Geometric Laplacians on self-conformal fractal curves in the plane

Naotaka Kajino (RIMS, Kyoto University) 梶野 直孝(京都大学・数理解析研究所)

French Japanese Conference on Probability & Interactions @Centre de conférences Marilyn et James Simons, IHES



Pictures from: N. Kajino, Adv. Stud. Pure Math., vol. 87, 2021, pp. 293–314.

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Constr./Analysis of "B.M." SG: Barlow–Perkins '88, Goldstein '87, Kusuoka '87 SC: Barlow–Bass '89, '99

Problem.

Construction & Analysis of "Laplacian" & "B.M." which respect given geometry?

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(Goldstein '87, Kusuoka '87, Barlow-Perkins '88, ...) 2/12 cf. Standard Dirichlet form and B.M. on the S.G.



Dirichlet form & B.M. on self-similar SCs

A self-similar regular Dirichlet form (*E*, *F*) exists.
 (Barlow–Bass '89, '99, Kusuoka–Zhou '92)



BB '89: $\exists 1 \tau > 1$, $\{ Law(\{B_{\tau^n t}^{ref, D_n}\}_{t \ge 0})\}_{n=0}^{\infty}$ is tight.

• Such a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ is unique. (Barlow–Bass–Kumagai–Teplyaev '10)



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Problem.

Construction & Analysis of "Laplacian" & "B.M." which respect given geometry?

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Constr./Analysis of "B.M." SG: Barlow–Perkins '88, Goldstein '87, Kusuoka '87 SC: Barlow–Bass '89, '99 Thm(K.). Construction & Analysis of "Laplacian" & "B.M." which respect given geometry!

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3.Quasi-conformal deformations of round SCs?

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3.Quasi-conformal deformations of round SCs? 4.Self-conformal quasi-arcs $\partial \subset \partial_{\mathbb{C}} U$ (later this talk)

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2.SLE_{κ}-curve, $\kappa \in (0, 4]$? (*cf.* Lawler–Rezaei '15) 3.Quasi-conformal deformations of round SCs? 4.Self-conformal quasi-arcs $\partial \subset \partial_{\mathbb{C}} U$ (later this talk)



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2 Results for Apollonian gasket: $K_{\alpha,\beta,\gamma_{embedding}}^{harmonic}$ Thm(K., *cf.* Teplyaev '04). $\exists^1(\mathcal{E}^{\alpha,\beta,\gamma},\mathcal{F}_{\alpha,\beta,\gamma})$: non-zero, str. local, regular symmetric Dirichlet form over $K_{\alpha,\beta,\gamma}$, Re, Im are $\mathcal{E}^{\alpha,\beta,\gamma}$ -harmonic on $K_{\alpha,\beta,\gamma} \setminus V_0$ Rmk. Choice of a reference measure is irrelevant: α $\mathcal{C}_{\alpha,\beta,\gamma} := \mathcal{F}_{\alpha,\beta,\gamma} \cap \mathcal{C}(K_{\alpha,\beta,\gamma}) \text{ and } \mathcal{E}^{\alpha,\beta,\gamma}|_{\mathcal{C}_{\alpha,\beta,\gamma}} \text{ are unique.}$ Thm(K.). LIP|_{K_{\alpha,\beta,\beta,\beta}} is a core of ($\mathcal{E}^{\alpha,\beta,\gamma}, \mathcal{F}_{\alpha,\beta,\gamma}$), and}

2 Results for Apollonian gasket: $K_{\alpha,\beta,\gamma_{embedding}}$ Thm(K., *cf.* Teplyaev '04). $\exists^1(\mathcal{E}^{\alpha,\beta,\gamma},\mathcal{F}_{\alpha,\beta,\gamma})$: non-zero, str. local, regular symmetric Dirichlet form over $K_{\alpha,\beta,\gamma}$, Re, Im are $\mathcal{E}^{\alpha,\beta,\gamma}$ -harmonic on $K_{\alpha,\beta,\gamma} \setminus V_0$ **Rmk.** Choice of a reference measure is irrelevant: $\mathcal{C}_{\alpha,\beta,\gamma} := \mathcal{F}_{\alpha,\beta,\gamma} \cap \mathcal{C}(K_{\alpha,\beta,\gamma})$ and $\mathcal{E}^{\alpha,\beta,\gamma}|_{\mathcal{C}_{\alpha,\beta,\gamma}}$ are unique.

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2 Results for Apollonian gasket: $K_{\alpha,\beta,\gamma_{embedding}}$ Thm(K., *cf.* Teplyaev '04). $\exists^1(\mathcal{E}^{\alpha,\beta,\gamma}, \mathcal{F}_{\alpha,\beta,\gamma})$: non-zero, str. local, regular symmetric Dirichlet form over $K_{\alpha,\beta,\gamma}$, Re, Im are $\mathcal{E}^{\alpha,\beta,\gamma}$ -harmonic on $K_{\alpha,\beta,\gamma} \setminus V_0$ **Rmk.** Choice of a reference measure is irrelevant: $\mathcal{C}_{\alpha,\beta,\gamma} := \mathcal{F}_{\alpha,\beta,\gamma} \cap \mathcal{C}(K_{\alpha,\beta,\gamma})$ and $\mathcal{E}^{\alpha,\beta,\gamma}|_{\mathcal{C}_{\alpha,\beta,\gamma}}$ are unique. Thm(K.). LIP $|_{K_{\alpha,\beta,\gamma}}$ is a core of $(\mathcal{E}^{\alpha,\beta,\gamma},\mathcal{F}_{\alpha,\beta,\gamma})$, and $\forall u \in \text{LIP}, \ \mathcal{E}^{\alpha,\beta,\gamma}(u,u) = \sum_{C \subset \operatorname{arc} K_{\alpha,\beta,\gamma}} \operatorname{rad}(C) \int_{C} |\nabla_{C} u|^{2} d \operatorname{vol}_{C}.$ (NOT doubling!) $\triangleright \mu^{\alpha,\beta,\gamma} := \sum_{C \subset \operatorname{arc} K_{\alpha,\beta,\gamma}} \operatorname{rad}(C) \operatorname{vol}_{C} : \operatorname{volume} \operatorname{meas}.$ $\mu^{lpha,eta,\gamma}$ $= 2 \operatorname{Area}()$

