathbb{C}^n, \quad 0 \neq \zeta \in \mathbb{C}^n.$$

Example $\Phi(x) = \frac{|x|^2}{2} = \frac{x \cdot \overline{x}}{2}, \quad x \in \mathbb{C}^n.$

Associated to Φ we introduce the Bargmann space

$$H_\Phi(\mathbb{C}^n) = \mathrm{Hol}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, e^{-2\Phi/h} L(dx)).$$

Here $h \to 0^+$ is the semiclassical parameter (Planck's constant) and $L(dx)$ is the Lebesgue measure on $\mathbb{C}^n$.

Original idea of V. Bargmann (1961) : express Quantum Mechanics directly in phase space $T^*\mathbb{R}^n \simeq \mathbb{C}^n$. V. Fock (1928).

Toeplitz quantization

Given a measurable function $p : \mathbb{C}^n \to \mathbb{C}$, let us consider the Toeplitz operator with symbol p,

$$\mathrm{Top}(p) = \Pi_\Phi \circ p \circ \Pi_\Phi : H_\Phi(\mathbb{C}^n) \to H_\Phi(\mathbb{C}^n),$$

equipped with the natural (maximal) domain

$$\mathcal{D}(\mathrm{Top}(p)) = \{u \in H_\Phi(\mathbb{C}^n);\ pu \in L^2(\mathbb{C}^n, e^{-2\Phi/h} L(dx))\}.$$

Here

$$\Pi_\Phi : L^2(\mathbb{C}^n, e^{-2\Phi/h} L(dx)) \to H_\Phi(\mathbb{C}^n)$$

is the orthogonal (Bergman) projection.

Example. Let $\Phi(x) = \frac{|x|^2}{2}$. We have

$$\mathrm{Top}(x_j) = x_j, \quad \mathrm{Top}(\overline{x}_j) = h\partial_{x_j}, \quad 1 \le j \le n.$$

The Bergman projection is given by

$$\Pi_\Phi u(x) = \frac{C}{h^n} \int e^{2\Psi(x,\overline{y})/h} u(y) e^{-2\Phi(y)/h} L(dy), \quad C = C_\Phi > 0.$$

Here Ψ is the polarization of Φ : the unique holomorphic quadratic form on $\mathbb{C}^{2n}_{x,y}$ such that $\Psi(x,\overline{x}) = \Phi(x)$, $x \in \mathbb{C}^n$.

Example. $\Phi(x) = \frac{|x|^2}{2} \Longrightarrow \Psi(x,y) = \frac{x \cdot y}{2}$.

J. Sjöstrand (1995), ...

Toeplitz vs Weyl quantization I

Let $h = 1$ and let $p \in L^\infty(\mathbb{C}^n)$, say. We have

$$\mathrm{Top}(p) = a^w(x, D_x).$$

Here the Weyl quantization $a^w(x, D_x)$ of $a \in C^\infty(\Lambda_\Phi)$ is given by

$$a^w(x, D_x)\, u(x) = \frac{1}{(2\pi)^n} \iint_{\Gamma_{\Phi(x)}} e^{i(x-y)\cdot\theta} a\left(\frac{x+y}{2}, \theta\right) u(y)\, dy \wedge d\theta,$$

where

$$\Lambda_\Phi = \left\{ \left(x, \frac{2}{i}\frac{\partial \Phi}{\partial x}(x)\right), x \in \mathbb{C}^n \right\} \subset \mathbb{C}^{2n} = \mathbb{C}^n_x \times \mathbb{C}^n_\xi$$

and $\Gamma_\Phi(x) \subset \mathbb{C}^{2n}_{y,\theta}$ is the natural contour of integration,

$$\theta = \frac{2}{i}\frac{\partial \Phi}{\partial x}\left(\frac{x+y}{2}\right).$$

Toeplitz vs Weyl quantization II

The Weyl symbol $a \in C^\infty(\Lambda_\Phi)$ is given by

$$a\left(x, \frac{2}{i}\frac{\partial \Phi}{\partial x}(x)\right) = \left(\exp\left(\frac{1}{4}(\Phi''_{x\overline{x}})^{-1}\partial_x \cdot \partial_{\overline{x}}\right) p\right)(x), \quad x \in \mathbb{C}^n.$$

The symbol of $(\Phi''_{x\overline{x}})^{-1}\partial_x \cdot \partial_{\overline{x}}$ is

$$-\frac{1}{4}(\Phi''_{x\overline{x}})^{-1}\overline{\zeta} \cdot \zeta < 0, \quad 0 \neq \zeta \in \mathbb{C}^n \simeq \mathbb{R}^{2n} \Longrightarrow$$

the Weyl symbol a is given by the forward heat flow acting on p.

V. Guillemin (1985), . . . , J. Sjöstrand (1994), ...

When is a Toeplitz operator bounded on $H_\Phi(\mathbb{C}^n)$?

Example (C. Berger – L. Coburn, 1994). Let $\Phi(x) = |x|^2/2$ and let

$$p(x) = \exp(\lambda |x|^2), \quad \lambda \in \mathbb{C}, \quad \mathrm{Re}\,\lambda < 1/2.$$

Explicit computations show that

$$\mathrm{Top}(p) \in \mathcal{L}(H_\Phi(\mathbb{C}^n), H_\Phi(\mathbb{C}^n)) \Longleftrightarrow |1-\lambda| \geq 1.$$

The Weyl symbol a can be computed by exact stationary phase and we see that

$$|1-\lambda| \geq 1 \Longleftrightarrow a \in L^\infty(\Lambda_\Phi).$$

The Berger-Coburn Conjecture

Conjecture (C. Berger – L. Coburn, 1994) For any "reasonable" Toeplitz symbol p [1], we have

$$\mathrm{Top}(p) \in \mathcal{L}(H_\Phi(\mathbb{C}^n), H_\Phi(\mathbb{C}^n)) \Longleftrightarrow \text{the Weyl symbol } a \in L^\infty(\Lambda_\Phi).$$

The conjecture still stands.

