Equivalence of fluctuations between SHE and KPZ equation in weak disorder regime

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1. Overview
2. Previous research on SHE and KPZ equation
3. Main result

An experiment for wetting region

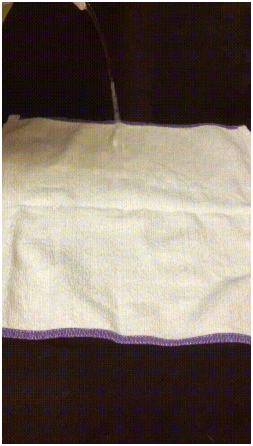


Figure 1: Experiment

KPZ universality

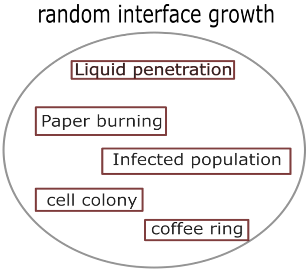


Figure 2: Random interface growth

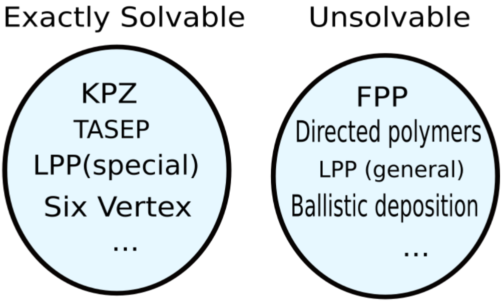
Karder-Parisi-Zhang (KPZ) universality conjecture (1986)

The above phenomena are described by the following equation:

$$\partial_t h = \frac{1}{2}\Delta h + \frac{1}{2}|\nabla h|^2 + \beta\xi.$$

KPZ universality II

$d = 1$(KPZ Universality Class)



$d \geq 2$

- Any model is so far unsolvable.
- Construction of a solution to KPZ equation is harder.

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KPZ equation

Fix $\beta > 0$. The following is called the KPZ equation:

For $t \in [0, \infty)$ and $x \in \mathbb{R}^d$,

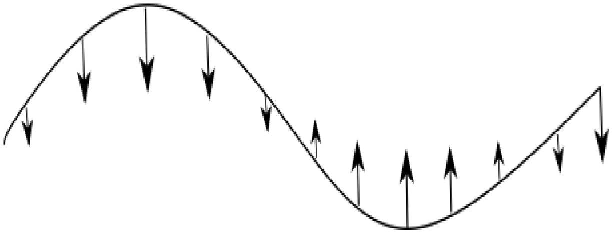
$$\begin{aligned}\partial_t h(t,x) &= \frac{1}{2}\Delta h(t,x) + \frac{1}{2}\left|\nabla h(t,x)\right|^2 + \beta\xi(t,x) \\ &= \frac{1}{2}\sum_{i=1}^{d}\partial_{x_i}^2 h(t,x) + \frac{1}{2}\sum_{i=1}^{d}(\partial_{x_i} h(t,x))^2 + \beta\xi(t,x),\end{aligned}$$

where $\xi(t,x)$ is a space-time white noise on $[0,\infty) \times \mathbb{R}^d$.

The meaning of each term

$$\partial_t h(t,x) = \frac{1}{2}\underbrace{\Delta h(t,x)}_{\text{relaxation}} + \frac{1}{2}\underbrace{|\nabla h(t,x)|^2}_{\text{lateral growth}} + \underbrace{\beta\xi(t,x)}_{\text{noise}}$$

$\Delta h(t,x)$: relaxation



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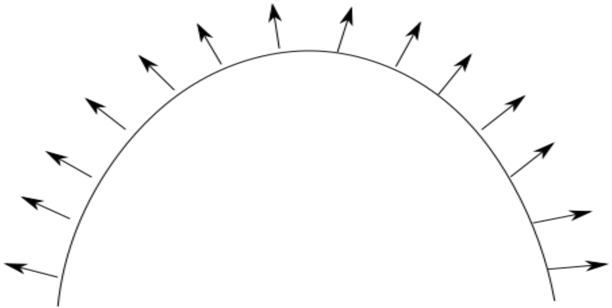
$\frac{1}{2}|\nabla h(t,x)|^2$: lateral growth