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Remarks on Thm 1

• $\mu^{\alpha,\beta,\gamma} \perp \mathcal{H}^{d_{\mathsf{AG}}}|_{K_{\alpha,\beta,\gamma}}$ (for $\mathcal{H}^{d_{\mathsf{AG}}}(\operatorname{arcs}) = 0$ by $d_{\mathsf{AG}} > 1$). • $\sum_{n} e^{-t\lambda_{n}^{\alpha,\beta,\gamma}} = \int p_{t}^{K_{\alpha,\beta,\gamma}}(x,x) d\mu^{\alpha,\beta,\gamma}(x) \stackrel{t\downarrow 0}{\sim} \stackrel{\mathcal{H}^{d_{\mathsf{AG}}}(K_{\alpha,\beta,\gamma})}{\frac{t^{d_{\mathsf{AG}}}(K_{\alpha,\beta,\gamma})}{ct^{d_{\mathsf{AG}}/2}}}$ \Leftrightarrow Thm 1, BUT $p_{t}^{K_{\alpha,\beta,\gamma}}(x,x) \asymp c_{x}t_{x}t_{x}^{-1/2}$ for $\mu^{\alpha,\beta,\gamma}$ -a.e. x!

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7/12 **3** Some Kleinian groups G_m with $\partial_{\infty}G_m$ a RSC $> m > 6 \left(\frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{m} < \pi \right)$ $\triangleright \{\ell_k\}_{k=1}^3$: \mathbb{B}^2 -geodesics, form \triangle , angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{m}$ $arprop \Gamma_{oldsymbol{m}} := ig\langle \{ \mathbf{Inv}_{oldsymbol{\ell}_{oldsymbol{k}}} \}_{oldsymbol{k}=1}^{oldsymbol{3}}ig angle$ $\sim \mathbb{B}^2 = \bigcup_{\tau \in \Gamma_m} \tau(\triangle_{\ell_1 \ell_2 \ell_3})$ $= G = G_m := \langle \Gamma_m, \operatorname{Inv}_S \rangle$ $\sim \partial_{\infty}G_m$ is a round SC.

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angle$ $\sim \mathbb{B}^2 = \bigcup_{\tau \in \Gamma_m} \tau(\triangle_{\ell_1 \ell_2 \ell_3})$ $\bullet S = S_m := \partial B_{\mathbb{R}^2}(0, \exists 1 r_m):$ angle $(S, \ell_2) = \frac{\pi}{3}$. $\triangleright G = G_m := \langle \Gamma_m, \operatorname{Inv}_S \rangle$ $\rightsquigarrow \partial_{\infty} G_m$ is a round SC.

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 $>m>6\left(\frac{\pi}{2}+\frac{\pi}{3}+\frac{\pi}{m}<\pi\right)$ $\triangleright \{\ell_k\}_{k=1}^3$: \mathbb{B}^2 -geodesics, form \triangle , angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{m}$ $arprop \Gamma_{m} := ig\langle \{ \operatorname{Inv}_{\ell_{k}} \}_{k=1}^{3} ig
angle$ $\sim \mathbb{B}^2 = \bigcup_{\tau \in \Gamma_m} \tau(\triangle_{\ell_1 \ell_2 \ell_3})$ $\bullet S = S_m := \partial B_{\mathbb{R}^2}(0, \exists 1 r_m):$ angle $(S, \ell_2) = \frac{\pi}{3}$. $\triangleright G = G_m := \langle \Gamma_m, \operatorname{Inv}_S \rangle$ $\rightsquigarrow \partial_{\infty} G_m$ is a round SC.

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 $> m > 6 \left(\frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{m} \sqrt[8]{\pi}\right)$ $\triangleright \{\ell_k\}_{k=1}^3$: \mathbb{B}^2 -geodesics, form \triangle , angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{m}$ $ho \Gamma_{m} := \left\langle \{ \operatorname{Inv}_{\ell_{k}} \}_{k=1}^{3}
ight
angle$ $\sim \mathbb{B}^2 = \bigcup_{\tau \in \Gamma_m} \tau(\triangle_{\ell_1 \ell_2 \ell_3})$ $lackslash S = S_m := \partial B_{\mathbb{R}^2}(0, \exists 1 r_m)$ $\operatorname{angle}(S, \ell_2) = \frac{\pi}{3}.$ $\ell_2 \triangleright G = G_m := \langle \Gamma_m, \operatorname{Inv}_S \rangle$ $\partial_{\infty}G_m := \overline{\bigcup_{g \in G_m} g(\partial \mathbb{B}^2)}$ $\rightsquigarrow \partial_{\infty} G_m$ is a round SC. • $G \cap \mathbb{H}^3$ has no parabolics (so hyperbl.), $\partial_{\mathrm{Grmv}} G \simeq \partial_{\infty} G$

 $> m > 6 \left(\frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{m} \frac{2}{\pi}\right)$ $\triangleright \{\ell_k\}_{k=1}^3$: \mathbb{B}^2 -geodesics, form \triangle , angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{m}$ $ho \Gamma_{m} := \left\langle \{ \operatorname{Inv}_{\ell_{k}} \}_{k=1}^{3}
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angle$ $\sim \mathbb{B}^2 = \bigcup_{\tau \in \Gamma_m} \tau(\triangle_{\ell_1 \ell_2 \ell_3})$ $\bullet S = S_m := \partial B_{\mathbb{R}^2}(0, \exists 1 r_m):$ $\operatorname{angle}(S, \ell_2) = \frac{\pi}{3}.$ $\ell_2 \triangleright G = G_m := \langle \Gamma_m, \operatorname{Inv}_S \rangle$ $\partial_{\infty}G_m := \overline{\bigcup_{g \in G_m} g(\partial \mathbb{B}^2)}$ $\rightsquigarrow \partial_{\infty} G_m$ is a round SC. • $G \cap \mathbb{H}^3$ has no parabolics (so hyperbl.), $\partial_{\mathrm{Grmv}} G \simeq \partial_{\infty} G$ •(Sullivan '79) $d = d(m) := \dim_{\text{Haus}} \partial_{\infty} G \in (1,2)$ &