C. Berger – L. Coburn, 1994 : some partial results towards the conjecture.

1. such that $p\, e^{2\Psi(\cdot,\overline{y})} \in L^2_\Phi$, for all $y \in \mathbb{C}^n$. Here Ψ is the polarization of Φ.

Theorem (L. Coburn – J. Sjöstrand – M. H., 2019, 2023)

Let Φ be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ and let q be a complex valued quadratic form on $\mathbb{C}^n$. Assume that

$$\mathrm{Re}\, q(x) < \Phi_{\mathrm{herm}}(x) := \frac{1}{2}\left(\Phi(x) + \Phi(ix)\right), \quad 0 \neq x \in \mathbb{C}^n,$$

and that

$$\det \partial_{\overline{x}} \partial_x \left(2\Phi - q\right) \neq 0.$$

The Toeplitz operator

$$\mathrm{Top}(e^q) : H_\Phi(\mathbb{C}^n) \to H_\Phi(\mathbb{C}^n)$$

is bounded if and only if the Weyl symbol $a \in C^\infty(\Lambda_\Phi)$ of $\mathrm{Top}(e^q)$ satisfies $a \in L^\infty(\Lambda_\Phi)$. Furthermore, $\mathrm{Top}(e^q)$ is compact precisely when the Weyl symbol a vanishes at infinity.

More Toeplitz surprises

The composition problem : For which Toeplitz symbols f, g is there an h such that

$$\mathrm{Top}(f)\,\mathrm{Top}(g) = \mathrm{Top}(h)?$$

Example (L. Coburn, 2001) : let $\Phi(x) = \frac{|x|^2}{2}$. There exists $\lambda_0 \in \mathbb{C}$ with $0 < \mathrm{Re}\,\lambda_0 < 1/2$, $|\lambda_0 - 1| = 1$, such that the unitary operator

$$\mathrm{Top}(e^{\lambda_0 |x|^2}) : H_\Phi(\mathbb{C}^n) \to H_\Phi(\mathbb{C}^n)$$

satisfies : $\left(\mathrm{Top}(e^{\lambda_0 |x|^2})\right)^2$ is not a Toeplitz operator.

Question (G. Rozenblum, 2023) : Assume that $f, g \in L^\infty(\mathbb{C}^n)$. Does the composition property hold then ?

Semiclassical asymptotics for Bergman kernels

It would be nice to understand the Bergman projection for more general domains $\Omega \subset \mathbb{C}^n$ and non-quadratic exponential weights.

Let $\Omega \subset \mathbb{C}^n$ be open pseudoconvex and let $\Phi \in C^\infty(\Omega; \mathbb{R})$ be strictly plurisubharmonic,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \Phi(x)}{\partial x_j \partial \overline{x}_k} \zeta_j \overline{\zeta}_k > 0, \quad x \in \Omega, \quad 0 \neq \zeta \in \mathbb{C}^n.$$

Associated to Φ we introduce the Bergman space

$$H_\Phi(\Omega) = \mathrm{Hol}(\Omega) \cap L^2(\Omega, e^{-2\Phi/h} L(dx)).$$

Here $h \to 0^+$ is the semiclassical parameter and $L(dx)$ is the Lebesgue measure on $\mathbb{C}^n$.

We would like to understand the orthogonal (Bergman) projection

$$\Pi_\Phi : L^2(\Omega, e^{-2\Phi/h} L(dx)) \to H_\Phi(\Omega)$$

in the semiclassical limit $h \to 0^+$.

Weighted L^2 spaces of holomorphic functions I

Complex geometry/Toeplitz quantization : spaces of the form $H_\Phi(\Omega)$ serve as local models for the space of holomorphic sections of a high power of a holomorphic line bundle over a complex manifold.

F. Berezin (1975), . . . , G. Tian (1990), T. Bouche (1990), D. Catlin (1999), S. Zelditch (1998), ... R. Berman – B. Berndtsson – J. Sjöstrand (2008).

C. Fefferman (1974), L. Boutet de Monvel – J. Sjöstrand (1975) (asymptotics of the Bergman and Szegő kernels for strictly pseudoconvex smooth domains $\Subset \mathbb{C}^n$), . . . , M. Kashiwara (1977), A. Deleporte (2023) (the Szegő kernel for domains with analytic boundary).

Weighted L^2 spaces of holomorphic functions II

Let $\mathcal{L}$ be a complex line bundle over a complex compact n-dimensional manifold X, and assume that $\mathcal{L}$ is equipped with a C^∞ metric. The curvature is given by the $(1,1)$–form $\partial\overline{\partial}\Phi$, where locally $|s| = e^{-\Phi}$, for some local non-vanishing holomorphic section s of $\mathcal{L}$. We assume that the curvature form is strictly positive :

$$i\partial\overline{\partial}\Phi = i\sum_{j,k=1}^{n} \frac{\partial^2\Phi(x)}{\partial x_j \partial \overline{x}_k} dx_j \wedge d\overline{x}_k > 0,$$

so that the local weight Φ is strictly plurisubharmonic.

Weighted L^2 spaces of holomorphic functions III

Let $\mathcal{L}^k = \overbrace{\mathcal{L} \otimes \ldots \otimes \mathcal{L}}^{k \text{ times}}$. The Bergman projection is the orthogonal projection

$$\Pi_k : L^2(X; \mathcal{L}^k) \to \left(L^2 \cap \mathrm{Hol}\right)(X; \mathcal{L}^k).$$

Locally, we take a non-vanishing section s as before and represent general sections of $\mathcal{L}^k$ as us^k, where u is scalar. The asymptotic analysis of Π_k is therefore locally equivalent to the study of the orthogonal projection

$$\Pi_\Phi : L^2(\Omega, e^{-2\Phi/h} L(dx)) \to H_\Phi(\Omega).$$

Here the semiclassical parameter $h \to 0^+$ is the inverse of a high power $k \to \infty$ of the line bundle $\mathcal{L}$, $h = \dfrac{1}{k}$.

