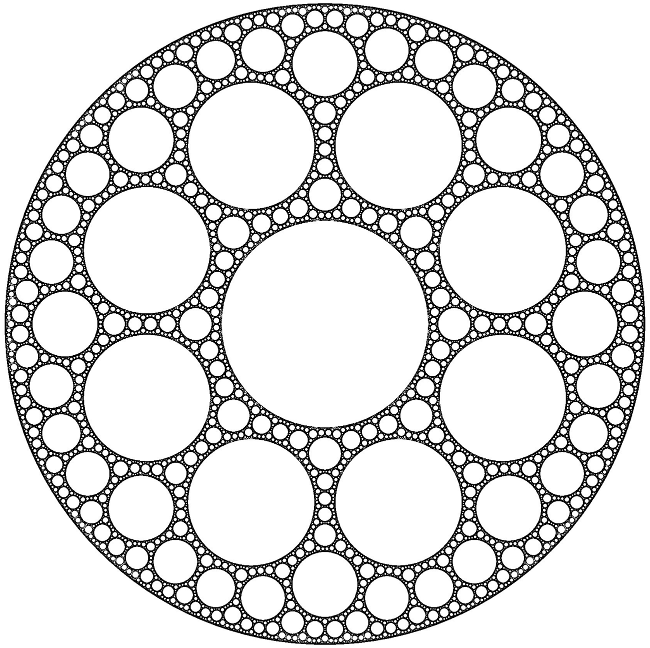
Geometric Laplacians on self-conformal fractal curves in the plane

Naotaka Kajino (RIMS, Kyoto University)
梶野 直孝（京都大学・数理解析研究所）

French Japanese Conference on Probability & Interactions
@Centre de conférences Marilyn et James Simons, IHES



08 March 2024
15:10–16:00

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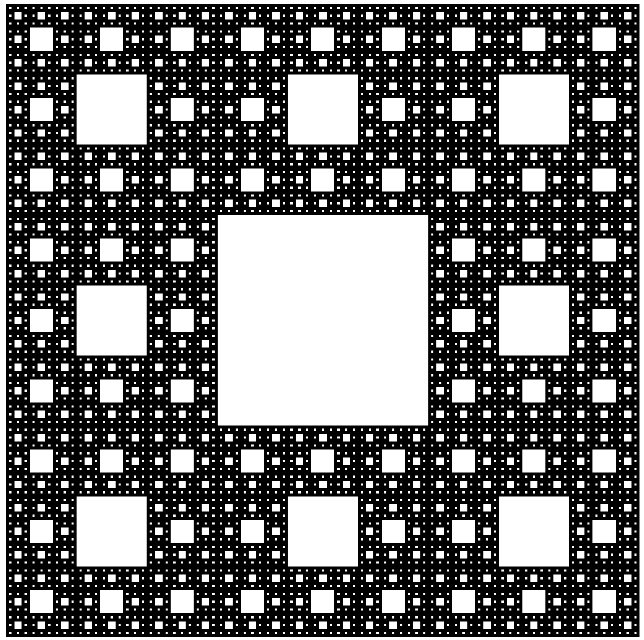

Pictures from: N. Kajino, Adv. Stud. Pure Math., vol. 87, 2021, pp. 293–314.

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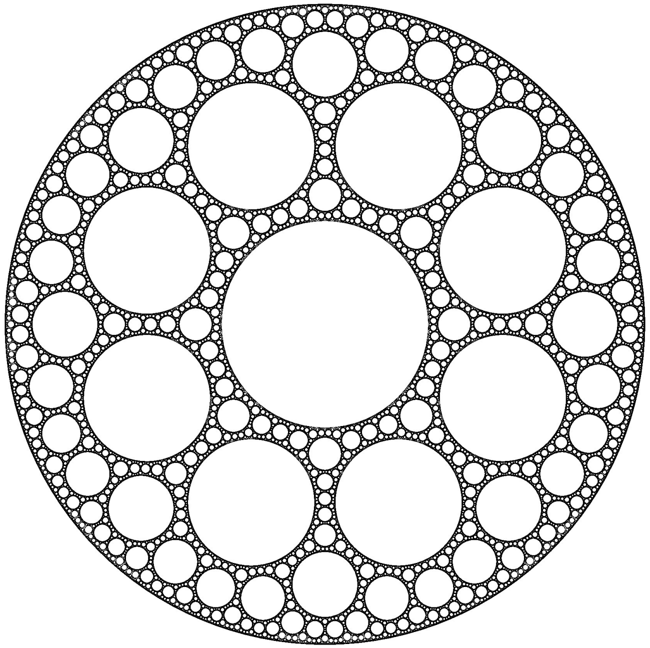

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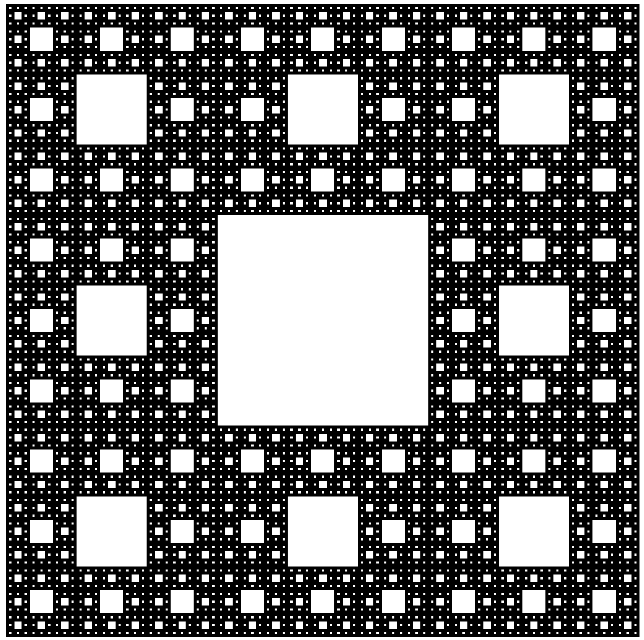

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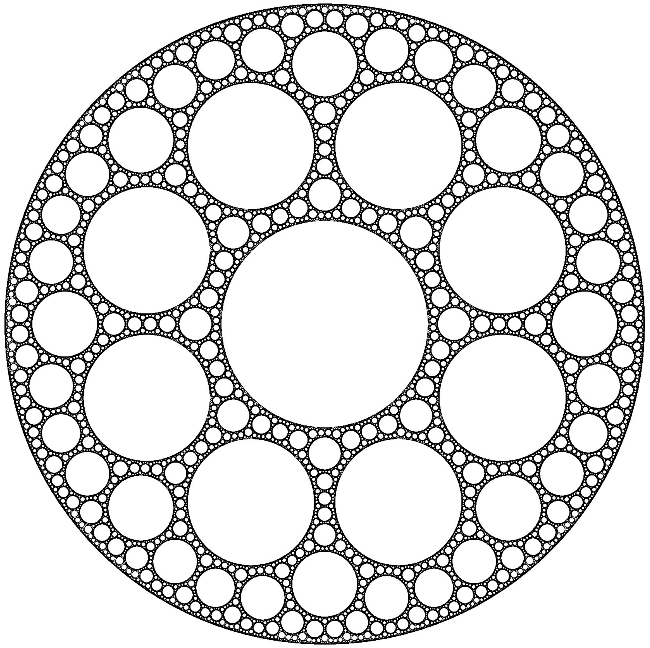

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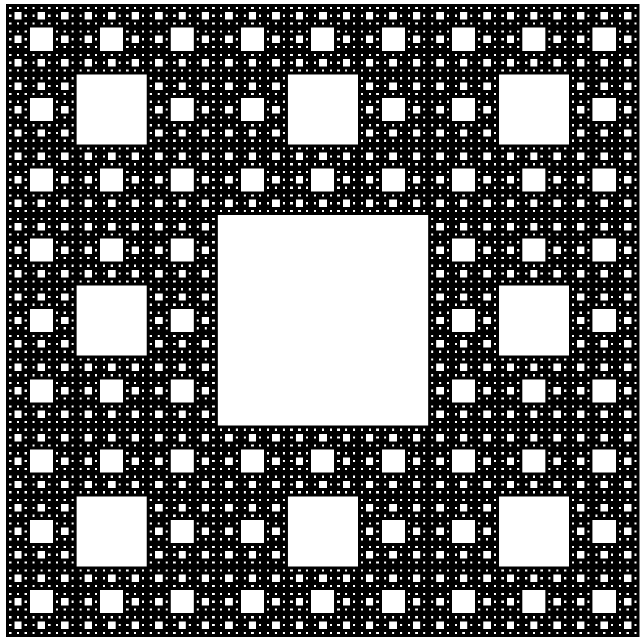