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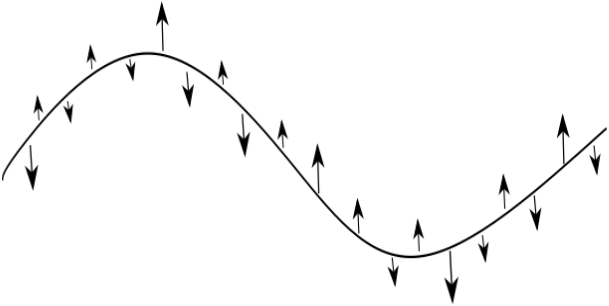
$\beta\xi(t,x)$: noise



Problem of the construction of solutions to the KPZ eq.

KPZ equation

$$\partial_t h(t,x) = \frac{1}{2}\Delta h(t,x) + \frac{1}{2}\left|\nabla h(t,x)\right|^2 + \beta\xi(t,x).$$

Problem

- ξ is not a function $\Rightarrow$ $h(t,x)$ may not be a function.
- How to make sense of $\left|\nabla h(t,x)\right|^2$?

Problem of the construction of solutions to the KPZ eq.

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- ξ is not a function $\Rightarrow$ $h(t,x)$ may not be a function.
- How to make sense of $|\nabla h(t,x)|^2$?

Scheme

- Mollify the noise.
- Consider a (smooth) solution to the mollified KPZ equation.
- We switch off the mollification and consider the limit of solutions.

Regularization scheme

Let ϕ be a smooth and compactly supported function on $\mathbb{R}^d$.
Mollify the white noise $\xi(t,x)$ in space on scale ϵ:

$$\xi^\epsilon(t,x) := \int \phi^\epsilon(x-y)\xi(t,y)\mathrm{d}y \Rightarrow \xi(t,x),$$

where $\phi^\epsilon(x) := \epsilon^{-d}\phi(\epsilon^{-1}x)$.

Let us consider:

$$\partial_t h_\epsilon(t,x) = \frac{1}{2}\Delta h_\epsilon(t,x) + \frac{1}{2}\left|\nabla h_\epsilon(t,x)\right|^2 + \beta\xi^\epsilon(t,x)$$

Q. h_ϵ converges as $\epsilon \to 0$?

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Q. h_ϵ converges as $\epsilon \to 0$?

A. No! We have to modify the equation more.

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where $\phi^\epsilon(x) := \epsilon^{-d}\phi(\epsilon^{-1}x)$.

We introduce new parameters and consider the regularized KPZ:

$$\partial_t h_\epsilon(t,x) = \frac{1}{2}\Delta h_\epsilon(t,x) + \frac{1}{2}\left|\nabla h_\epsilon(t,x)\right|^2 + \beta_\epsilon \xi^\epsilon(t,x) - C_\epsilon.$$

Can we take β_ϵ and C_ϵ so that h_ϵ converges as $\epsilon \to 0$?

Choice of β_ϵ and C_ϵ

Recall that $\xi^\epsilon(t,x) := \int \phi^\epsilon(x-y)\xi(t,y)\mathrm{d}y$.

The choice of β_ϵ

For fixed $\hat{\beta} \in (0,\infty)$, we choose

$$\beta_\epsilon = \begin{cases} \hat{\beta} & (d=1) \\ \hat{\beta}\sqrt{\frac{2\pi}{\log \epsilon^{-1}}} & (d=2) \\ \hat{\beta}\epsilon^{\frac{d-2}{2}} & (d\geq 3). \end{cases}$$

The choice of C_ϵ

We choose $C_\epsilon = \beta_\epsilon^2 \epsilon^{-d} \|\phi\|_{L^2}^2$.