Weighted L^2 spaces of holomorphic functions IV

Exponentially weighted spaces of holomorphic functions occur naturally also in analytic microlocal analysis, in connection with FBI transforms.

Let $\varphi \in \mathrm{Hol}\left(\mathrm{neigh}\left((x_0, y_0), \mathbb{C}^{2n}\right)\right)$, $y_0 \in \mathbb{R}^n$, be such that

$$-\mathrm{Im}\, \varphi'_y(x_0, y_0) = 0, \quad \mathrm{Im}\, \varphi''_{yy}(x_0, y_0) > 0, \quad \det \varphi''_{xy}(x_0, y_0) \neq 0.$$

Associated to φ is the FBI transform

$$Tu(x; h) = h^{-3n/4} \int e^{i\varphi(x,y)/h} \chi(y)\, u(y),\, dy, \quad x \in \mathrm{neigh}(x_0, \mathbb{C}^n),$$

where $u \in L^2(\mathbb{R}^n)$, $\chi \in C_0^\infty(\mathrm{neigh}(y_0, \mathbb{R}^n))$, $\chi = 1$ near y_0.

J. Sjöstrand (1982).

Weighted L^2 spaces of holomorphic functions V

The FBI transform

$$Tu(x;h) = h^{-3n/4} \int e^{i\varphi(x,y)/h} \chi(y)\, u(y),\, dy, \quad x \in \mathrm{neigh}(x_0, \mathbb{C}^n),$$

satisfies

$$T = \mathcal{O}(1) : L^2(\mathbb{R}^n) \to H_\Phi(\Omega),$$

where $\Omega \subset \mathbb{C}^n$ is a small neighborhood of x_0 and the weight

$$\Phi(x) = \sup_{y \in \mathrm{neigh}(y_0, \mathbb{R}^n)} (-\mathrm{Im}\, \varphi(x,y))$$

is strictly plurisubharmonic.

The Catlin-Zelditch expansion

Introducing the Schwartz kernel of Π_Φ, let us write

$$\Pi_\Phi u(x) = \int_\Omega K(x,\overline{y})u(y)e^{-2\Phi(y)/h}\, L(dy),$$

where $K(x,\widetilde{y}) \in \mathrm{Hol}(\Omega \times \overline{\Omega})$. The existence of a complete asymptotic expansion for the Bergman kernel K, as $h \to 0^+$, has been established by D. Catlin and S. Zelditch.

Let $x_0 \in \Omega$ and let $\Psi \in C^\infty\left(\mathrm{neigh}\left((x_0,\overline{x_0}),\mathbb{C}^{2n}\right)\right)$ be a polarization of Φ, i.e.

$$\Psi(x,\overline{x}) = \Phi(x),$$

and $\forall N$,

$$\left(\partial_{\overline{x}}\Psi\right)(x,y) = \mathcal{O}_N(|x-\overline{y}|^N), \quad \left(\partial_{\overline{y}}\Psi\right)(x,y) = \mathcal{O}_N(|x-\overline{y}|^N).$$

In other words, Ψ is an almost holomorphic extension of Φ. L. Hörmander (1969).

Theorem (D. Catlin 1999, S. Zelditch 1998)

There exists a classical elliptic symbol of the form

$$a(x,\widetilde{y};h) \sim \sum_{j=0}^{\infty} a_j(x,\widetilde{y})h^j, \quad (x,\widetilde{y}) \in \mathrm{neigh}\left((x_0,\overline{x_0}),\mathbb{C}^{2n}\right),$$

with $a_j \in C^\infty$ *holomorphic to* ∞*-order along the anti-diagonal, such that on the level of effective kernels, i.e. for the kernel of the operator* $e^{-\Phi/h} \circ \Pi_\Phi \circ e^{\Phi/h}$, *we have*

$$e^{-\Phi(x)/h}\left(K(x,\overline{y}) - \frac{1}{h^n}e^{2\Psi(x,\overline{y})/h}a(x,\overline{y};h)\right)e^{-\Phi(y)/h} = \mathcal{O}(h^\infty).$$

Remark. The original proofs of Catlin and Zelditch rely on a reduction to the main result of Boutet – Sjöstrand, which depends, in turn, on the full fledged machinery of the theory of Fourier integral operators with complex phase functions by A. Melin – J. Sjöstrand (1974).

A direct approach to Bergman projections

It would be nice to have a direct approach to the Bergman kernel asymptotics, not relying on any heavy machinery, which would also be more self-contained and explicit.

One such approach has been developed by Berman – Berndtsson – Sjöstrand (2008).

Direct approach by Berman-Berndtsson-Sjöstrand

Main idea : Express the identity operator on $H_\Phi(\Omega)$ in a nice way so that it automatically becomes the (asymptotic) Bergman projection.

Starting point : write the identity as a semiclassical pseudodifferential operator on H_Φ : let $U \Subset V \Subset \Omega$ be small open neighborhoods of $x_0 \in \Omega$. We have

$$u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma(x)} e^{\frac{i}{h}(x-y)\cdot\eta} u(y)\, dy\, d\eta + \mathcal{O}(1) || u ||_{H_\Phi(V)} e^{\frac{1}{h}(\Phi(x)-\eta)}, \quad x \in U, \quad u \in H_\Phi(V), \quad (1)$$

for some $\eta > 0$.