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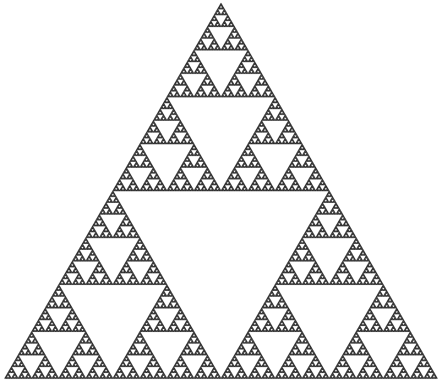
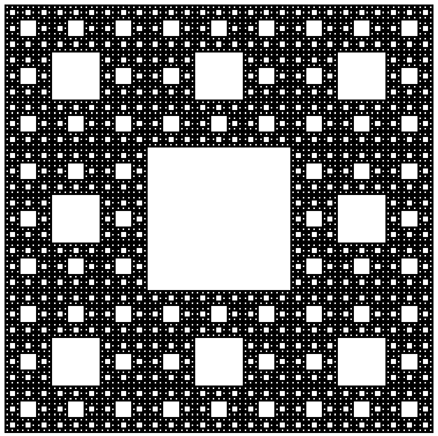
1 Fractals in self-similar & self-conformal geometries

(self-similar) SG
(Sierpiński Gasket)



(self-similar) SC
(Sierpiński Carpet)



Constr./Analysis of "B.M."

(SG: Barlow–Perkins '88,
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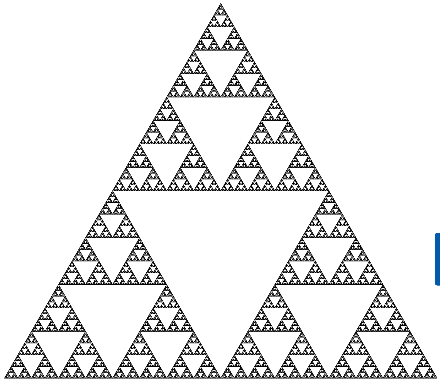

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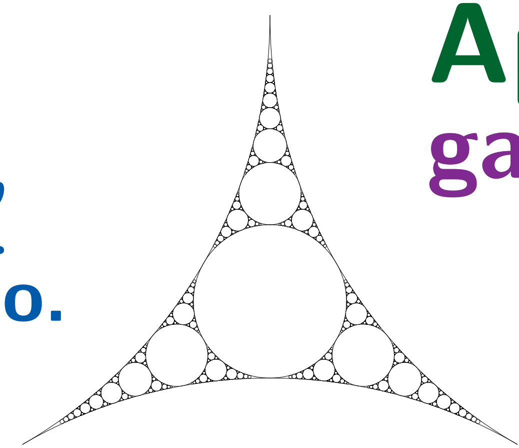

Apollonian
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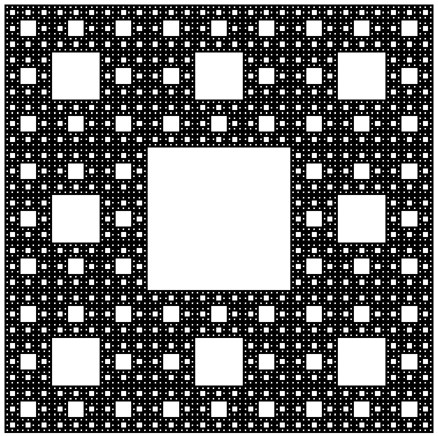

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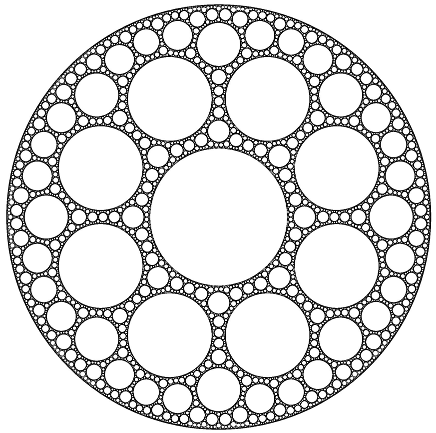

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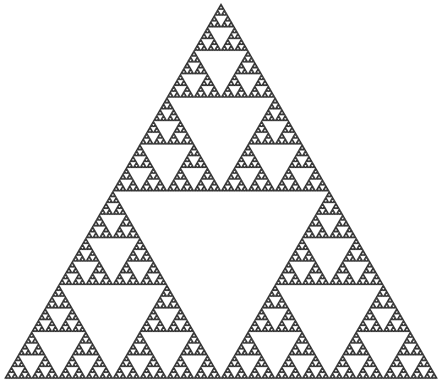

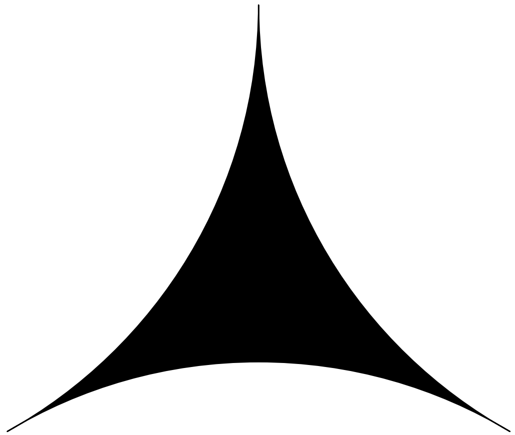

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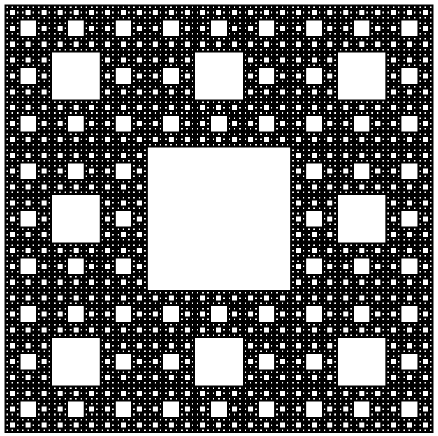

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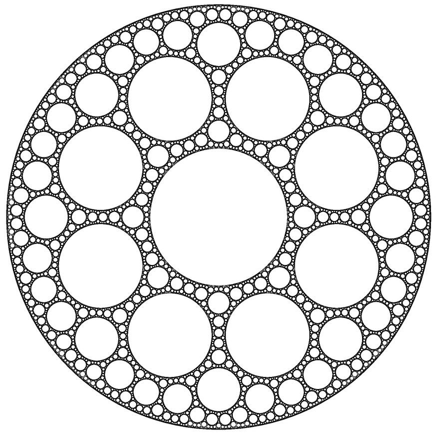

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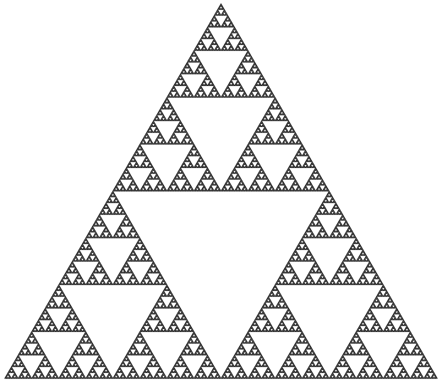

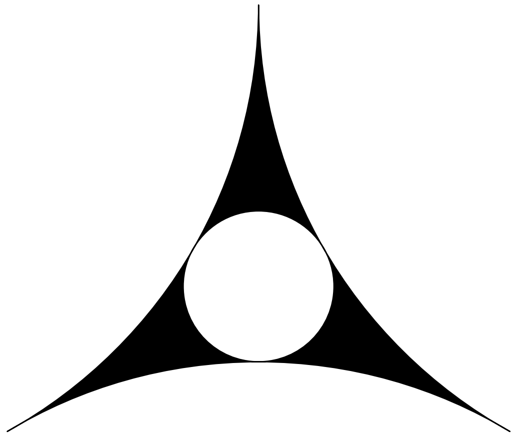

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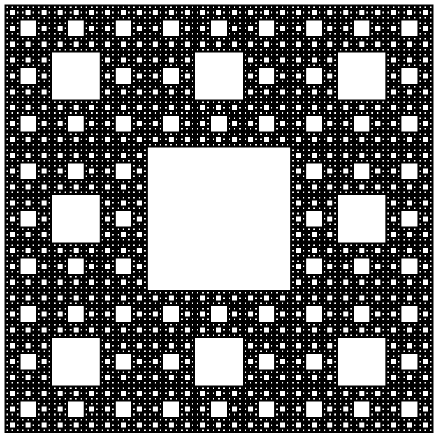