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Remark

- There are other choices (e.g., Family-Vicsek scaling).
- This choice is related to directed polymers explained later.

Previous researches ($d = 1$)

We consider the solution h_ϵ to the mollified KPZ:

$$\partial_t h_\epsilon(t,x) = \frac{1}{2}\Delta h_\epsilon(t,x) + \frac{1}{2}\left|\nabla h_\epsilon(t,x)\right|^2 + \beta_\epsilon \xi^\epsilon(t,x) - C_\epsilon.$$

Theorem (Bertini-Giacomin)

When $d = 1$, for any $\hat{\beta} \geq 0$, $h_\epsilon(t,x) \to \exists h(t,x)$ as $\epsilon \to 0$.

Previous researches ($d = 2$)

Theorem (Caravenna-Sun-Zygouras)

When $d = 2$,

$$h_\epsilon(t,x) \xrightarrow{d} \begin{cases} \exists h(t,x) \in \mathbb{R}, & (\hat{\beta} < 1) \\ -\infty. & (\hat{\beta} > 1) \end{cases}$$

They have studied more:

- If $\hat{\beta} < 1$, then the fluctuation converges to EW limit;
- Even if $\hat{\beta} = 1$, a non-trivial limit of $\exp(h_\epsilon)$ still exists in distributional sense (Critical Stochastic Heat Flow).

KPZ $\Rightarrow$ Directed Polymer (via Feynman-Kac formula)

We define $u_\epsilon(t,x) := \exp(h_\epsilon(t,x))$, $V(x) := \int \phi(x-y)\phi(y)\mathrm{d}y$.

Proposition

The following are equivalent:

- $\partial_t h_\epsilon(t,x) = \frac{1}{2}\Delta h_\epsilon(t,x) + \frac{1}{2}|\nabla h_\epsilon(t,x)|^2 + \beta_\epsilon \xi^\epsilon(t,x) - C_\epsilon$
- $\partial_t u_\epsilon = \frac{1}{2}\Delta u_\epsilon + \beta_\epsilon \xi^\epsilon(t,x) u_\epsilon$

If u_ϵ is the solution to the above, then u_ϵ can be written as

$$u_\epsilon(t,x) = \mathrm{E}_x^{\mathrm{BM}}\left[u_\epsilon(0,B_t)\exp\left(\beta_\epsilon \int_0^t \xi^\epsilon(t-s,B_s)ds - A_\epsilon(t)\right)\right]$$

$$\stackrel{d}{=} \mathrm{E}_x^{\mathrm{BM}}\left[u_\epsilon(0,B_t)\exp\left(\beta_\epsilon \int_0^t \xi^\epsilon(s,B_s)ds - A_\epsilon(t)\right)\right],$$

where $A_\epsilon(t) = \frac{\hat{\beta}^2\epsilon^{-2}tV(0)}{2}$ (Itô correction).

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$$\stackrel{d}{=} \mathrm{E}_x^{\mathrm{BM}}\left[u_\epsilon(0,B_t)\underbrace{\exp\left(\beta_\epsilon \int_0^t \xi^\epsilon(s,B_s)ds - A_\epsilon(t)\right)}_{\textit{Hamiltonian of a directed polymer}}\right].$$

Remark on the expression of u_ϵ

In this slide, we assume $u_\epsilon(0,x) = 1$ and take $T = T_\epsilon = \epsilon^{-2}$.

- Using the definitions of β_ϵ, ξ^ϵ, we obtain

$$u_\epsilon(1,x) \overset{d}{=} \mathrm{E}^{\mathrm{BM}}_{\sqrt{T}x}\left[e^{\hat{\beta}\int_0^T (\xi(s,\cdot)*\phi)(B_s)ds) - \frac{\hat{\beta}^2 TV(0)}{2}} \right],$$
$$=: \mathrm{E}^{\mathrm{BM}}_{\sqrt{T}x}\left[\Phi_T\right].$$