How do we choose the contour of integration in (1) ?

Good contours

We say that a $2n$-dimensional contour $\Gamma(x) \subset \mathbb{C}^{2n}_{y,\eta}$ is a good contour for the plurisubharmonic function

$$(y,\eta) \mapsto -\mathrm{Im}\,((x-y)\cdot\eta) + \Phi(y)$$

if it passes through the critical point $(y,\eta) = (x, \frac{2}{i}\partial_x \Phi(x))$ and we have

$$-\mathrm{Im}\,((x-y)\cdot\eta) + \Phi(y) \leq \Phi(x) - \frac{1}{C}\mathrm{dist}\left((y,\eta), \left(x, \frac{2}{i}\partial_x\Phi(x)\right)\right)^2,$$

along $\Gamma(x)$. J. Sjöstrand (1982). Any good contour works in (1).

Example. The contour

$$\eta = \frac{2}{i}\frac{\partial\Phi}{\partial x}\left(\frac{x+y}{2}\right) + \frac{i}{C}\overline{(x-y)}, \quad C > 1,$$

is good.

Non-standard phase

We would like to express the identity operator with a non-standard phase,

$$u(x) = \frac{1}{h^n} \iint_{\widetilde{\Gamma}(x)} e^{\frac{2}{h}(\Psi(x,\theta)-\Psi(y,\theta))} a(x,y,\theta;h) u(y)\, dy\, d\theta \\ + \mathcal{O}(1) \| u \|_{H_\Phi(V)} e^{\frac{1}{h}(\Phi(x)-\eta)}, \quad x \in U, \quad u \in H_\Phi(V), \quad (2)$$

observing that for this new representation, the contour $\theta = \overline{y}$ is good,

$$u(x) = \frac{1}{h^n} \int e^{\frac{2}{h}\Psi(x,\overline{y})} a(x,y,\overline{y};h) u(y) e^{-\frac{2}{h}\Phi(y)} dy\, d\overline{y} \\ + \mathcal{O}(1) \| u \|_{H_\Phi(V)} e^{\frac{1}{h}(\Phi(x)-\eta)}, \quad (3)$$

in view of the basic estimate

$$2\mathrm{Re}\, \Psi(x,\overline{y}) - \Phi(x) - \Phi(y) \asymp - |x-y|^2 ,$$

valid near the diagonal.

Kuranishi trick

To see that the representations (1) and (2) are equivalent, we use the Kuranishi trick. Assume for simplicity that Φ is real analytic and write, by Taylor's formula,

$$\frac{2}{i}\left(\Psi(x,\theta)-\Psi(y,\theta)\right)=(x-y)\cdot\eta(x,y,\theta).$$

We can therefore pass between (x,y,θ) and (x,y,η) by a change of variables, obtaining the representation (3) for the identity operator,

$$u(x)=\frac{1}{h^n}\int e^{\frac{2}{h}\Psi(x,\overline{y})}a(x,y,\overline{y};h)u(y)e^{-\frac{2}{h}\Phi(y)}dy\,d\overline{y}$$

$$+\mathcal{O}(1)||\,u\,||_{H_\Phi(V)}e^{\frac{1}{h}(\Phi(x)-\eta)}.$$

Eliminating the y dependence in the amplitude

The representation (3) looks almost like the Bergman projection, but we still need to eliminate the y dependence in a. To this end we show that there exists a symbol $b(x,y,\theta;h)$ with values in $(n-1)$-forms in θ such that for some

$$\widetilde{a}(x,\theta;h) = a(x,x,\theta;h) + \mathcal{O}(h)$$

uniquely determined, we have with $\mathcal{O}(h^\infty)$–errors,

$$e^{\frac{2}{h}(\Psi(x,\theta)-\Psi(y,\theta))}\left(a(x,y,\theta;h) - \widetilde{a}(x,\theta;h)\right)\, d\theta$$
$$= h d_\theta \left(e^{\frac{2}{h}(\Psi(x,\theta)-\Psi(y,\theta))} b(x,y,\theta;h)\right).$$

This can be accomplished by successive division procedures.

Asymptotic Bergman projection

We finally get an approximate reproducing property in the C^∞ setting,

$$u(x) = \underbrace{\frac{1}{h^n} \int e^{\frac{2}{h}\Psi(x,\overline{y})} \widetilde{a}(x,\overline{y};h) u(y) e^{-\frac{2}{h}\Phi(y)} dy\, d\overline{y}}_{\widetilde{\Pi} u}$$

$$+ Ku, \quad x \in U, \quad u \in H_\Phi(V),$$

where

$$K = \mathcal{O}(h^\infty) : H_\Phi(V) \to L^2(U, e^{-2\Phi/h} L(dx)).$$

To show that the operator $\widetilde{\Pi}$ is an approximation of the honest orthogonal projection Π_Φ, as $h \to 0^+$, we use Hörmander's L^2–estimates for $\overline{\partial}$.

Analytic weights

Let us recall the Catlin-Zelditch expansion,

$$K(x, \overline{y}) = \frac{1}{h^n} e^{\frac{2}{h}\Psi(x,\overline{y})} a(x, \overline{y}; h) + \mathcal{O}(h^\infty) e^{\frac{1}{h}(\Phi(x)+\Phi(y))}, \quad x, y \in \mathrm{neigh}(x_0, \mathbb{C}^n). \quad (4)$$

Assume now that the weight Φ is real analytic. We can then choose the polarization Ψ to be holomorphic. We expect the amplitude in (4) to be a classical analytic symbol and the remainder in (4) to be exponentially small.