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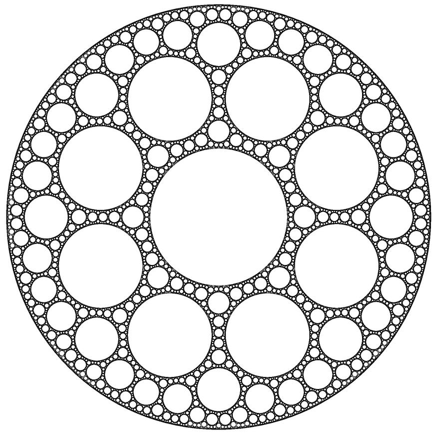

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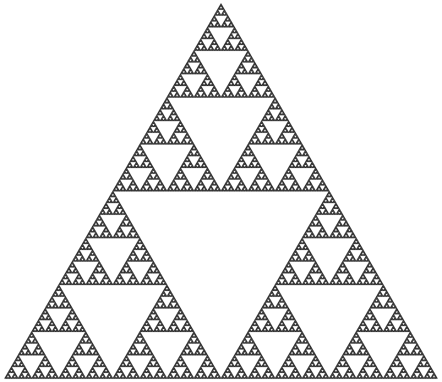

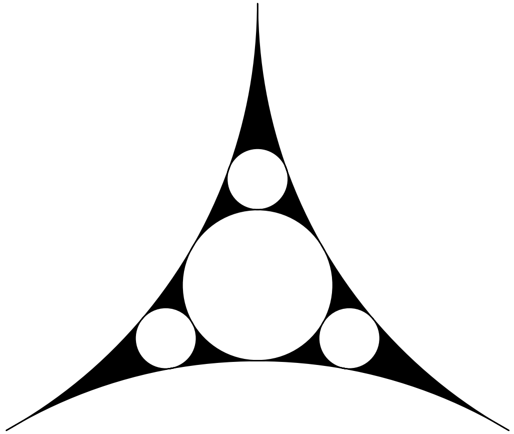

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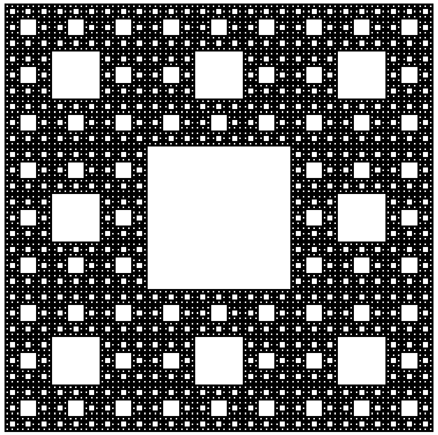

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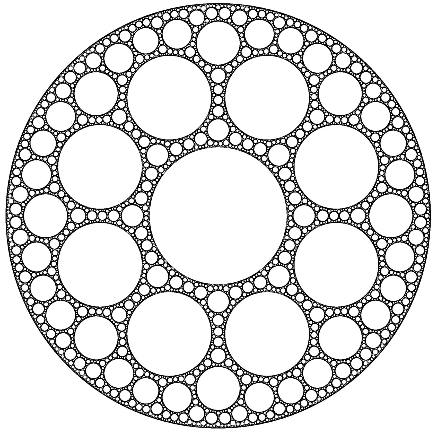

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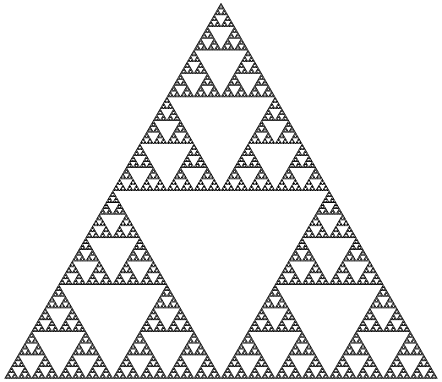

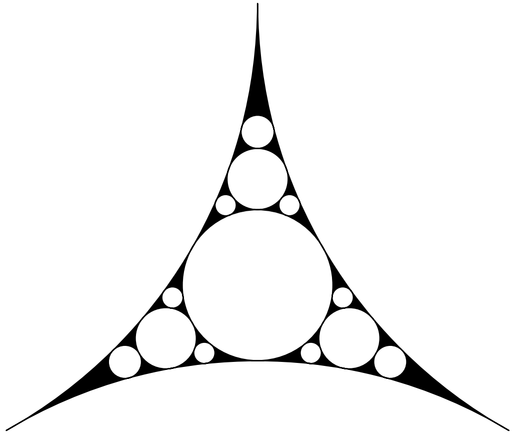

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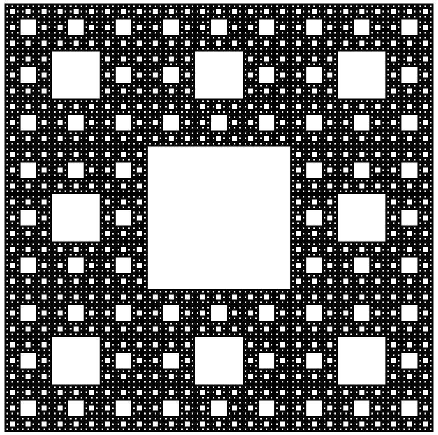

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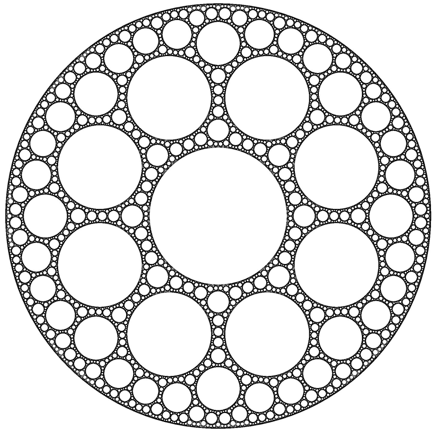

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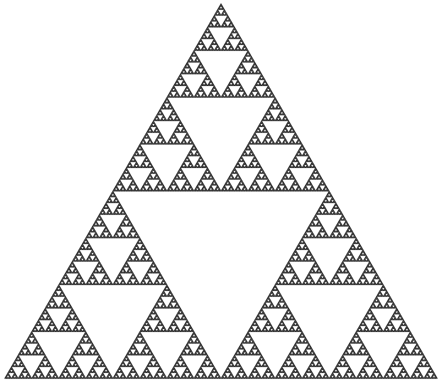

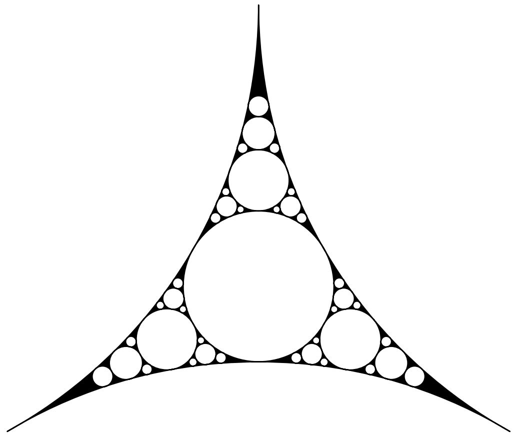

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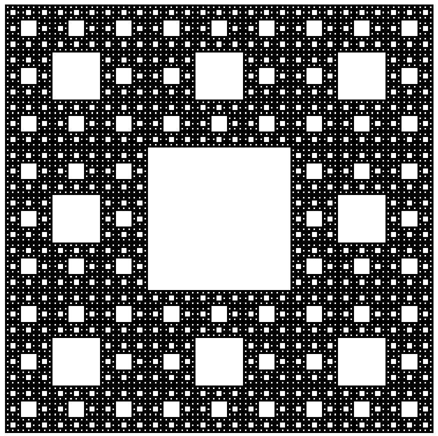