- Φ_s is a martingale w.r.t. $\sigma(\xi(r,x) : \ r \leq s, \, x \in \mathbb{R}^d)$.
- Let $\mathcal{Z}_T := \mathrm{E}^{\mathrm{BM}}_0[\Phi_T]$. Then, $(\mathcal{Z}_T)_{T\geq 0}$ is also a martingale. Hence, by the Martingale convergence theorem,

$$\mathcal{Z}_T \to \exists \mathcal{Z}_\infty \text{ a.s.}$$

Subcritical and L^2 region

When $d \geq 3$, there are critical parameters $0 < \hat{\beta}_{L^2} \leq \hat{\beta}_c$,

$\hat{\beta} < \hat{\beta}_{L^2}$	$\hat{\beta} < \hat{\beta}_c$	$\hat{\beta} > \hat{\beta}_c$
$(\mathcal{Z}_T)_T$ is bounded in L^2	$\mathcal{Z}_T \to \mathcal{Z}_\infty > 0$ a.s.	$\mathcal{Z}_T \to 0$ a.s.
L^2-region	weak disorder	strong disorder

- It is believed that $\hat{\beta}_{L^2} < \hat{\beta}_c$
  (discrete case: Birkner-Greven-den Hollander 11).

Law of large numebers of u_ϵ for $d \geq 3$

Recall that u_ϵ is the solution to

$$\partial_t u_\epsilon = \frac{1}{2}\Delta u_\epsilon + \beta_\epsilon u_\epsilon \xi^\epsilon.$$

Let $u_\epsilon(0,\cdot) \equiv u_0(\cdot) \in \mathcal{C}_b(\mathbb{R}^d)$. We define $\bar{u}(t,x) = E_x^{\mathrm{BM}}[u_0(B_t)]$.

Theorem (Mukherjee-Shamov-Zeitouni, Cosco-Nakashima-N)

1. *For all* $\hat{\beta} < \hat{\beta}_c$, $f \in \mathcal{C}_c(\mathbb{R}^d)$ *and* $u_0 \in \mathcal{C}_b(\mathbb{R}^d)$, *as* $\epsilon \to 0$,

$$\int_{\mathbb{R}^d} f(x) u_\epsilon(t,x)\mathrm{d}x \xrightarrow{L^1} \int_{\mathbb{R}^d} f(x)\bar{u}(t,x)\mathrm{d}x.$$

2. *For all* $\hat{\beta} > \hat{\beta}_c$, $f \in \mathcal{C}_c(\mathbb{R}^d)$ *and* $u_0 \in \mathcal{C}_b(\mathbb{R}^d)$, *as* $\epsilon \to 0$,

$$u_\epsilon(t,x) \xrightarrow{P} 0.$$

Law of large numebers of u_ϵ for $d \geq 3$

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$$\int_{\mathbb{R}^d} f(x) u_\epsilon(t,x) \mathrm{d}x \xrightarrow{L^1} \int_{\mathbb{R}^d} f(x) \bar{u}(t,x) \mathrm{d}x.$$

2. *For all* $\hat{\beta} > \hat{\beta}_c$ *and* $u_0 \in \mathcal{C}_b(\mathbb{R}^d)$, *as* $\epsilon \to 0$,

$$u_\epsilon(t,x) \xrightarrow{P} 0.$$

The limit of u_ϵ (also h_ϵ) is just a function (NOT random)!

Fluctuation of u_ϵ (SHE)

For all $\hat{\beta} < \hat{\beta}_c$,

$$\int_{\mathbb{R}^d} f(x) u_\epsilon(t,x) \mathrm{d}x \xrightarrow{L^1} \int_{\mathbb{R}^d} f(x) \bar{u}(t,x) \mathrm{d}x.$$

Theorem (Gu-Ryzhik-Zeitouni, Cosco-Nakashima-N)