Classical analytic symbols

Let $U \subset \mathbb{C}^n$ be open. A (formal) classical analytic symbol is given by

$$a(x;h) = \sum_{j=0}^{\infty} a_j(x) h^j,$$

where $a_j \in \mathrm{Hol}(U)$ are such that for each $\widetilde{U} \Subset U$ there exists $C = C_{\widetilde{U}} > 0$ such that

$$|a_j(x)| \leq C^{j+1} j^j, \quad j = 0, 1, 2, \dots, x \in \widetilde{U}.$$

L. Boutet de Monvel – P. Krée (1967), J. Sjöstrand (1982).

We have a realization of a on $\widetilde{U}$ obtained by performing "la sommation au plus petit terme",

$$a_{\widetilde{U}}(x;h) = \sum_{0 \leq j \leq \left(C_{\widetilde{U}} e h\right)^{-1}} a_j(x) h^j \in \mathrm{Hol}(\widetilde{U}).$$

Theorem (O. Rouby – J. Sjöstrand – S. Vu Ngoc 2020, A. Deleporte 2021)

In the analytic case, there exists a classical analytic symbol $a(x, \widetilde{y}; h)$ *defined in a neighborhood of* $(x_0, \overline{x_0})$*, such that, taking a realization of* a*, we have*

$$e^{-\Phi(x)/h}\left(K(x,\overline{y}) - \frac{1}{h^n}e^{2\Psi(x,\overline{y})/h}a(x,\overline{y};h)\right)e^{-\Phi(y)/h} = \mathcal{O}(1)\, e^{-1/Ch}, \quad C > 0.$$

Alternative proofs have been given by L. Charles (2021) and H. Hezari – H. Xu (2021).

One word about the proof by Rouby – Sjöstrand – Vu Ngoc

- This proof follows some of the main ideas of the approach by Berman – Berndtsson – Sjöstrand, and in particular, the Kuranishi trick is still an essential ingredient.
- Much more care is needed to be able to pass between the Bergman and the pseudodifferential representations, without dropping out of the class of classical analytic symbols $\Rightarrow$ the use of the Kuranishi trick leads to some difficulties.

Getting rid of the Kuranishi trick

The Kuranishi trick is very nice and useful when it can be applied easily but may also break down, particularly in situations when the Levi form $i\partial\overline{\partial}\Phi \geq 0$ of Φ becomes degenerate or nearly degenerate.

Example. (J. Sjöstrand – M.H., work in progress). We develop a heat evolution approach to second microlocalization with respect to a real analytic hypersurface of the form

$$\Sigma = r^{-1}(0) \subset \mathbb{R}^{2n},$$

where

$$r(y, \eta) = \sum_{j,k=1}^{n} g^{jk}(y)\eta_j\eta_k - 1,$$

$(g^{jk}) = (g_{jk})^{-1}$, $g = (g_{jk})$ is a real analytic Riemannian metric on $\mathbb{R}^n$.

Passing to the FBI transform side,

$$T : L^2(\mathbb{R}^n) \to H_{\Phi_0}(\Omega),$$

we study the heat evolution semigroup

$$e^{-tP^2/2h} = \mathcal{O}(1) : H_{\Phi_0} \to H_{\Phi_t}, \quad t \geq 0.$$

Here

$$P = P(x, hD_x; h) = T \circ (-h^2\Delta_g - 1) \circ T^{-1} : H_{\Phi_0} \to H_{\Phi_0}.$$

The time evolution of the exponential weight $\Phi_t(x)$ is governed by the real Hamilton-Jacobi equation,

$$\partial_t \Phi_t(x) + \frac{1}{2} p^2 \left(x, \frac{2}{i} \partial_x \Phi_t(x) \right) = 0, \quad \Phi_{t=0} = \Phi_0,$$

which remains well-posed for all $t \geq 0$ (p is the leading symbol of P).

When performing a second microlocalization in an $\mathcal{O}(\mu)$–neighborhood of Σ, where $\mu \asymp h^\delta$, $\frac{1}{2} < \delta < 1$, we consider large times

$$t = \frac{1}{\mu},$$

for which the deformed weight Φ_t starts exhibiting some degenerate behavior as $\mu \to 0^+$.

The model weight corresponding to $p(x,\xi) = \xi_n$:

$$\Phi_t(x) = \frac{1}{2}(\mathrm{Im}\, x')^2 + \frac{\mu}{2(1+\mu)}(\mathrm{Im}\, x_n)^2, \quad x = (x', x_n) \in \mathbb{C}^{n-1} \times \mathbb{C} = \mathbb{C}^n.$$

The weight Φ_t is strictly plurisubharmonic but not uniformly as $\mu \to 0^+$,

$$\Phi''_{t,\overline{x}x}(x)\zeta \cdot \overline{\zeta} \geq \frac{\mu\, |\zeta|^2}{\mathcal{O}(1)}.$$

A direct approach to Bergman projections, not relying upon the Kuranishi trick, has been developed with A. Deleporte and J. Sjöstrand, in the strictly plurisubharmonic real analytic case. It has then been adapted to the C^∞ case with M. Stone.

Theorem (A. Deleporte – J. Sjöstrand – M. H. 2023, analytic case)

Let $\Omega \subset \mathbb{C}^n$ be open and let Φ be strictly plurisubharmonic in Ω such that Φ is real analytic near $x_0 \in \Omega$. There exist a unique classical analytic symbol $a(x, \widetilde{y}; h)$ defined near $(x_0, \overline{x}_0)$, solving

$$Aa = 1 + \mathcal{O}(e^{-1/Ch}), \quad C > 0,$$

near $(x_0, \overline{x}_0)$, where A is an explicit elliptic analytic Fourier integral operator, and small open neighborhoods $U \Subset V \Subset \Omega$ of x_0, such that the operator

$$\widetilde{\Pi} u(x) = \frac{1}{h^n} \int_V e^{\frac{2}{h}\Psi(x,\overline{y})} a(x, \overline{y}; h) u(y) e^{-\frac{2}{h}\Phi(y)} L(dy)$$

satisfies

$$\widetilde{\Pi} - 1 = \mathcal{O}(1) e^{-\frac{1}{Ch}} : H_\Phi(V) \to H_\Phi(U), \quad C > 0.$$

Here $\Psi \in Hol(V \times \overline{V})$ is the polarization of Φ.