 $> m > 6 \left(\frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{m} \frac{2}{\pi}\right)$ $\triangleright \{\ell_k\}_{k=1}^3$: \mathbb{B}^2 -geodesics, form \triangle , angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{m}$ $arprop \Gamma_{oldsymbol{m}} := ig\langle \{ \mathrm{Inv}_{oldsymbol{\ell}_{oldsymbol{k}}} \}_{oldsymbol{k}=1}^{oldsymbol{3}}ig
angle$ $\sim \mathbb{B}^2 = \bigcup_{\tau \in \Gamma_m} \tau(\triangle_{\ell_1 \ell_2 \ell_3})$ $lackslash S = S_m := \partial B_{\mathbb{B}^2}(0, \exists 1 r_m)$ $\operatorname{angle}(S, \ell_2) = \frac{\pi}{3}.$ $\ell_2 \triangleright G = G_m := \langle \Gamma_m, \operatorname{Inv}_S \rangle$ $\partial_{\infty}G_m := \overline{\bigcup_{g \in G_m} g(\partial \mathbb{B}^2)}$ $\rightsquigarrow \partial_{\infty} G_m$ is a round SC. • $G \cap \mathbb{H}^3$ has no parabolics (so hyperbl.), $\partial_{\mathrm{Grmv}} G \simeq \partial_{\infty} G$ •(Sullivan '79) $d = d(m) := \dim_{\text{Haus}} \partial_{\infty} G \in (1, 2)$ & $\mathcal{H}^{d}(B_{r}(x) \cap \partial_{\infty}G) \asymp r^{d}$, so $\mathcal{H}^{d}|_{\partial_{\infty}G}$ is finite & full supp.

 $> m > 6 \left(\frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{m} \frac{2}{\pi}\right)$ $\triangleright \{\ell_k\}_{k=1}^3$: \mathbb{B}^2 -geodesics, form \triangle , angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{m}$ $ho \Gamma_{m} := \left\langle \{ \operatorname{Inv}_{\ell_{k}} \}_{k=1}^{3}
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angle$ $\sim \mathbb{B}^2 = \bigcup_{\tau \in \Gamma_m} \tau(\triangle_{\ell_1 \ell_2 \ell_3})$ $ullet S = S_m := \partial B_{\mathbb{R}^2}(0, \exists 1 r_m)$ angle $(S, \ell_2) = \frac{\pi}{3}$. $\ell_2 \triangleright G = G_m := \langle \Gamma_m, \operatorname{Inv}_S \rangle$ $\partial_{\infty}G_m := \overline{\bigcup_{g \in G_m} g(\partial \mathbb{B}^2)}$ $\sim \partial_{\infty} G_m$ is a round SC. •(Sullivan '79) $d = d(m) := \dim_{\text{Haus}} \partial_{\infty} G \in (1,2)$ & $\mathcal{H}^{d}(B_{r}(x) \cap \partial_{\infty}G) \asymp r^{d}$, so $\mathcal{H}^{d}|_{\partial_{\infty}G}$ is finite & full supp.

 $> m > 6 \left(\frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{m} \frac{2}{\pi}\right)$ $\triangleright \{\ell_k\}_{k=1}^3$: \mathbb{B}^2 -geodesics, form \triangle , angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{m}$ $arprop \Gamma_{oldsymbol{m}} := ig\langle \{ \mathrm{Inv}_{oldsymbol{\ell}_{oldsymbol{k}}} \}_{k=1}^{3} ig
angle$ $\rightsquigarrow \mathbb{B}^2 = \bigcup_{\tau \in \Gamma_m} \tau(\bigtriangleup_{\ell_1 \ell_2 \ell_3})$ $ullet S = S_m := \partial B_{\mathbb{B}^2}(0, \exists 1 r_m)$ $\operatorname{angle}(S, \ell_2) = \frac{\pi}{3}.$ $\ell_2 \triangleright G = G_m := \langle \Gamma_m, \operatorname{Inv}_S \rangle$, $\partial_{\infty}G_m := \overline{\bigcup_{g \in G_m}}g(\partial \mathbb{B}^2)$ $\swarrow \leadsto \partial_{\infty} G_{m}$ is a round SC. •(Sullivan '79) $d = d(m) := \dim_{\text{Haus}} \partial_{\infty} G \in (1,2)$ & $\mathcal{H}^{d}(B_{r}(x) \cap \partial_{\infty}G) \asymp r^{d}$, so $\mathcal{H}^{d}|_{\partial_{\infty}G}$ is finite & full supp. $\mathsf{Thm}\ \mathsf{2}\ (\mathsf{K}.). \lim_{\lambda \to \infty} \#\{n \in \mathbb{N} \mid \lambda_n^U \leq \lambda\} / \lambda^{d/2} \, = \, c_m \mathfrak{H}^d \! (U).$

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3.Quasi-conformal deformations of round SCs?

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3. $f(\partial_{\infty}\tilde{G})(\partial_{\infty} \text{ of qc deform. } f\tilde{G}f^{-1} \text{ of } \tilde{G}_{fin}^{-1})?$

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3. $f(\partial_{\infty} \tilde{G}) (\partial_{\infty} \text{ of qc deform. } f \tilde{G} f^{-1} \text{ of } \tilde{G}_{\text{fin}} G^{-1})?$ 4.Self-conformal quasi-arcs $\partial \subset \partial_{\mathbb{C}} U$ (in progress):

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2.SLE_{κ}-curve, $\kappa \in (0, 4]$? (*cf.* Lawler–Rezaei '15) 3. $f(\partial_{\infty} \tilde{G})$ (∂_{∞} of qc deform. $f \tilde{G} f^{-1}$ of $\tilde{G} \subset G_m$)? 4.Self-conformal quasi-arcs $\partial \subset \partial_{\mathbb{C}} U$ (in progress):

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4 Case of non-circle-packing self-conformal fractals?