Suppose $u_\epsilon(0,\cdot) \equiv u_0 \in \mathcal{C}_b$. *For all* $\hat{\beta} < \hat{\beta}_{L^2}$,

$$\epsilon^{-\frac{d-2}{2}} \int_{\mathbb{R}^d} f(x)(u_\epsilon(t,x) - \bar{u}(t,x)) \mathrm{d}x \xrightarrow{(d)} \int f(x) \mathcal{U}_1(t,x) \mathrm{d}x,$$

where $\mathcal{U}_1(t,x)$ *is the solution of* $\mathcal{U}_1(0,x) \equiv 0$ *and*

$$\partial_t \mathcal{U}_1(t,x) = \frac{1}{2} \Delta \mathcal{U}_1(t,x) + \gamma(\hat{\beta}) \bar{u}(t,x) \xi(t,x). \quad \textit{(EW equation)}$$

Note that $\gamma(\hat{\beta}) \to \infty$ as $\hat{\beta} \to \hat{\beta}_{L^2}$.

Fluctuation of h_ϵ (KPZ)

Let $h_\epsilon(t,x) := \log u_\epsilon(t,x)$.

Theorem (MU, DGRZ, LZ, CNN)

Suppose $\|\log u_0\|_\infty < \infty$. *For all* $\hat\beta < \hat\beta_{L^2}$,

$$\epsilon^{-\frac{d-2}{2}} \int_{\mathbb{R}^d} f(x)(h_\epsilon(t,x) - \mathbb{E}h_\epsilon(t,x))\mathrm{d}x \overset{(d)}{\to} \int f(x)\mathcal{U}_2(t,x)\mathrm{d}x,$$

where $\mathcal{U}_2(t,x)$ *is the solution of* $\mathcal{U}_2(0,x) \equiv 0$ *and*

$$\partial_t \mathcal{U}_2(t,x) = \frac{1}{2}\Delta\mathcal{U}_2(t,x) + \nabla \log \bar{u}(t,x) \cdot \nabla\mathcal{U}_2(t,x) + \gamma(\hat\beta)\xi(t,x).$$

Some remarks

- We discuss a relation between SHE and KPZ equation in weak disorder regime.
- Out main result is only stated in discrete directed polymer.
- From now on, we only consider the discrete model.
- However, we believe that the same techniques work for continuum models.

Discrete polymer model

- Let $(\omega(i,n))_{i\in\mathbb{Z}^d, n\in\mathbb{N}}$ be i.i.d. random variables.
- We assume $\lambda(\beta) := \log \mathbb{E}^{\beta\omega(i,n)} \in \mathbb{R}$ for any $\beta \in \mathbb{R}$.
- Let us define the partition function $Z_N(x)$ for $x \in \mathbb{Z}^d$ of directed polymers:

$$Z_N(x) := Z_{\omega,N,\beta}(x) := \mathrm{E}_x\left[e^{\beta\sum_{k=1}^N \omega(k,X_k) - N\lambda(\beta)}\right],$$

where E_x denotes the expectation of the simple random walk starting at x.
- For simplicity of notation, we write $Z_N := Z_N(0)$.
- We call $\{\beta \geq 0 | \ \mathbb{P}(\liminf_N Z_N > 0) = 1\}$ the weak disorder regime.

Discrete Polymer vs. Continuum Polymer

Recall that

- $u_\epsilon(t,x) \overset{d}{=} \mathrm{E}^{\mathrm{BM}}\left[\exp\left(\hat\beta\int_0^T \phi*\xi(s,B_s)ds - TA(\hat\beta)\right)\right]$ with $T=\epsilon^{-2}$.
- $h_\epsilon(t,x) = \log u_\epsilon(t,x)$ solves $\partial_t h = \frac{1}{2}\Delta_x h + \frac{1}{2}|\nabla_x h|^2 + \beta_\epsilon \xi^\epsilon - C_\epsilon$.