Theorem (M. Stone – M. H. 2022, C^∞ case)

Let $\Omega \subset \mathbb{C}^n$ be open and let $\Phi \in C^\infty(\Omega)$ be strictly plurisubharmonic in Ω. Let $x_0 \in \Omega$. There exists a classical C^∞ symbol $a(x, \widetilde{y}; h) \sim \sum_{j=0}^\infty a_j(x, \widetilde{y}) h^j$, defined near $(x_0, \overline{x}_0)$, with $a_j \in C^\infty$ holomorphic to ∞–order along the anti-diagonal $\widetilde{y} = \overline{x}$, solving

$$(Aa)(x, \overline{x}; h) = 1 + \mathcal{O}(h^\infty),$$

near x_0, where A is an explicit elliptic Fourier integral operator, and small open neighborhoods $U \Subset V \Subset \Omega$ of x_0, such that the operator

$$\widetilde{\Pi} u(x) = \frac{1}{h^n} \int_V e^{\frac{2}{h}\Psi(x, \overline{y})} a(x, \overline{y}; h) u(y) e^{-\frac{2}{h}\Phi(y)} L(dy)$$

satisfies

$$\widetilde{\Pi} - 1 = \mathcal{O}(h^\infty) : H_\Phi(V) \to L^2(U, e^{-2\Phi/h} L(dx)).$$

Here $\Psi \in C^\infty(V \times \overline{V})$ is a polarization of Φ.

A couple of words about the proofs

To fix the ideas, let us discuss the analytic case first.

We would like to construct an operator of the form

$$\widetilde{\Pi} u(x) = \frac{1}{h^n} \int_V e^{\frac{2}{h}\Psi(x,\overline{y})} a(x,\overline{y};h) u(y) e^{-\frac{2}{h}\Phi(y)} L(dy), \quad u \in H_\Phi(V),$$

where V is a small neighborhood of x_0, enjoying the reproducing property $\widetilde{\Pi} u = u$, at least approximately.

Main idea : demand that the reproducing property should hold in the weak formulation,

$$(\widetilde{\Pi} u, v)_{H_\Phi} = (u,v)_{H_\Phi} + \mathcal{O}(e^{-1/Ch}) \|u\|_{H_\Phi} \|v\|_{H_\Phi},$$

for some $C > 0$ and all $u, v \in H_\Phi(V)$. This should imply that

$$\widetilde{\Pi} - 1 = \mathcal{O}(1)\, e^{-1/Ch} : H_\Phi(V) \to H_\Phi(V).$$

Switching to the polarized form

Let us write in the polarized form,

$$\widetilde{\Pi}u(x) = \frac{1}{h^n} \iint_{\Gamma} e^{\frac{2}{h}(\Psi(x,\widetilde{y}) - \Psi(y,\widetilde{y}))} a(x, \widetilde{y}; h) u(y)\, dy\, d\widetilde{y}, \quad u \in H_{\Phi},$$

where $\Gamma = \{(y, \widetilde{y}); \widetilde{y} = \overline{y}\} \subset \mathbb{C}^{2n}_{y,\widetilde{y}}$ is the anti-diagonal, and similarly for the scalar product in H_{Φ},

$$(u, v)_{H_{\Phi}} = \int u(x)\overline{v(x)} e^{-\frac{2}{h}\Phi(x)}\, dx\, d\overline{x} = \iint_{\Gamma} u(x) v^{*}(\widetilde{x}) e^{-\frac{2}{h}\Psi(x,\widetilde{x})} dx\, d\widetilde{x}.$$

Here $\Gamma = \{(x, \widetilde{x}); \widetilde{x} = \overline{x}\} \subset \mathbb{C}^{2n}_{x,\widetilde{x}}$ and $v^{*}(\widetilde{x}) = \overline{v(\overline{\widetilde{x}})}$.

Reshuffling the order of integration

Suppressing the contours of integration, we get by a formal application of Fubini's theorem,

$$(\widetilde{\Pi}u, v)_{H_\Phi} =$$

$$\iint \left(\overbrace{\frac{1}{h^n} \iint e^{\frac{2}{h}(\Psi(x,\widetilde{y})-\Psi(y,\widetilde{y}))} a(x,\widetilde{y};h) u(y)\, dy\, d\widetilde{y}}^{\widetilde{\Pi}u} \right) v^*(\widetilde{x}) e^{-\frac{2}{h}\Psi(x,\widetilde{x})} dx\, d\widetilde{x}$$

$$= \frac{1}{h^n} \iint \left(\iint e^{\frac{2}{h}(\Psi(x,\widetilde{y})-\Psi(y,\widetilde{y})-\Psi(x,\widetilde{x}))} a(x,\widetilde{y};h)\, dx\, d\widetilde{y} \right) u(y) v^*(\widetilde{x}) dy\, d\widetilde{x}.$$

This expression agrees with $(u, v)_{H_\Phi}$ provided that

$$\frac{1}{h^n} \iint e^{\frac{2}{h}(\Psi(x,\widetilde{y})-\Psi(y,\widetilde{y})-\Psi(x,\widetilde{x}))} a(x,\widetilde{y};h)\, dx\, d\widetilde{y} = e^{-\frac{2}{h}\Psi(y,\widetilde{x})}. \tag{5}$$

A special canonical transformation

Let us rewrite (5) as follows,

$$\frac{1}{h^n}\iint e^{\frac{2}{h}\varphi(y,\widetilde{x};x,\widetilde{y})}a(x,\widetilde{y};h)\,dx\,d\widetilde{y}=1, \tag{6}$$

where

$$\varphi(y,\widetilde{x};x,\widetilde{y})=\Psi(x,\widetilde{y})-\Psi(y,\widetilde{y})-\Psi(x,\widetilde{x})+\Psi(y,\widetilde{x})$$

is holomorphic near $(x_0,\overline{x_0};x_0,\overline{x_0})\in\mathbb{C}^{4n}$.