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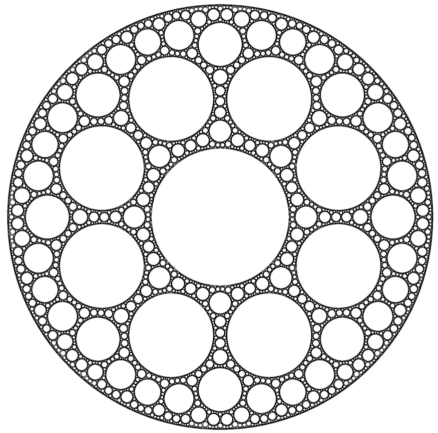

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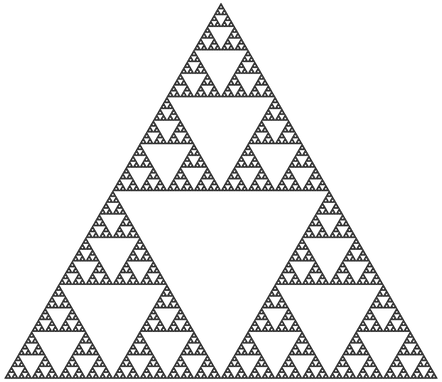

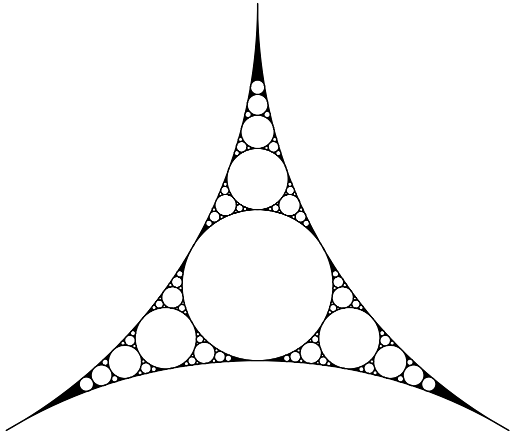

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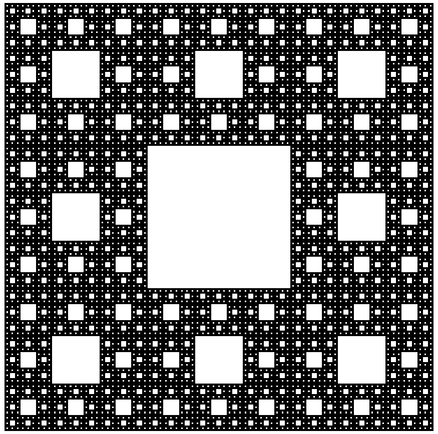

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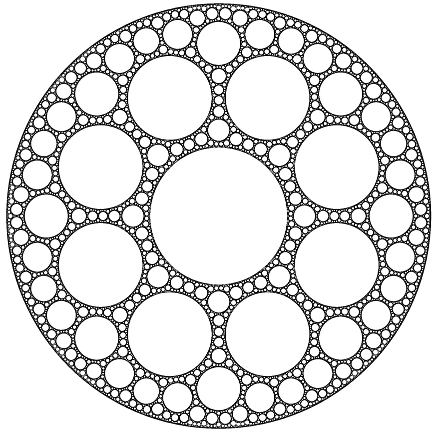

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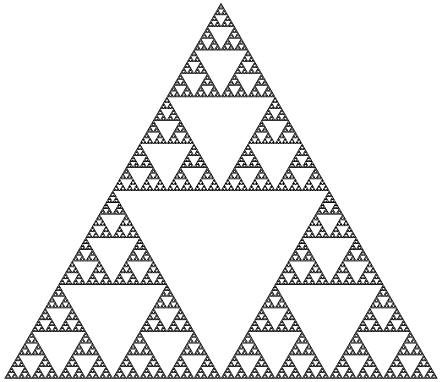

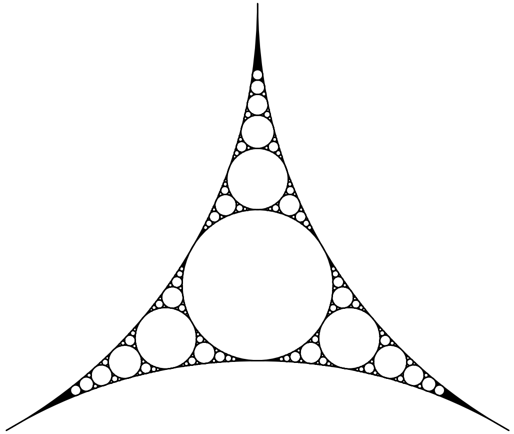

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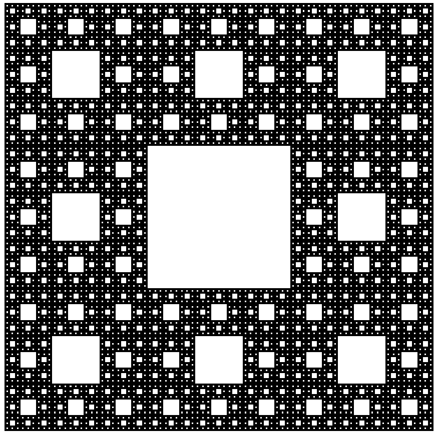

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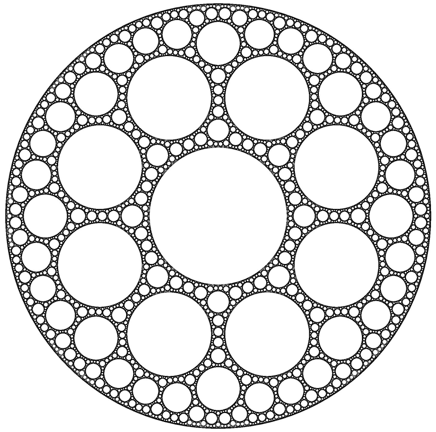

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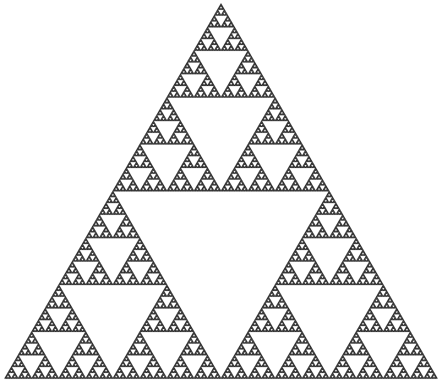

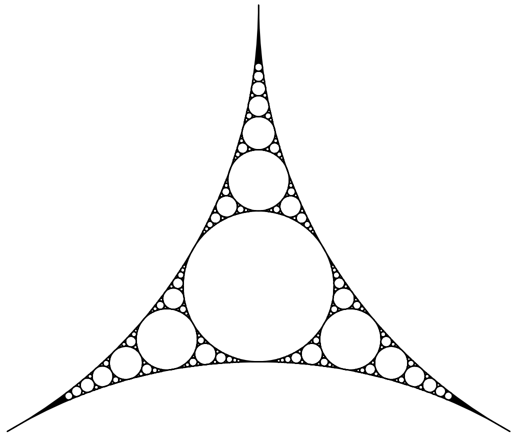

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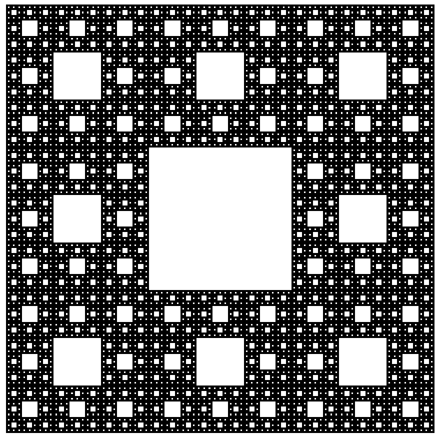

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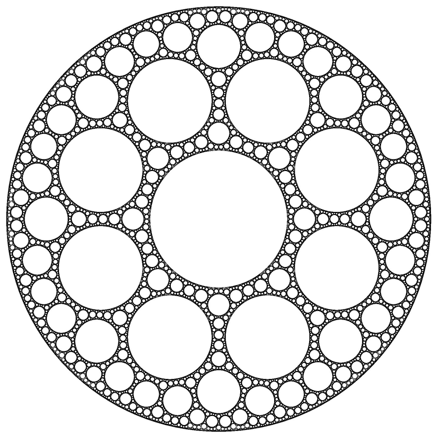