- 1.CLE_{κ}-carpet, $\kappa \in (\frac{8}{3}, 4]$, of Sheffield–Werner '12? 2.SLE_{κ}-curve, $\kappa \in (0, 4]$? (*cf.* Lawler–Rezaei '15) 3. $f(\partial_{\infty}\tilde{G})$ (∂_{∞} of qc deform. $f\tilde{G}f^{-1}$ of $\tilde{G}\subset G_m$)?
- 4.Self-conformal quasi-arcs $\partial \subset \partial_{\mathbb{C}} U$ (in progress):

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4.Self-conformal quasi-arcs $\partial \subset \partial_{\mathbb{C}} U$ (in progress): F1: expanding, conformal 22 **J**2 **ə**s $F_1(U_1)$ $\begin{aligned} F_{j}(\partial_{j}) = \bigcup_{\substack{k \in A_{j} \\ k \in A_{j}}} \{F_{j}|_{\partial_{j}}\}_{j=1}^{N} \text{ top. mixing on } \partial = \bigcup_{\substack{j=1 \\ j=1}}^{N} \partial_{j} \\ F_{j}(\bigcup_{j} \cap U) \subset U, F_{j}(\bigcup_{j} \cap \partial U) \subset \partial U \end{aligned}$ DN-2 $\partial N-1$

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Exmp. \bullet (Bowen '79) ∂_{∞} of quasi-Fuchsian groups without parabolics

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Exmp. \bullet (Bowen '79) ∂_{∞} of quasi-Fuchsian groups without parabolics \bullet (*cf.* Makarov '90) Julia $(z^2 + c)$ for $c \in \mathbb{C}$ with |c| > 0 small

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4.Self-conformal quasi-arcs $\partial \subset \partial_{\mathbb{C}} U$ (in progress): $\triangleright \mathcal{E}^{\partial}(u,v) := \int_{\partial} (du/d\omega_{U,q_0}) (dv/d\omega_{U,q_0}) d\omega_{U,q_0}$



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4.Self-conformal quasi-arcs $\partial \subset \partial_{\mathbb{C}} U$ (in progress): $\triangleright \mathcal{E}^{\partial}(u, v) := \int_{\partial} (du/d\omega_{U,q_0}) (dv/d\omega_{U,q_0}) d\omega_{U,q_0}$ $\bullet^{\exists 1} d_{w} \geq 2$, with $d_{f} := \dim_{H} \partial$, $\exists \mu_{\partial}$: meas. on ∂ , $\mu_{\partial}([x, y]_{\partial}) \asymp |x - y|^{d_{f}d_{w}} / \omega_{U,q_0}([x, y]_{\partial}) . (cf. Makarov '90)$ $\bullet Assume d_{f} > 1 (\Leftrightarrow d_{w} > 2; cf. Przytycki-Urbański-Zdunik '89)$

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4.Self-conformal quasi-arcs $\partial \subset \partial_{\mathbb{C}} U$ (in progress): $\triangleright \mathcal{E}^{\partial}(u,v) := \int_{\partial} (du/d\omega_{U,q_0}) (dv/d\omega_{U,q_0}) \, d\omega_{U,q_0}$ • $\exists 1 d_w > 2$, with $d_f := \dim_H \partial$, $\exists \mu_{\partial}$: meas. on ∂ , $\mu_{\partial}([x,y]_{\partial}) \, \asymp \, |x-y|^{d_{\mathrm{f}}d_{\mathrm{w}}} / \omega_{U,q_0}([x,y]_{\partial}) \, . \, (cf.$ Makarov '90) • Assume $d_{\rm f} > 1$ ($\Leftrightarrow d_{\rm w} > 2$; cf. Przytycki–Urbański–Zdunik '89) \rightsquigarrow For $-\Delta_{(\partial,\mu_{\partial},\mathcal{E}^{\partial},W^{1,2}(\partial,\omega_{U,q_{\Omega}}))}$, we have:
4 Case of non-circle-packing self-conformal fractals?

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4.Self-conformal quasi-arcs $\partial \subset \partial_{\mathbb{C}} U$ (in progress): $\triangleright \mathcal{E}^{\partial}(u,v) := \int_{\partial} (du/d\omega_{U,q_0}) (dv/d\omega_{U,q_0}) d\omega_{U,q_0}$ • $\exists^{1} d_{w} > 2$, with $d_{f} := \dim_{H} \partial$, $\exists \mu_{\partial}$: meas. on ∂ , $\mu_{\partial}([x,y]_{\partial}) \, \asymp \, |x-y|^{d_{\mathrm{f}}d_{\mathrm{w}}} / \omega_{U,q_0}([x,y]_{\partial}) \, . \, (cf.$ Makarov '90) • Assume $d_{
m f} > 1$ ($\Leftrightarrow d_{
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ightarrow\infty}\!\lambda^{-1/d_{\mathrm{w}}}\#\{n\in\mathbb{N}\mid\lambda_{n}^{(x,y)_{\partial}}\!\leq\!\lambda\}\!=\!c_{\partial}\mathfrak{H}^{d_{\mathrm{f}}}((x,y)_{\partial}).$

4 Case of non-circle-packing self-conformal fractals?

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4.Self-conformal quasi-arcs $\partial \subset \partial_{\mathbb{C}} U$ (in progress): $ho \mathcal{E}^{\partial}(u,v) := \int_{\partial} (du/d\omega_{U,q_0}) (dv/d\omega_{U,q_0}) \, d\omega_{U,q_0}$ • $\exists^{1} d_{w} \geq 2$, with $d_{f} := \dim_{H} \partial$, $\exists \mu_{\partial}$: meas. on ∂ , $\mu_{\partial}([x,y]_{\partial}) \,{\asymp}\, |x-y|^{d_{\mathrm{f}}d_{\mathrm{w}}} / \omega_{U,q_0}([x,y]_{\partial})$. (cf. Makarov '90) • Assume $d_{
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m w} > 2$; cf. Przytycki–Urbański–Zdunik '89) \rightsquigarrow For $-\Delta_{(\partial,\mu_{\partial},\mathcal{E}^{\partial},W^{1,2}(\partial,\omega_{U,q_{0}}))}$, we have: Thm 3 (K.). For nonlinear ∂ , $\exists c_{\partial} \in (0, \infty)$, $\forall x \neq y \in \partial$, $\lim_{\lambda
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Exmp. \bullet (Bowen '79) ∂_{∞} of quasi-Fuchsian groups without parabolics " \bullet " (*cf.* Zdunik '90) Julia($z^2 + c$) for $c \in \mathbb{C}$ with |c| > 0 small

 $\begin{array}{l} & \overset{10/12}{}\\ \mathsf{Thm}\,1\,(\mathsf{K}.). \ ^{\exists}c_{0} \in (0,\infty), \ ^{\forall}\alpha,\beta,\gamma \in (0,\infty), \ \ (d:=d_{\mathsf{AG}}) \\ & \lim_{\lambda \to \infty} \#\{n \in \mathbb{N} \mid \lambda_{n}^{\alpha,\beta,\gamma} \leq \lambda\}/\lambda^{d_{\mathsf{AG}}/2} \!=\! c_{0}\mathcal{H}^{d_{\mathsf{AG}}}(K_{\alpha,\beta,\gamma}). \end{array}$

Prf. To follow Kigami–Lapidus' method [CMP '93], we use Kesten's renewal thm for Markov chains [Ann. Prob. '74].