We have

$$\det\varphi''_{(y,\widetilde{x}),(x,\widetilde{y})}(x_0,\overline{x_0};x_0,\overline{x_0})\neq 0 \quad\Longrightarrow$$

φ is a generating function for the canonical transformation

$$\kappa:\left(x,\widetilde{y};-\frac{2}{i}\partial_x\varphi,-\frac{2}{i}\partial_{\widetilde{y}}\varphi\right)\mapsto\left(y,\widetilde{x};\frac{2}{i}\partial_y\varphi,\frac{2}{i}\partial_{\widetilde{x}}\varphi\right).$$

The operator

$$(Aa)(y, \widetilde{x}; h) = \frac{1}{h^n} \iint e^{\frac{2}{h}\varphi(y,\widetilde{x};x,\widetilde{y})} a(x, \widetilde{y}; h)\, dx\, d\widetilde{y}$$

is therefore a (formal) elliptic analytic Fourier integral operator quantizing κ, which takes functions of $(x, \widetilde{y})$ to functions of $(y, \widetilde{x})$.

What are the critical points of the phase function $(x, \widetilde{y}) \mapsto \varphi(y, \widetilde{x}; x, \widetilde{y})$? We have a unique critical point given by $(x, \widetilde{y}) = (y, \widetilde{x})$, which is non-degenerate,

$$\det \varphi''_{(x,\widetilde{y}),(x,\widetilde{y})}(x_0, \overline{x_0}; x_0, \overline{x_0}) \neq 0,$$

with the critical value $0 \Longrightarrow$

Main observation : the canonical transformation κ maps the zero section $\{\eta = 0\} \subset \mathbb{C}^{4n}_{y,\eta}$ to the zero section $\{\xi = 0\} \subset \mathbb{C}^{4n}_{x,\xi}$.

A non-degenerate saddle point and a good contour

It follows that there exists a good contour $\Gamma(y,\widetilde{x}) \subset \mathbb{C}^{2n}_{x,\widetilde{y}}$ for the pluriharmonic function

$$(x,\widetilde{y}) \mapsto \mathrm{Re}\,\varphi(y,\widetilde{x};x,\widetilde{y}),$$

passing through the critical point $(x,\widetilde{y}) = (y,\widetilde{x})$ and such that

$$\mathrm{Re}\,\varphi(y,\widetilde{x};x,\widetilde{y}) \leq -\frac{1}{C}\mathrm{dist}\left((x,\widetilde{y}),(y,\widetilde{x})\right)^2,$$

along $\Gamma(y,\widetilde{x})$.

Remark. The following explicit choice of a good contour $\Gamma(y,\widetilde{x})$ is particularly pleasant to work with in the C^∞ case :

$$\Gamma(y,\widetilde{x}) = (y,\widetilde{x}) + i\Lambda = \{y+z, \widetilde{x}-\overline{z}; z \in \mathrm{neigh}(0,\mathbb{C}^n)\},$$

where $\Lambda = \{(z,\overline{z}); z \in \mathbb{C}^n\} \subset \mathbb{C}^{2n}$ is the anti-diagonal.

Solving $Aa = 1$

The corresponding realization of the Fourier integral operator A,

$$(A_\Gamma a)(y, \widetilde{x}; h) = \frac{1}{h^n} \iint_{\Gamma(y,\widetilde{x})} e^{\frac{2}{h}\varphi(y,\widetilde{x};x,\widetilde{y})} a(x, \widetilde{y}; h)\, dx\, d\widetilde{y}$$

maps the space of classical analytic symbols defined near $(x_0, \overline{x_0})$ to itself, and since it is elliptic, the same holds for its microlocal inverse. There exists therefore a unique classical analytic symbol $a(x, \widetilde{y}; h)$ such that

$$A_\Gamma a = 1 + \mathcal{O}(e^{-1/Ch}), \quad C > 0.$$

Justifying the formal computation

Our sincere hope is that the classical analytic symbol

$$a(x,\widetilde{y};h) \sim \sum_{j=0}^{\infty} a_j(x,\widetilde{y})h^j,$$

that we have just constructed is the amplitude of the asymptotic Bergman projection, but this has to be justified. Indeed, an application of Fubini's theorem in the beginning of the proof was purely formal, and we still need to justify the change of the order of integration, using good contours.

Basic difficulty : the natural contour $\Gamma = \{(x,\overline{x}); x \in \mathbb{C}^n\}$ occurring in the polarized expression for the H_Φ scalar product,

$$(u,v)_{H_\Phi} = \iint_\Gamma u(x)v^*(\widetilde{x})e^{-\frac{2}{h}\Psi(x,\widetilde{x})}dx\,d\widetilde{x},$$

is not quite good :

$$\left(\Phi(x) + \Phi(\overline{\widetilde{x}}) - 2\mathrm{Re}\,\Psi(x,\widetilde{x})\right)|_\Gamma = 0.$$

A resolution of the identity in H_Φ

To overcome this difficulty, in the actual proof we work with a "coherent states decomposition" of $v \in H_\Phi$ of the form

$$v(x) = \int_{\mathrm{neigh}(x_0,\mathbb{C}^n)} v_y(x) L(dy) + \mathcal{O}(1)\|v\|_{H_\Phi} e^{\frac{1}{h}\left(\Phi(x) - \frac{1}{C}\right)},$$

where

$$v_y \in H_{\Phi_y}, \quad \Phi_y(x) \le \Phi(x) - |x-y|^2/C.$$

Such a decomposition can be obtained by the Fourier inversion formula in the space H_Φ (cf. with the approach by Berman-Berndtsson–Sjöstrand). Working with the individual coherent states v_y, sufficiently localised near y, we regain the good contour property and the contour deformation argument can then be justified using Stokes' formula.