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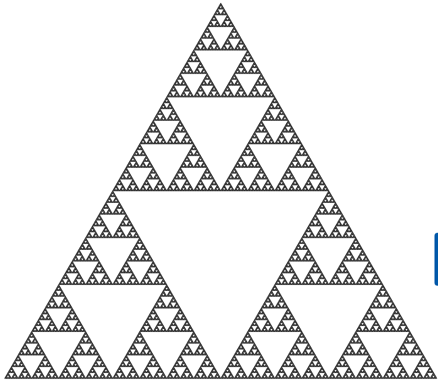

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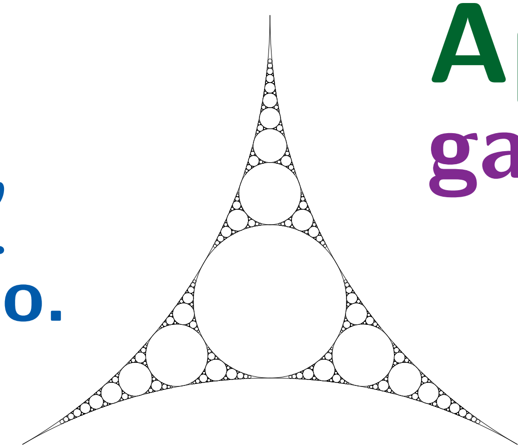

Apollonian
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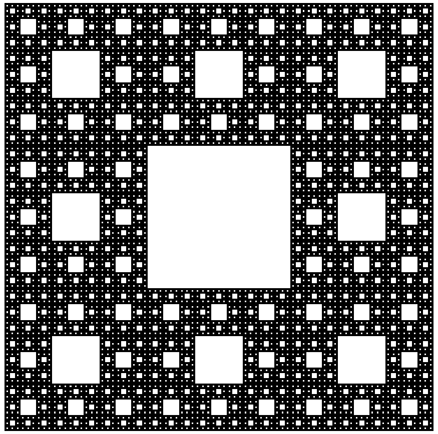

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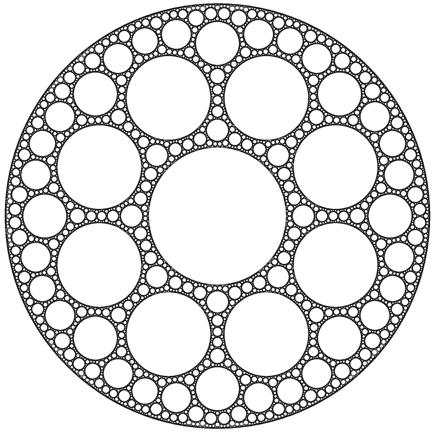

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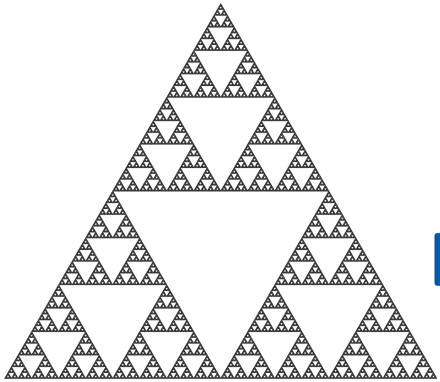

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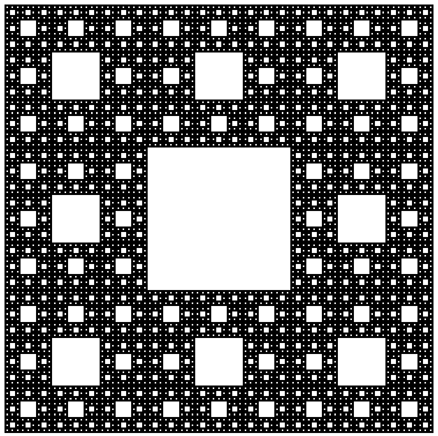

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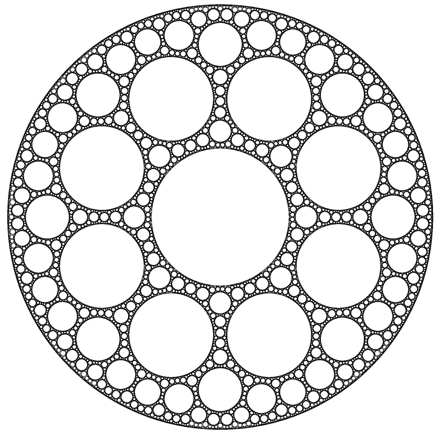

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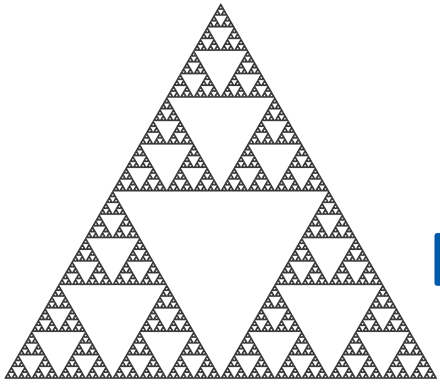

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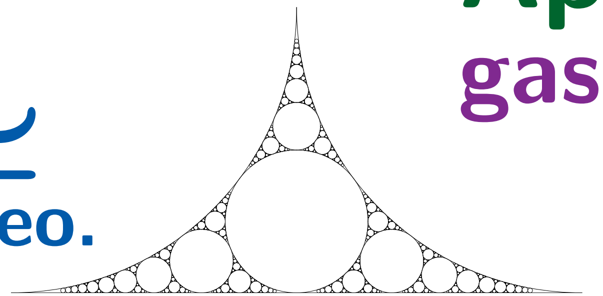

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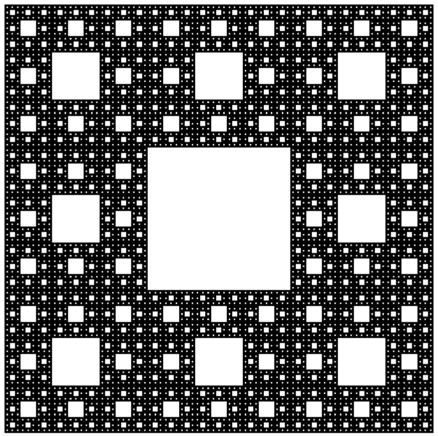

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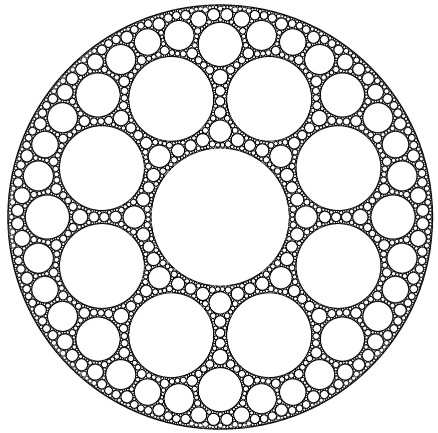

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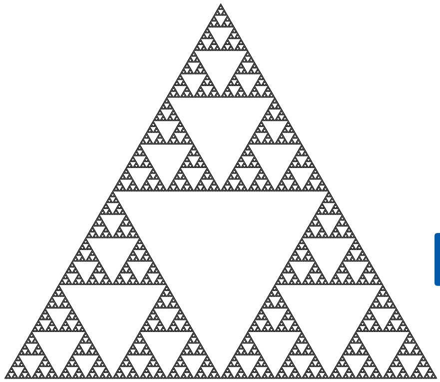

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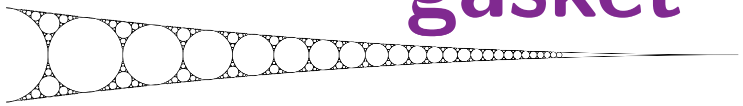

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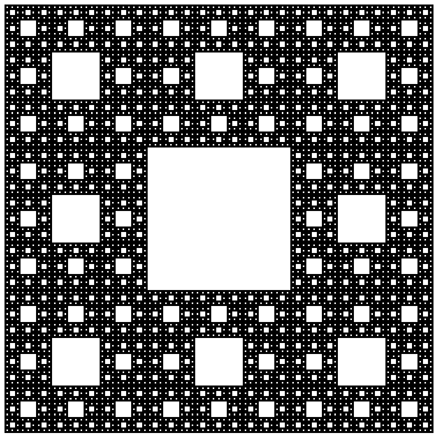