 $arapprox K_x \setminus V_0 = igcup_{k=1}^6 igcup_{l=1}^\infty K_{arphi_{k,l}(x)}$ $arapprox \Gamma := \{x_{=(lpha,eta,\gamma)} | \mathcal{H}^d(K_x) = 1\}$ (the space of "Euc. shapes" of AGs)

 $arproptole \{[X_n]\}_{n=0}^{\infty}$: Markov chain on Γ , $x \sim [arphi_{k,l}(x)]$ w.prob. $\mathcal{H}^d(K_{arphi_{k,l}(x)})$ (note $\sum_{k,l} \mathcal{H}^d(K_{arphi_{k,l}(x)}) = 1$) $arproptole V_n := -rac{1}{d} \log \mathcal{H}^d(K_{X_n})$ Thm 1 (K.). $\exists c_0 \in (0,\infty), \forall \alpha, \beta, \gamma \in (0,\infty), (d := d_{\mathsf{AG}})$ $\lim_{\lambda o\infty}\#\{n\in\mathbb{N}\mid\lambda_n^{lpha,eta,\gamma}\!\leq\!\lambda\}/\lambda^{d_{\mathsf{AG}}/2}\!=\!c_0\mathfrak{H}^{d_{\mathsf{AG}}}(K_{lpha,eta,\gamma}).$

Prf. To follow Kigami–Lapidus' method [CMP '93], we use Kesten's renewal thm for Markov chains [Ann. Prob. '74].

 $K_x = (lpha, eta, \gamma)$ $K_{\varphi_{2,4}(x)} \overset{(\kappa)}{\longrightarrow} K_{\varphi_{1,4}(x)}$ $K_{arphi_{2,3}(x)}$ $K_{arphi_{1,3}(x)}$ $K_{\varphi_{2,2}(x)} \land K_{\varphi_{1,2}(x)}$ $K_{arphi_{2,1}(x)}$ $K_{arphi_{1,1}(x)}$ $K_{arphi_{3,1}(x)}$ $\setminus K_{arphi_{6,1}(x)}$ $K_{arphi_{6,2}(x)} arapsilon V_n := -rac{1}{2} \log \mathcal{H}^d(K_{X_n})$ $K_{arphi_{3,2}(x)}$ $K_{arphi_{3,3}(x)}$ $\mathbf{A}_{arphi_{6,3}(x)}$ $K_{\varphi_{5,2}(x)}$ $\varphi_{4,3}(x)$

 $arapprox K_{oldsymbol{x}} ackslash V_0 \!=\! igcup_{k=1}^6 igcup_{l=1}^\infty K_{arphi_{oldsymbol{k},l}(oldsymbol{x})}$

cf. p. 30, Figure 3 of R. D. Mauldin & M. Urbański, Adv. Math. 136 (1998), 26–38 $\begin{array}{l} \overset{10/12}{\mathsf{Thm}} 1 (\mathsf{K}.). \overset{\exists}{} c_0 \in (0,\infty), \, ^\forall \alpha,\beta,\gamma \in (0,\infty), \ \, (d:=d_{\mathsf{AG}}) \\ \underset{\lambda \to \infty}{\overset{\#}{=}} w \mid \lambda_n^{\alpha,\beta,\gamma} \leq \lambda \} / \lambda^{d_{\mathsf{AG}}/2} = c_0 \mathcal{H}^{d_{\mathsf{AG}}}(K_{\alpha,\beta,\gamma}). \end{array}$

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 $\triangleright K_x \setminus V_0 = \bigcup_{k=1}^6 \bigcup_{l=1}^\infty K_{\varphi_{k,l}(x)}$ $\triangleright \Gamma := \{ x_{=(\alpha,\beta,\gamma)} | \mathcal{H}^d(K_x) = 1 \}$ (the space of "Euc. shapes" of AGs)

 $\left\{ \begin{bmatrix} X_n \end{bmatrix} \right\}_{n=0}^{\infty} : \text{Markov chain on } \Gamma, \\ x \sim [\varphi_{k,l}(x)] \text{w.prob. } \mathcal{H}^d(K_{\varphi_{k,l}(x)}) \\ (\text{note } \sum_{k,l} \mathcal{H}^d(K_{\varphi_{k,l}(x)}) = 1) \\ \sum_{k=0}^{k} \mathcal{H}^d(K_{\varphi_{k,l}(x)}) = 1 \end{bmatrix}$

cf. **p. 30, Figure 3** of R. D. Mauldin & M. Urbański, Adv. Math. 136 (1998), 26–38

$$\begin{array}{l} \overset{10/12}{\operatorname{\mathsf{Thm}}} {\operatorname{\mathsf{I}}}({\mathsf{K}}.). \overset{\exists}{} c_0 \in (0,\infty), \, {}^{\forall} \alpha,\beta,\gamma \in (0,\infty), \ (d:=d_{\mathsf{AG}}) \\ \lim_{\lambda \to \infty} \#\{n \in \mathbb{N} \mid \lambda_n^{\alpha,\beta,\gamma} \leq \lambda\}/\lambda^{d_{\mathsf{AG}}/2} \!=\! c_0 \mathcal{H}^{d_{\mathsf{AG}}}(K_{\alpha,\beta,\gamma}). \end{array}$$

Prf. To follow Kigami–Lapidus' method [CMP '93], we use Kesten's renewal thm for Markov chains [Ann. Prob. '74].