One word about the C^∞ case

Working systematically with almost holomorphic extensions, we still have to solve the equation

$$(A_\Gamma a)(x, \overline{x}; h) = 1,$$

with an $\mathcal{O}(h^\infty)$ error. Using complex stationary phase, we find that there exists an elliptic symbol

$$a(x, \widetilde{y}; h) \sim \sum_{j=0}^{\infty} a_j(x, \widetilde{y}) h^j$$

in the C^∞ sense, defined near $(x_0, \overline{x}_0)$, with $a_j \in C^\infty$ holomorphic to ∞–order along the anti-diagonal, such that

$$(A_\Gamma a)(x, \overline{x}; h) = 1 + \mathcal{O}(h^\infty), \quad x \in \mathrm{neigh}(x_0, \mathbb{C}^n).$$

We have $a_0(x, \overline{x}) = A_n \det\left(\Phi''_{x\overline{x}}(x)\right) \neq 0$, $A_n \neq 0$.

From asymptotic to exact Bergman projections I

In this talk, we have been constructing asymptotic Bergman projections, but how do we know that they really approximate the honest orthogonal projection

$$\Pi_\Phi : L^2(V, e^{-2\Phi/h} L(dx)) \to H_\Phi(V)$$

in the semiclassical limit $h \to 0^+$?

To this end, we first observe that in the C^∞ case, say, we have

$$\overline{\partial} \circ \widetilde{\Pi} = \mathcal{O}(h^\infty) : L^2(V, e^{-2\Phi/h} L(dx)) \to L^2(V, e^{-2\Phi/h} L(dx)).$$

An application of Hörmander's L^2–estimates for $\overline{\partial}$ gives therefore that

$$\Pi_\Phi \widetilde{\Pi} - \widetilde{\Pi} = \mathcal{O}(h^\infty) : L^2(V, e^{-2\Phi/h} L(dx)) \to L^2(V, e^{-2\Phi/h} L(dx)).$$

Here we tacitly assume that the domain V is pseudoconvex, which is OK since we work locally.

From asymptotic to exact Bergman projections II

Introducing the adjoint $\widetilde{\Pi}^*$ of $\widetilde{\Pi}$ in $L^2(V, e^{-2\Phi/h}L(dx))$, we also get the same conclusion for $\widetilde{\Pi}^*$,

$$\Pi_\Phi \widetilde{\Pi}^* - \widetilde{\Pi}^* = \mathcal{O}(h^\infty) : L^2(V, e^{-2\Phi/h}L(dx)) \to L^2(V, e^{-2\Phi/h}L(dx)).$$

Taking the adjoints of this relation we get

$$\widetilde{\Pi}\Pi_\Phi - \widetilde{\Pi} = \mathcal{O}(h^\infty) : L^2(V, e^{-2\Phi/h}L(dx)) \to L^2(V, e^{-2\Phi/h}L(dx)).$$

Combining this with the reproducing property for $\widetilde{\Pi}$,

$$\widetilde{\Pi}\Pi_\Phi - \Pi_\Phi = \mathcal{O}(h^\infty) : L^2(V, e^{-2\Phi/h}L(dx)) \to L^2(U, e^{-2\Phi/h}L(dx)).$$

we obtain

$$\widetilde{\Pi} - \Pi_\Phi = \mathcal{O}(h^\infty) : L^2(V, e^{-2\Phi/h}L(dx)) \to L^2(U, e^{-2\Phi/h}L(dx)).$$

Perspectives I

- Attack the case of Gevrey exponential weights : assume that $\Phi \in \mathcal{G}^s(\Omega)$, for some $s > 1$, in the sense that for each $\widetilde{\Omega} \Subset \Omega$ there exists $C = C_{\widetilde{\Omega}} > 0$ such that for all α, $\beta \in \mathbb{N}^n$ we have

$$\left|\partial_x^\alpha \partial_{\overline{x}}^\beta \Phi(x)\right| \leq C^{1+|\alpha|+\beta}(\alpha!\beta!)^s, \quad x \in \widetilde{\Omega}.$$

There exists in this case a polarization $\Psi \in \mathcal{G}^s$ of Φ and therefore,

$$\left|\overline{\partial}_{x,y}\Psi(x,y)\right| \leq \mathcal{O}(1) \exp\left(-\frac{1}{C}\left|y - \overline{x}\right|^{-\frac{1}{s-1}}\right).$$

L. Carleson (1961), J. Mather (1971).

Perspectives II

- Can we show that the amplitude of the asymptotic Bergman projection

$$a(x, \widetilde{y}; h) \sim \sum_{j=0}^{\infty} a_j(x, \widetilde{y}) h^j$$

is a Gevrey symbol in the natural sense, so that

$$|\partial^\alpha a_j| \leq C^{1+|\alpha|+j} (\alpha!)^s (j!)^s.$$

H. Xu (2018), Y. Guedes Bonthonneau – M. Jézéquel (2020), R. Lascar – J. Sjöstrand – M. Zerzeri – M. H. (2023) (H_Φ–techniques for semiclassical Gevrey operators). The Gevrey WKB method ? R. Lascar – I. Moyano (2023).

- Apply the direct approach to degenerate situations.

THANK YOU VERY MUCH FOR YOUR ATTENTION !