	Discrete Polymer	**Continuum Polymer**
Energy	$H_N = \sum_{k=1}^N \omega(k,S_k)$	$\mathcal{H}_T = \int_0^T \phi*\xi(s,B_s)ds$
Partition	$\mathrm{E}^{\mathrm{SRW}}\left[\exp\left(\beta H_N - N\lambda(\beta)\right)\right]$	$\mathrm{E}^{\mathrm{BM}}\left[\exp\left(\hat\beta\mathcal{H}_T - TA(\hat\beta)\right)\right]$
Critical	$\sup\{\beta>0 \mid \lim Z_N > 0\}$	$\sup\{\hat\beta>0 \mid \lim \mathcal{Z}_T > 0\}$

Tail exponent

We suppose that the distribution of $\omega(i,n)$ has a compact support.
Let $p_*(\beta) := \sup\{p \geq 0 | \ \sup_N \mathbb{E}[Z_N(\beta)^p] < \infty\}$ (tail exponent).

Theorem (Junk 22+)

In the weak disorder regime, it holds

$$p_*(\beta) \geq \frac{d+2}{d}.$$

Moreover, for a certain class of weight distribution (finite support),

$$\beta < \beta_c \Leftrightarrow p_*(\beta) > \frac{d+2}{d}.$$

Fluctuation for SHE in weak disorder regime

We define $p_* := p_*(\beta)$ as before.
Let $\xi := \xi(\beta) := -\frac{d}{2} + \frac{d+2}{2(p_* \wedge 2)}$ (fluctuation exponent).

Theorem (Junk 22+)

Given a compactly supported function $f : \mathbb{R}^d \to \mathbb{R}$, we define the fluctuation of discrete SHE as

$$\chi_N(f) := N^{-d/2} \sum_{x \in \mathbb{Z}^d} f(x/\sqrt{N})(Z_N(x) - \mathbb{E}Z_N(x)).$$

In the weak disorder regime, for any $\epsilon > 0$ and $f \not\equiv 0$, it holds

$$\lim_{N \to \infty} \mathbb{P}(n^{-\xi-\epsilon} < |\chi_N(f)| < n^{-\xi+\epsilon}) = 1.$$

Equivalence of fluctuation between SHE and KPZ

From now on, we assume the following:

- $p_*(\beta) > \frac{d+2}{d}$.
- $\omega(i,x)$ has a compact support.

Given a compactly supported function $f : \mathbb{R}^d \to \mathbb{R}$, we define the fluctuation of discrete KPZ as

$$\kappa_N(f) := N^{-d/2} \sum_{x \in \mathbb{Z}^d} f(x/\sqrt{N})(\log Z_N(x) - \mathbb{E} \log Z_N(x)).$$

Theorem (Junk-N 23+)

There exists $\delta > 0$ *such that*

$$\lim_{n\to\infty} \mathbb{P}(|\chi_N(f) - \kappa_N(f)| \leq N^{-\xi-\delta}) = 1.$$

Theorem (Junk-N 23+)

There exists $\delta > 0$ *such that*

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Together with the result of [Junk 22] ($|\chi_N(f)| = N^{-\xi+o(1)}$), we conclude:

Corollary (Junk-N 23+)

For any $\epsilon > 0$ *and* $f \not\equiv 0$,

$$\mathbb{P}(N^{-\xi-\epsilon} \leq |\kappa_N(f)| \leq N^{-\xi+\epsilon}) \to 1.$$

Moreover, there exists $\delta > 0$ *such that*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{|\chi_N(f) - \kappa_N(f)|}{\min\{|\chi_N(f)|, |\kappa_N(f)|\}} \leq N^{-\delta}\right) = 1.$$

Why I was surprised

- For $\beta \in (\beta_2, \beta_c)$, it is believed that

$$\chi_N(f) := N^{-d/2} \sum_{x \in \mathbb{Z}^d} f(x/\sqrt{N})(Z_N(x) - \mathbb{E}Z_N(x)) \to \mathrm{Stable}.$$

- We can prove that
  - $\sup_{N \in \mathbb{N}} \mathbb{E}[(\log Z_N)^2] < \infty$
  - $\lim_{|x-y| \to \infty} \sup_{N \in \mathbb{N}} \mathbb{E}[(\log Z_N(x) - \mathbb{E} \log Z_N(x))(\log Z_N(y) - \mathbb{E} \log Z_N(y))] = 0.$