 $arapprox K_{oldsymbol{x}} ackslash V_0 \!=\! igcup_{k=1}^6 igcup_{l=1}^\infty K_{arphi_k,l}(oldsymbol{x})$ $\triangleright \Gamma := \{ x_{=(lpha,eta,\gamma)} | \mathcal{H}^d(K_x) = 1 \}$ (the space of "Euc. shapes" of AGs)

 $K_{\varphi_{1,2}(x)} \qquad \triangleright \{ [X_n] \}_{n=0}^{\infty} : \text{Markov chain on } \Gamma,$ $K_{arphi_{1,1}(x)} \quad x \sim [arphi_{k,l}(x)] \text{w.prob.} \, \mathcal{H}^d(K_{arphi_{k,l}(x)})$ (note $\sum_{k,l} \mathfrak{H}^d(K_{\varphi_{k,l}(x)}) = 1$)

> cf. p. 30, Figure 3 of R. D. Mauldin & M. Urbański, Adv. Math. 136 (1998), 26–38

$$\begin{array}{l} \overset{10/12}{\mathsf{Thm}} 1 (\mathsf{K}.). \ ^{\exists} c_{0} \in (0,\infty), \ ^{\forall} \alpha,\beta,\gamma \in (0,\infty), \ \ (d:=d_{\mathsf{AG}}) \\ \lim_{\lambda \to \infty} \# \{ n \in \mathbb{N} \mid \lambda_{n}^{\alpha,\beta,\gamma} \leq \lambda \} / \lambda^{d_{\mathsf{AG}}/2} = c_{0} \mathcal{H}^{d_{\mathsf{AG}}}(K_{\alpha,\beta,\gamma}). \end{array}$$

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$$\triangleright K_{x} \setminus V_{0} = \bigcup_{k=1}^{6} \bigcup_{l=1}^{\infty} K_{\varphi_{k,l}(x)}$$

$$\triangleright \Gamma := \{ x_{=(\alpha,\beta,\gamma)} | \mathcal{H}^{d}(K_{x}) = 1 \}$$
(the space of "Euc. shapes" of AGs)
$$\triangleright \{ [X_{n}] \}_{n=0}^{\infty} : \text{Markov chain on } \Gamma,$$

$$x \sim [\varphi_{k,l}(x)] \text{ w. prob. } \mathcal{H}^{d}(K_{\varphi_{k,l}(x)})$$
(note $\sum_{k,l} \mathcal{H}^{d}(K_{\varphi_{k,l}(x)}) = 1$)

cf. p. 30, Figure 3 of R. D. Mauldin & M. Urbański, Adv. Math. 136 (1998), 26–38 Thm 1 (K.). $\exists c_0 \in (0,\infty), \forall \alpha, \beta, \gamma \in (0,\infty), (d := d_{\mathsf{AG}})$ $\lim_{\lambda o\infty} \#\{n\in\mathbb{N}\mid\lambda_n^{lpha,eta,\gamma}\!\leq\!\lambda\}/\lambda^{d_{\mathsf{AG}}/2}\!=\!c_0\mathfrak{H}^{d_{\mathsf{AG}}}(K_{lpha,eta,\gamma}).$ $arphi \{ [X_n] \}_{n=0}^{\infty}$: MC on $\Gamma_{(ext{shapes})}$, $x \rightsquigarrow [arphi_{k,l}(x)]$ w.prob. $\mathcal{H}^d(K_{arphi_{k,l}(x)})$ $\triangleright V_n := -rac{1}{d} \log \mathcal{H}^d(K_{X_n})$ (the changes in size along $\{[X_n]\}_{n=0}^{\infty}$) $arapprox K_{oldsymbol{x}} igararrow V_0 \!=\! igcup_{k,l} K_{oldsymbol{arphi}_{k,l}(oldsymbol{x})}
hinspace{-ds} \mathcal{N}_x^{ ext{Dir.}}(e^{2s})$ $K_x x = (\alpha, \beta, \gamma)$ $K_{\varphi_{2,4}(x)} \land K_{\varphi_{1,4}(x)}$ $K_{arphi_{2,3}(x)}$ $K_{arphi_{1,3}(x)}$ $K_{\varphi_{2,2}(x)} \land K_{\varphi_{1,2}(x)}$ $K_{arphi_{2,1}(x)}$ $K_{arphi_{1,1}(x)}$ $|\mathcal{K}_{arphi_{6,1}(x)}$ Need: ullet $|\mathcal{R}(x,s)| \leq c' e^{-c|s|^{lpha}}$. $K_{arphi_{3,1}(x)}$ $K_{arphi_{3,2}(x)}$ $\langle K_{arphi_{6,2}(x)}
angle$ $K_{arphi_{3,3}(x)}$ $K_{arphi_{6,3}(x)}$ cf. p. 30, Figure 3 of R. D. Mauldin & M. $K_{arphi_{5,2}(x)}$ Urbański, Adv. Math. 136 (1998), 26–38 $\varphi_{4,3}(x)$

$$\begin{aligned} & \operatorname{Thm} 1 (\mathsf{K}.). \stackrel{\exists}{=} c_0 \in (0, \infty), \forall \alpha, \beta, \gamma \in (0, \infty), \quad (d := d_{\mathsf{AG}}) \\ & \lim_{\lambda \to \infty} \# \{ n \in \mathbb{N} \mid \lambda_n^{\alpha,\beta,\gamma} \leq \lambda \} / \lambda^{d_{\mathsf{AG}}/2} = c_0 \mathcal{H}^{d_{\mathsf{AG}}}(K_{\alpha,\beta,\gamma}). \\ & \triangleright \{ [X_n] \}_{n=0}^{\infty} : \mathsf{MC} \text{ on } \Gamma_{(\mathsf{shapes})}, x \rightsquigarrow [\varphi_{k,l}(x)] \text{ w. prob. } \mathcal{H}^{d}(K_{\varphi_{k,l}(x)}) \\ & \triangleright V_n := -\frac{1}{d} \log \mathcal{H}^d(K_{X_n}) \quad (\mathsf{the changes in size along } \{ [X_n] \}_{n=0}^{\infty}) \\ & \triangleright K_x \setminus V_0 = \bigcup_{k,l} K_{\varphi_{k,l}(x)} \triangleright F(x,s) := e^{-ds} \mathcal{N}_x^{\mathrm{Dir.}}(e^{2s}) \\ & K_x x = (\alpha, \beta, \gamma) \\ & K_{\varphi_{2,2}(x)} K_{\varphi_{1,2}(x)} \\ & K_{\varphi_{2,1}(x)} K_{\varphi_{1,1}(x)} \\ & K_{\varphi_{3,1}(x)} \\ & K_{\varphi_{3,1}(x)} \\ & K_{\varphi_{3,2}(x)} \\ & K_{\varphi_{5,2}(x)} \\ & K_{\varphi_{5,2}(x)} \\ & K_{\varphi_{5,2}(x)} \\ & K_{\varphi_{5,2}(x)} \\ & (X_{\alpha,\beta,\gamma}) \\ & K_{\varphi_{5,2}(x)} \\ & (X_{\alpha,\gamma}) \\ & (X_{\alpha,\beta}) \\ & (X_{\alpha,\beta,\gamma}) \\ & (X_{\alpha,$$