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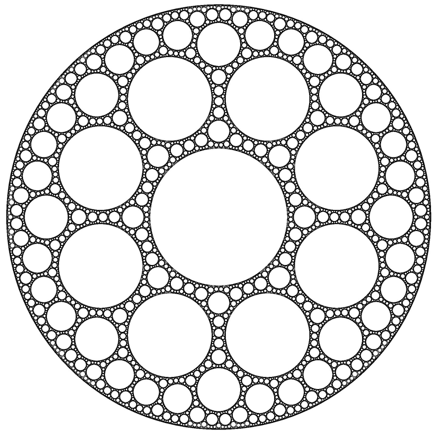


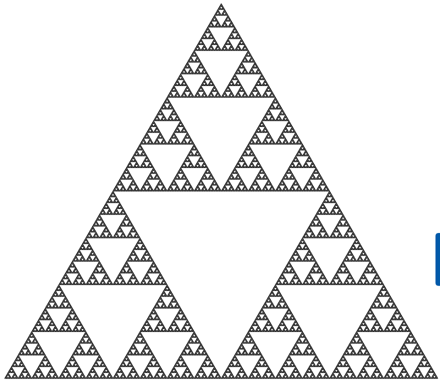
round SC

Constr./Analysis of "B.M."

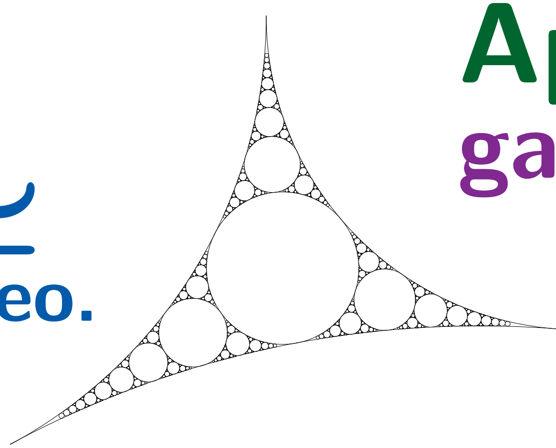
(
SG: Barlow–Perkins '88,
Goldstein '87, Kusuoka '87
SC: Barlow–Bass '89, '99
)

1 Fractals in self-similar & self-conformal geometries

(self-similar) SG
(Sierpiński Gasket)

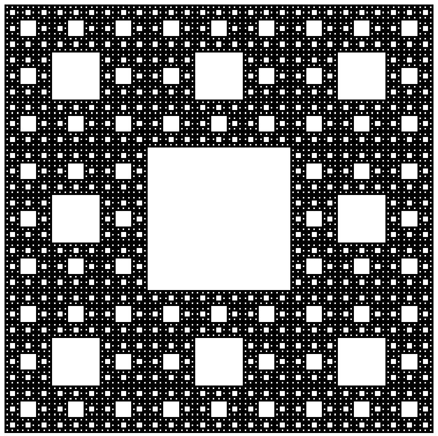


$\cong$
homeo.



Apollonian
gasket

(self-similar) SC
(Sierpiński Carpet)



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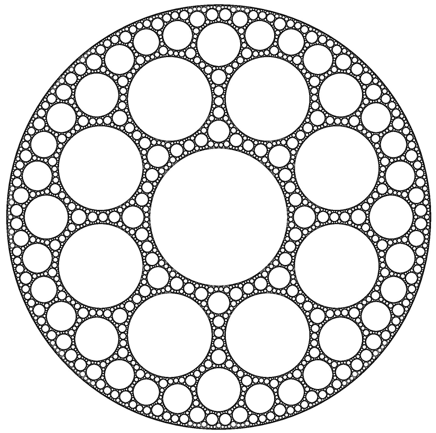

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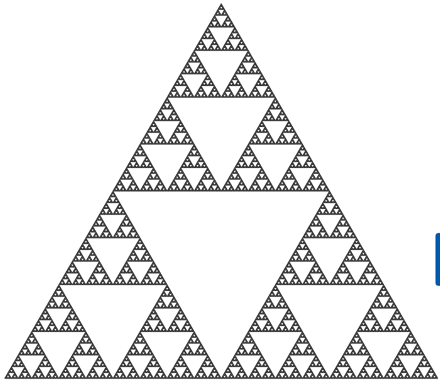
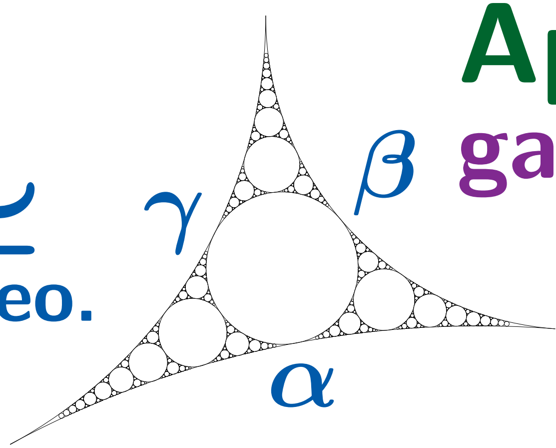
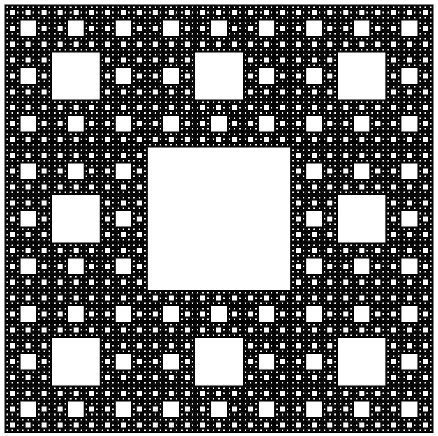


$\underset{\text{homeo.}}{\simeq}$



Apollonian gasket $K_{\alpha,\beta,\gamma}$

(self-similar) SC
(Sierpiński Carpet)



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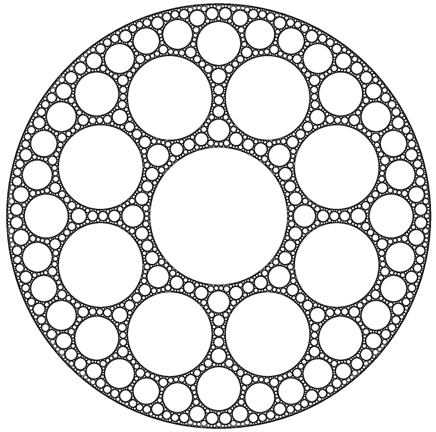

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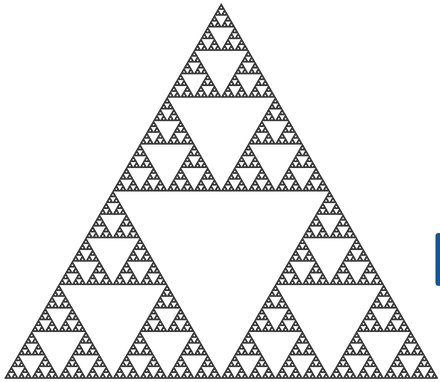
(Sierpiński Gasket)



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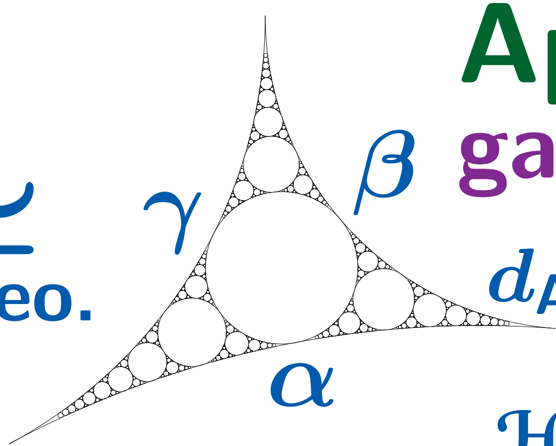
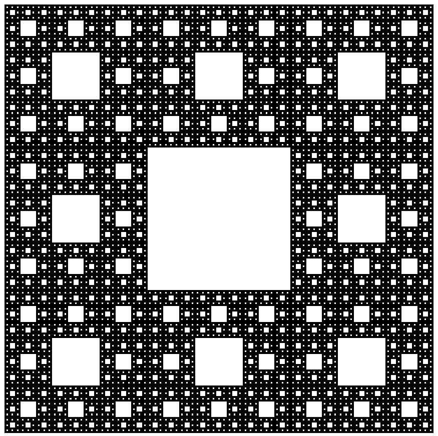