- Hence I believed that

$$\kappa_N(f) := N^{-d/2} \sum_{x \in \mathbb{Z}^d} f(x/\sqrt{N})(\log Z_N(x) - \mathbb{E} \log Z_N(x)) \to \mathrm{Gauss}.$$

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- Hence I believed that

$$\kappa_N(f) := N^{-d/2} \sum_{x \in \mathbb{Z}^d} f(x/\sqrt{N})(\log Z_N(x) - \mathbb{E} \log Z_N(x)) \to \text{Gauss}.$$

- It was not the case, and I was really reluctant..

Outline of the proof

Let $\ell_k := k^{1/8}$ and $E_k(y) := e^{\beta\omega(k,y)} - 1$. We define

$$\rho_N(f) := N^{-d/2} \sum_{x \in \mathbb{Z}^d} f(x/\sqrt{N}) \left[\sum_{k=1}^{N} \sum_{y \in \mathbb{Z}^d} \overleftarrow{Z}_{\ell_k}(k,y) E_k(y) p_N(x,y) \right].$$

We will prove that

$$\chi_N(f) \approx \rho_N(f) \quad \text{(Approximation for SHE)},$$
$$\kappa_N(f) \approx \rho_N(f) \quad \text{(Approximation for KPZ)}.$$

Sketch proof (Approximation for SHE)

Let $\ell_k := k^{1/8}$ and $E_k(y) := e^{\beta\omega(k,y)-\lambda(\beta)} - 1$. We have the following approximation for χ_n:

$$\chi_N(f) = N^{-d/2} \sum_{x\in\mathbb{Z}^d} f(x/\sqrt{N})(Z_N(x) - 1)$$

$$= N^{-d/2} \sum_{y\in\mathbb{Z}^d} \sum_{k=1}^{N} E_k(y) \left[\sum_{x\in\mathbb{Z}^d} f(x/\sqrt{N}) Z_k(x,y) p_k(x,y) \right]$$

$$\approx N^{-d/2} \sum_{y\in\mathbb{Z}^d} \sum_{k=1}^{N} E_k(y) \left[\sum_{x\in\mathbb{Z}^d} f(x/\sqrt{N}) \overleftarrow{Z}_{\ell_k}(k,y) p_N(x,y) \right] = \rho_N(f),$$

where we have used law of large numbers (similar to $Z_k \approx Z_{\ell_k}$).

Some ingredients

As before, we assume:
(1) $p_*(\beta) > \frac{d+2}{d}$, (2) $\omega(i, x)$ has a compact support.

The following play an important role in the proof below:

Theorem (Junk 23+)

Let $Z_N(x,y) := \mathrm{E}^x[e^{\beta \sum_{k=1}^{N-1} \omega(k,X_k)-(N-1)\lambda(\beta)} | X_N = y]$. *Let* $\ell_N := N^{1/8}$.
For any $p < p_*(\beta)$ *and* $c > 0$, *it holds*

$$\lim_{N\to\infty} \sup_{x,y\in[-N^{1-c},N^{1-c}]} \mathbb{E}|Z_N(x,y) - Z_{\ell_N} \overleftarrow{Z}_{\ell_N}(N,y)|^p = 0.$$

Theorem (Junk-N 23+)

There exists $C = C(\beta) > 0$ *such that for any* $u \geq 1$ *and* $N \in \mathbb{N}$,

$$\mathbb{P}(Z_N < 1/u) \leq Ce^{-(\log u)^2/C}.$$

Some ingredients

It holds:

$$\lim_{N\to\infty} \sup_{x,y\in[-N^{1-c},N^{1-c}]} \mathbb{E}|Z_N(x,y) - Z_{\ell_N} \overleftarrow{Z}_{\ell_N}(N,y)|^p = 0,$$

$$\mathbb{P}(Z_N < 1/u) \le Ce^{-(\log u)^2/C}.$$

We define the polymer measure:

$$\mu_N(x,y) := \frac{\mathrm{E}_x\left[e^{\beta\sum_{k=1}^{N-1}\omega(k,X_k)-n\lambda(\beta)}\mathbf{1}_{X_n=y}\right]}{Z_{N-1}(x)} = \frac{Z_N(x,y)p_N(x,y)}{Z_{N-1}(x)}.$$