$$\begin{array}{ll} \mathsf{Thm}\,\mathbf{1}\,(\mathsf{K}.). \stackrel{\exists}{=} c_{0} \in (0,\infty), \ \forall \alpha, \beta, \gamma \in (0,\infty), \ (d:=d_{\mathsf{AG}}) \\ \lim_{\lambda \to \infty} \#\{n \in \mathbb{N} \mid \lambda_{n}^{\alpha,\beta,\gamma} \leq \lambda\}/\lambda^{d_{\mathsf{AG}}/2} = c_{0}\mathcal{H}^{d_{\mathsf{AG}}}(K_{\alpha,\beta,\gamma}). \\ \triangleright \{[X_{n}]\}_{n=0}^{\infty}: \mathsf{MC} \text{ on } \Gamma_{(\mathsf{shapes})}, x \rightsquigarrow [\varphi_{k,l}(x)] \text{ w. prob. } \mathcal{H}^{d}(K_{\alpha,\beta,\gamma}). \\ \triangleright \{[X_{n}]\}_{n=0}^{\infty}: \mathsf{MC} \text{ on } \Gamma_{(\mathsf{shapes})}, x \rightsquigarrow [\varphi_{k,l}(x)] \text{ w. prob. } \mathcal{H}^{d}(K_{\varphi_{k,l}(x)}) \\ \triangleright V_{n} := -\frac{1}{d} \log \mathcal{H}^{d}(K_{X_{n}}) \text{ (the changes in size along } \{[X_{n}]\}_{n=0}^{\infty}) \\ \triangleright K_{x} \setminus V_{0} = \bigcup_{k,l} K_{\varphi_{k,l}(x)} \triangleright F(x,s) := e^{-ds} \mathcal{N}_{x}^{\mathrm{Dir.}}(e^{2s}) \\ K_{x} x = (\alpha, \beta, \gamma) \\ K_{\varphi_{2,2}(x)} \qquad K_{\varphi_{1,2}(x)} \qquad = \mathcal{R}(x,s) + \sum_{k,l} F(\varphi_{k,l}(x),s) \\ K_{\varphi_{2,1}(x)} \qquad K_{\varphi_{1,1}(x)} \qquad K_{\varphi_{6,1}(x)} \\ K_{\varphi_{6,1}(x)} \qquad K_{\varphi_{6,2}(x)} \\ K_{\varphi_{6,2}(x)} \qquad K_{\varphi_{6,2}(x)} \\ K_{\varphi_{6,2}(x)} \qquad K_{\varphi_{6,2}(x)} \\ K_{\varphi_{6,3}(x)} \qquad K_{\varphi_{6,2}(x)} \\ K_{\varphi_{6,3}(x)} \qquad K_{\varphi_{6,2}(x)} \\ K_{\varphi_{6,3}(x)} \qquad K_{\varphi_{6,2}(x)} \\ K_{\varphi_{6,3}(x)} \qquad K_{\varphi_{6,3}(x)} \\ K_{\varphi_{6,3}($$

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$$\begin{aligned} & \operatorname{Thm} 1 (\mathsf{K}.). \ \exists c_0 \in (0,\infty), \ \forall \alpha, \beta, \gamma \in (0,\infty), \ (d:=d_{\mathsf{AG}}) \\ & \lim_{\lambda \to \infty} \# \{n \in \mathbb{N} \mid \lambda_n^{\alpha,\beta,\gamma} \leq \lambda\} / \lambda^{d_{\mathsf{AG}}/2} = c_0 \mathcal{H}^{d_{\mathsf{AG}}}(K_{\alpha,\beta,\gamma}). \\ & \triangleright \{[X_n]\}_{n=0}^{\infty}: \mathsf{MC} \text{ on } \Gamma_{(\mathsf{shapes})}, x \rightsquigarrow [\varphi_{k,l}(x)] \text{ w. prob. } \mathcal{H}^{d}(K_{\varphi_{k,l}(x)}) \\ & \triangleright V_n := -\frac{1}{d} \log \mathcal{H}^d(K_{X_n}) \ (\mathsf{the changes in size along } \{[X_n]\}_{n=0}^{\infty}) \\ & \triangleright K_x \setminus V_0 = \bigcup_{k,l} K_{\varphi_{k,l}(x)} \mathrel{\triangleright} F(x,s) := e^{-ds} \mathcal{N}_x^{\mathsf{Dir.}}(e^{2s}) \\ & = \mathcal{R}(x,s) + \sum_{k,l} \mathcal{H}^d(K_{\varphi_{k,l}(x)}) \cdot \\ & K_{\varphi_{2,3}(x)} \qquad K_{\varphi_{1,3}(x)} \qquad F([\varphi_{k,l}(x)],s + \frac{1}{d} \log \mathcal{H}^d(K_{\varphi_{k,l}(x)})) \\ & K_{\varphi_{2,3}(x)} \qquad K_{\varphi_{1,2}(x)} \qquad K_{\varphi_{1,2}(x)} \qquad S \xrightarrow{\star \infty} \mathcal{N}_x ([X_n],s - V_n)] \\ & K_{\varphi_{3,1}(x)} \qquad K_{\varphi_{6,1}(x)} \qquad K_{\varphi_{6,1}(x)} \qquad K_{\varphi_{6,2}(x)} \\ & K_{\varphi_{3,2}(x)} \qquad K_{\varphi_{6,2}(x)} \qquad K_{\varphi_{6,2}(x)} \\ & K_{\varphi_{3,2}(x)} \qquad K_{\varphi_{6,2}(x)} \qquad K_{\varphi_{6,2}(x)} \\ & K_{\varphi_{3,2}(x)} \qquad K_{\varphi_{6,2}(x)} \qquad K_{\varphi_{6,2}(x)} \\ & K_{\varphi_{6,3}(x)} \qquad K_{\varphi_{6,3}(x)} \qquad K_{\varphi_{6,3}(x)} \\ & K_{\varphi_{6,3}(x)} \qquad K_{\varphi_{6,3}(x)} \qquad K_{\varphi_{6,3}(x)} \\ & K_{\varphi_{6,3}(x)} \qquad K_{\varphi_{6,3}(x)} \qquad$$