Apollonian gasket $K_{\alpha,\beta,\gamma}$

$d_{\mathsf{AG}} \in (1.3, 1.32)$ (Boyd '73)

$\mathcal{H}^{d_{\mathsf{AG}}}_{\alpha,\beta,\gamma} \in (0,\infty)$ (Sullivan '84)

(self-similar) SC
(Sierpiński Carpet)



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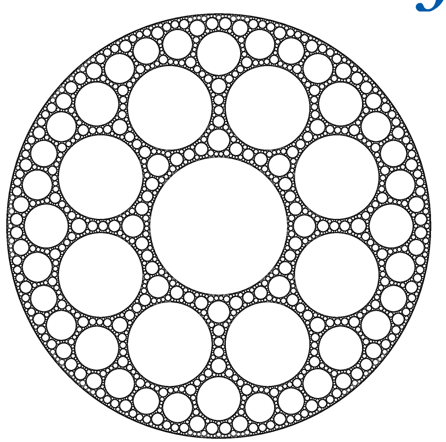

round SC

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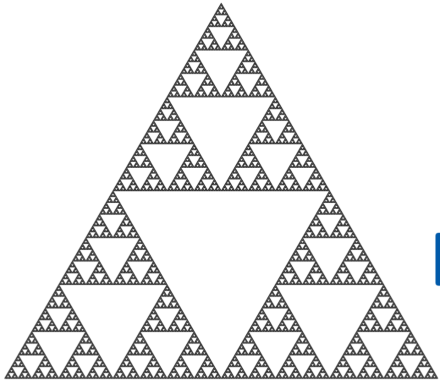

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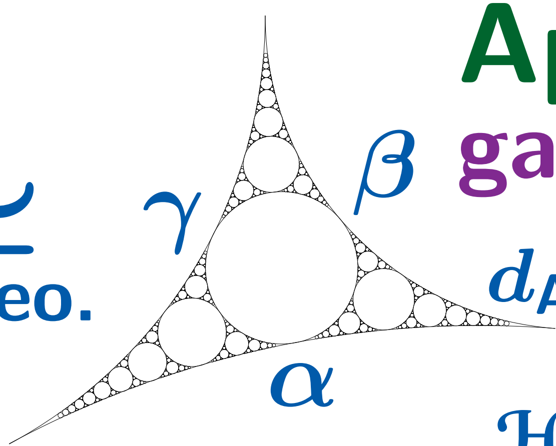

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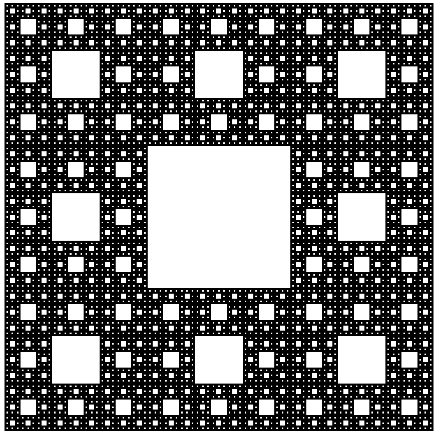

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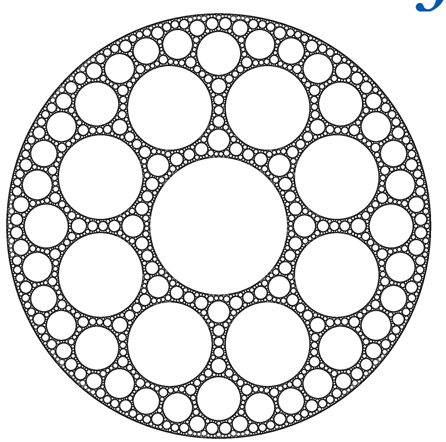


round SC

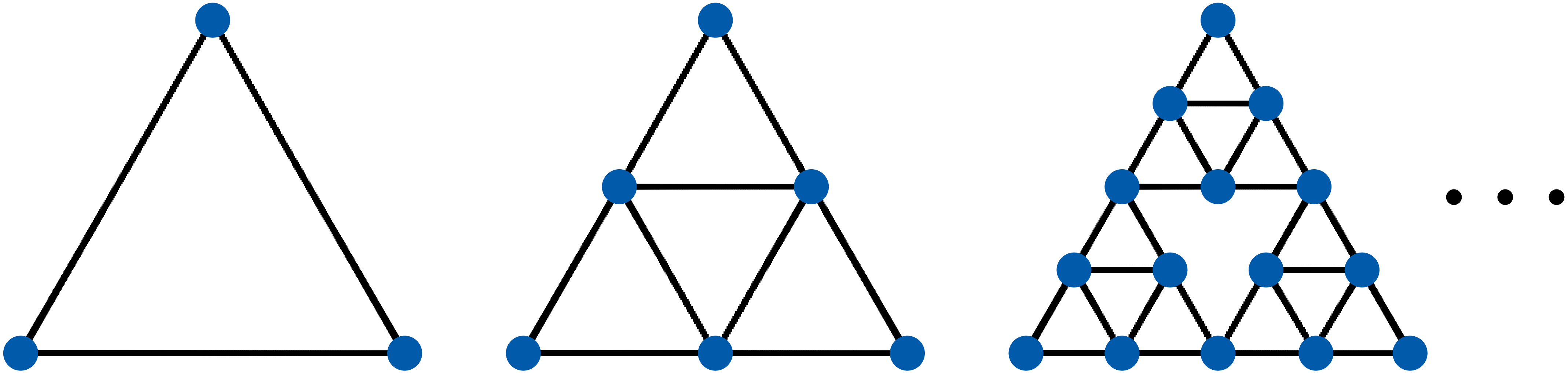
Constr./Analysis of "B.M."

(SG: Barlow–Perkins '88,
Goldstein '87, Kusuoka '87
SC: Barlow–Bass '89, '99)

Problem.
Construction & Analysis of "Laplacian" & "B.M." which respect given geometry?

(Goldstein '87, Kusuoka '87, Barlow-Perkins '88, ...)

cf. Standard Dirichlet form and B.M. on the S.G.

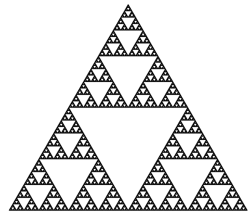


V_0 $\qquad$ V_1 $\qquad$ V_2 $\qquad \cdots$

$$\mathcal{E}_m(u,u) := (5/3)^m \sum_{x,y\in V_m,\, x \overset{m}{\sim} y} (u(x)-u(y))^2$$

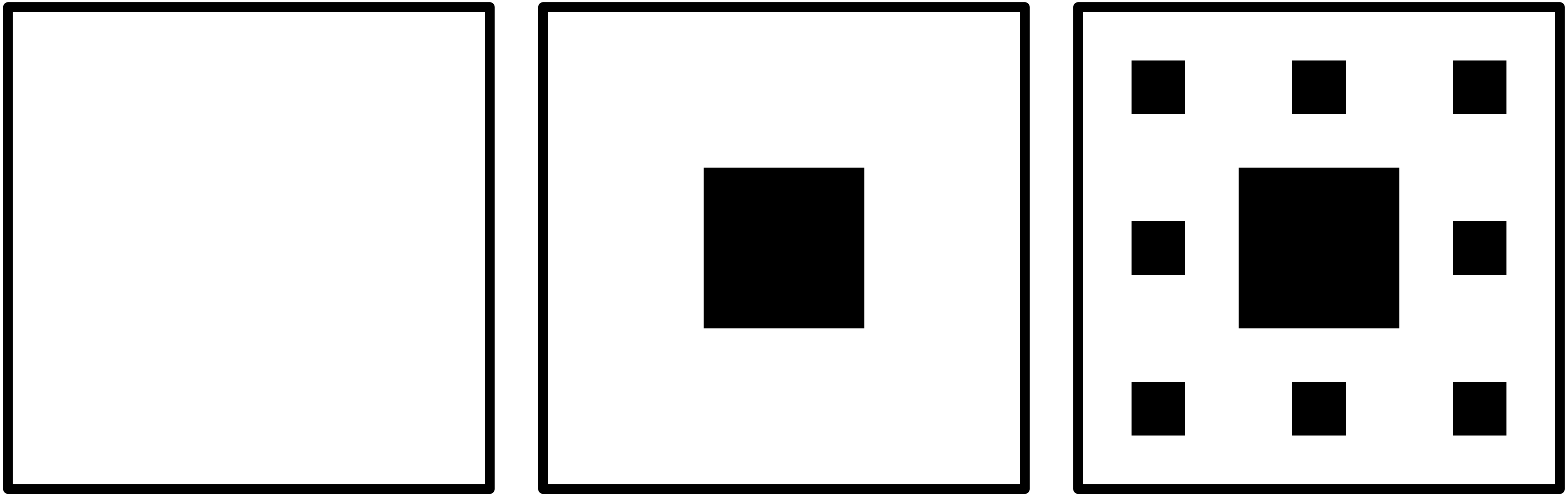
$$\Rightarrow \mathcal{E}_m(u,u) = \min\{\mathcal{E}_{m+1}(v,v) \mid v \in \mathbb{R}^{V_{m+1}},\ v|_{V_m} = u\}$$