By $Z_k(x,y) \approx Z_{\ell_k}(x)\overleftarrow{Z}_{\ell_k}(k,y)$ and $Z_{\ell_k}(x), Z_k(x) \to Z_\infty(x)$, they imply

$$\mu_k(x,y) = \frac{Z_k(x,y)p_k(x,y)}{Z_{k-1}(x)} \approx \frac{Z_{\ell_k}(x)\overleftarrow{Z}_{\ell_k}(k,y)p_k(x,y)}{Z_{k-1}(x)} \approx \overleftarrow{Z}_{\ell_k}(k,y)p_k(x,y).$$

Sketch proof (Approximation for KPZ I)

For κ_n, we consider the following Martingale decomposition:

$$\kappa_n(f) = N^{-d/2} \sum_{x\in\mathbb{Z}^d} f(x/\sqrt{N})(\log Z_N(x) - \mathbb{E}[\log Z_N(x)])$$

$$= \sum_{x\in\mathbb{Z}^d} \sum_{k=1}^{N} f(x/\sqrt{N})\left(\log \frac{Z_k(x)}{Z_{k-1}(x)} - \mathbb{E}\left[\log \frac{Z_k(x)}{Z_{k-1}(x)} \Big| \mathcal{F}_{k-1}\right]\right)$$

$$+ \sum_{x\in\mathbb{Z}^d} \sum_{k=1}^{N} f(x/\sqrt{N})\left(\mathbb{E}\left[\log \frac{Z_k(x)}{Z_{k-1}(x)} \Big| \mathcal{F}_{k-1}\right] - \mathbb{E}\left[\log \frac{Z_k(x)}{Z_{k-1}(x)}\right]\right)$$

$$\approx \sum_{x\in\mathbb{Z}^d} \sum_{k=1}^{N} f(x/\sqrt{N}) \sum_{y\in\mathbb{Z}^d} \mu_k(x,y) E_k(y),$$

where $\mu_k(x,y) := \frac{Z_k(x,y)p_k(x,y)}{Z_{k-1}(x)}$, $E_k(y) := e^{\beta\omega(k,y)-\lambda(\beta)} - 1$ and used

$$\log \frac{Z_k(x)}{Z_{k-1}(x)} = \log\left(1 + \sum_{y\in\mathbb{Z}^d} \mu_k(x,y)E_k(y)\right) \approx \sum_{y\in\mathbb{Z}^d} \mu_k(x,y)E_k(y).$$

Sketch proof (Approximation for KPZ II)

Let $\ell_k := k^{1/8}$ and $E_k(y) := e^{\beta\omega(k,y)} - 1$. Recall that

$$\mu_k(x,y) \approx \overleftarrow{Z}_{\ell_k}(k,y) p_k(x,y).$$

Hence, we have the following approximation for κ_n:

$$\kappa_N(f) \approx N^{-d/2} \sum_{x\in\mathbb{Z}^d} \sum_{k=1}^{N} f(x/\sqrt{N}) \sum_{y\in\mathbb{Z}^d} \mu_k(x,y) E_k(y)$$

$$\approx N^{-d/2} \sum_{x\in\mathbb{Z}^d} \sum_{k=1}^{N} f(x/\sqrt{N}) \sum_{y\in\mathbb{Z}^d} \left[\overleftarrow{Z}_{\ell_k}(k,y) E_k(y) p_N(x,y)\right] = \rho_N(f).$$

Therefore, we have $\chi_N(f) \approx \kappa_N(f)$.