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$$\exists c_0 \in (0, \infty), \forall \alpha, \beta, \gamma \in (0, \infty), (d := d_{AG})$$

 $\lim_{\lambda \to \infty} \# \{n \in \mathbb{N} \mid \lambda_n^{\alpha,\beta,\gamma} \leq \lambda\} / \lambda^{d_{AG}/2} = c_0 \mathcal{H}^{d_{AG}}(K_{\alpha,\beta,\gamma}).$
 $\triangleright \{[X_n]\}_{n=0}^{\infty}$: MC on $\Gamma_{(shapes)}, x \rightsquigarrow [\varphi_{k,l}(x)]$ w. prob. $\mathcal{H}^d(K_{\varphi_{k,l}(x)})$
 $\triangleright V_n := -\frac{1}{d} \log \mathcal{H}^d(K_{X_n})$ (the changes in size along $\{[X_n]\}_{n=0}^{\infty}$)
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 $K_{\varphi_{2,3}(x)} \qquad K_{\varphi_{1,3}(x)} \qquad F([\varphi_{k,l}(x)],s+\frac{1}{d} \log \mathcal{H}^d(K_{\varphi_{k,l}(x)})).$
 $K_{\varphi_{3,1}(x)} \qquad K_{\varphi_{6,1}(x)} \qquad F([\mathcal{R}(x,s)] \leq c' e^{-c|s|^{\alpha}}.$
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Cor (Special case of Oh–Shah '12). $\exists c > 0, \forall \alpha, \beta, \gamma \in (0, \infty),$ $\#\{C \subset_{\operatorname{circle}} K_{\alpha,\beta,\gamma} | \operatorname{ra}(C)^{-1} \leq \lambda\}^{\lambda \to \infty} c \cdot \mathcal{H}^d(K_{\alpha,\beta,\gamma}) \lambda^d.$ $> \{[X_n]\}_{n=0}^{\infty}: \mathsf{MC} \text{ on } \Gamma_{(\mathsf{shapes})}, x \sim [\varphi_{k,l}(x)] \text{ w. prob. } \mathcal{H}^d(K_{\varphi_{k,l}(x)}) \\ > V_n := -\frac{1}{d} \log \mathcal{H}^d(K_{X_n}) \text{ (the changes in size along } \{[X_n]\}_{n=0}^{\infty})$ $hinspace{-ds} K_x ackslash V_0 \!=\! igcup_{k,l} K_{arphi_{k,l}(x)}
hinspace{-ds} F(x,s) := e^{-ds} \mathbb{N}^{\mathrm{rad}}_x(e^s)$ $= \Re(x,s) + \sum_{k,l} \mathcal{H}^d(K_{\varphi_{k,l}(x)})$ $K_x x = (\alpha, \beta, \gamma)$ $F([\varphi_{k,l}(x)],s+\frac{1}{d}\log\mathcal{H}^{d}(K_{\varphi_{k,l}(x)}))$ $K_{arphi_{2,4}(x)} = K_{arphi_{1,4}(x)} = K_{arphi_{2,3}(x)} = K_{arphi_{1,3}(x)}$ $= \mathbb{E}_{\boldsymbol{x}}\left[\sum_{n=0}^{\infty} \mathcal{R}([X_n], s - V_n)\right]$ $K_{\varphi_{2,2}(x)} \land K_{\varphi_{1,2}(x)}$ $\xrightarrow{s \to \infty}_{\text{Kesten '74}} \int_{\Gamma \times \mathbb{R}} \Re(x,s) d\nu(x) ds!$ $K_{arphi_{1,1}(x)}$ $K_{\varphi_{2,1}(x)}$ Need: • $|\Re(x,s)| \leq c' e^{-c|s|^{\alpha}}$. $oldsymbol{K}_{arphi_{3,1}(x)}$ $ig K_{arphi_{6,1}(x)}$ $K_{arphi_{3,2}(x)}$ $\langle K_{arphi_{6,2}(x)}$ $ullet \{(X_n,V_n)\}_{n=0}^\infty$ unique. ergodic $K_{arphi_{3,3}(x)}$ cf. p. 30, Figure 3 of R. D. Mauldin & M. Urbański, Adv. Math. 136 (1998), 26–38 $K_{\varphi_{5,2}(x)}$ $\varphi_{4,3}(x)$

5 Proof ingredients for $G = G_m$ with $\partial_{\infty}G$ a RSC



5 Proof ingredients for $G = G_m$ with $\partial_{\infty}G$ a RSC

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- A "self-similar" decomp. ("fund. dom." for $G \cap \partial_{\infty} G$) (requires concrete knowledge of $G \cap \partial_{\infty} G$; NOT extend easily)
- A version of the "2-dimensional" Nash inequality (\rightsquigarrow the spectrum of Δ_{K_g} is discrete & $\exists p_t^{K_g}(x,y) \leq c_g t^{-1}$) • $\iota: K_g \hookrightarrow \mathbb{R}^2$ is \mathcal{E}^g -harmonic & $\Gamma_{\mathcal{E}^g}(\iota, \iota) = \nu^g$ ($\rightsquigarrow \int \langle X^g \rangle$), slower than $\int B^{\mathbb{R}} \downarrow \sim \mathbb{P}[\tau_{\mathbb{R}^d}(\iota, \iota) \leq t]$ small)

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- $\triangleright \nu^{g} := \sum_{C \subset \operatorname{arc} K_{g}} \operatorname{rad}(C) \operatorname{vol}_{C} (\operatorname{NOT doubling!})$ $\triangleright^{\forall} u \in \operatorname{LIP}|_{K_{g}}, \, \mathcal{E}^{g}(u, u) := \sum_{C \subset \operatorname{arc} K_{g}} \operatorname{rad}(C) \int_{C} |\nabla_{C} u|^{2} d\operatorname{vol}_{C} (cf. \operatorname{Osada'07})$

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 - $(\rightsquigarrow \{\langle X_t^g, z \rangle\}_t \text{ slower than } \{B_t^{\mathbb{R}}\}_t \rightsquigarrow \mathbb{P}_x[au_{B(x,r)} \leq t] \text{ small!})$

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- •(*cf.* Bonk '11) The circles in $\partial_{\infty}G$ are unif. rel. separated: $\forall j \neq k, \operatorname{dist}(C_j, C_k) \geq \delta_m \min\{\operatorname{rad}(C_j), \operatorname{rad}(C_k)\}.$