$$\mathcal{F}_{\mathrm{st}} := \Bigl\{ u \in C(K) \Bigm| \mathcal{E}_{\mathrm{st}}(u) := \uparrow\lim_{m\to\infty} \mathcal{E}_m(u|_{V_m}, u|_{V_m}) < \infty \Bigr\}$$

$(\mathcal{E}_{\mathrm{st}}, \mathcal{F}_{\mathrm{st}})$ on $L^2(K, \mathcal{H}^{\log_2 3}|_K) \rightleftharpoons$ B.M. on $K =$ 

Dirichlet form & B.M. on self-similar SCs

- A self-similar regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ exists.
  (Barlow–Bass '89, '99, Kusuoka–Zhou '92)

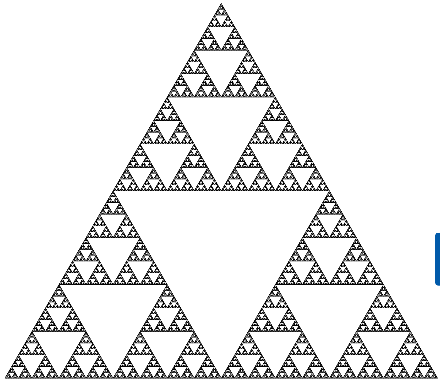


BB '89: ${}^{\exists 1}\tau > 1$, $\{\mathrm{Law}(\{B^{\mathrm{ref}, D_n}_{\tau^n t}\}_{t \geq 0})\}_{n=0}^{\infty}$ is tight.

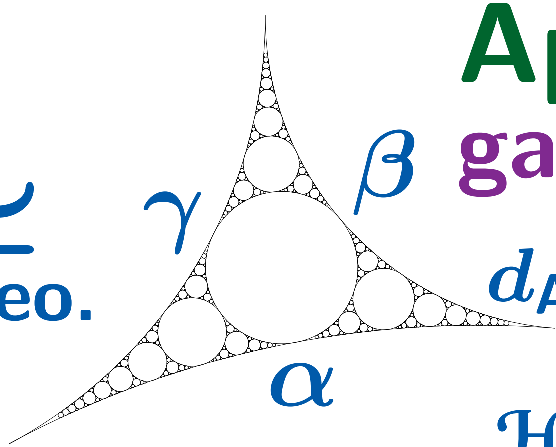
- Such a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ is unique.
  (Barlow–Bass–Kumagai–Teplyaev '10)

1 Fractals in self-similar & self-conformal geometries

(self-similar) SG
(Sierpiński Gasket)



$\overset{\simeq}{\text{homeo.}}$

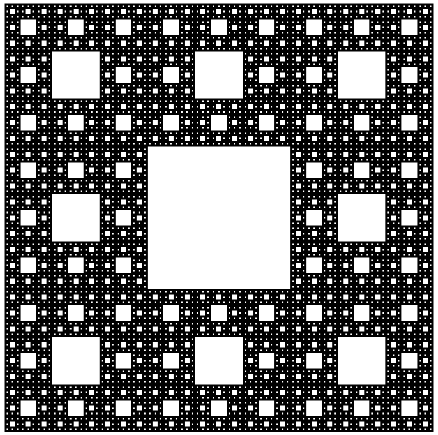


Apollonian gasket $K_{\alpha,\beta,\gamma}$

$d_{\mathsf{AG}} \in (1.3, 1.32)$ (Boyd '73)

$\mathcal{H}^{d_{\mathsf{AG}}}_{\alpha,\beta,\gamma} \in (0,\infty)$ (Sullivan '84)

(self-similar) SC
(Sierpiński Carpet)



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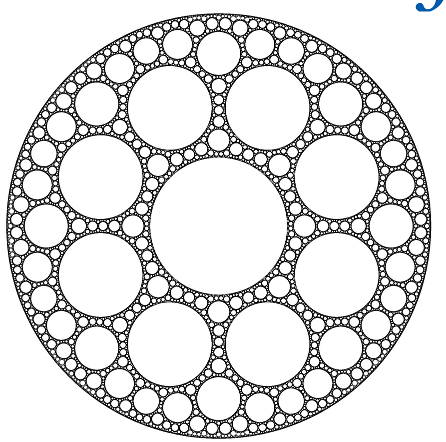

round SC

Constr./Analysis of "B.M."

(SG: Barlow–Perkins '88, Goldstein '87, Kusuoka '87
SC: Barlow–Bass '89, '99)

Problem.
Construction & Analysis of "Laplacian" & "B.M." which respect given geometry?

1 Fractals in self-similar & self-conformal geometries

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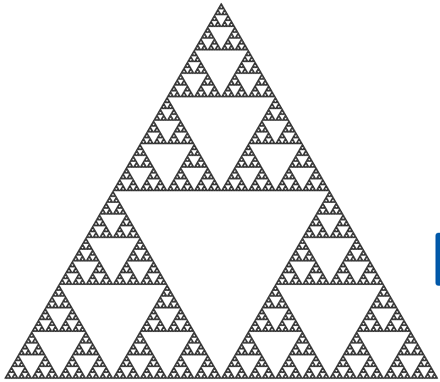
(Sierpiński Gasket)



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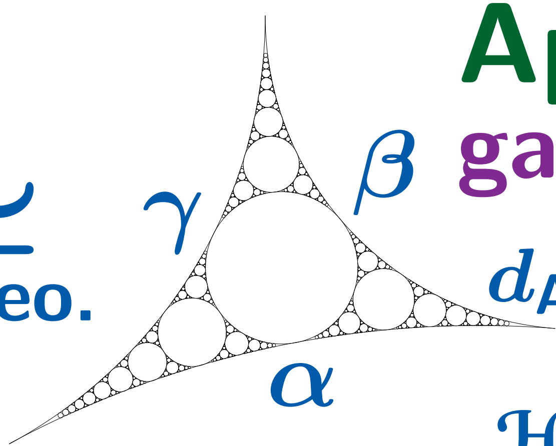

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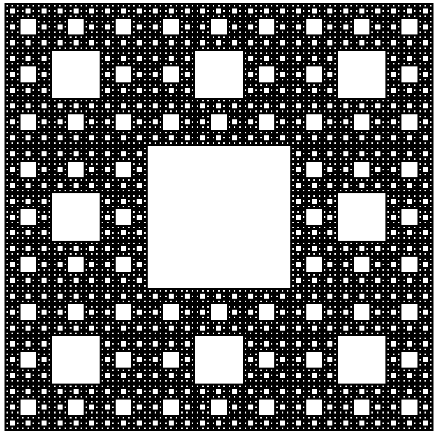

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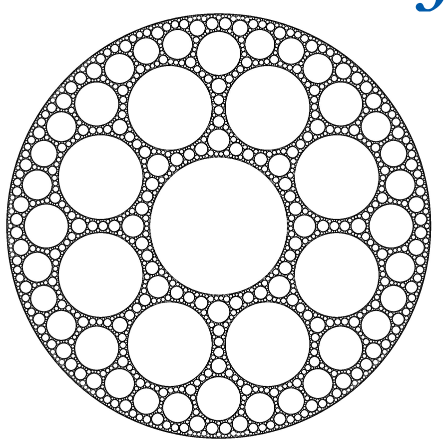

round SC

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(SG: Barlow–Perkins '88, Goldstein '87, Kusuoka '87
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Thm (K.).
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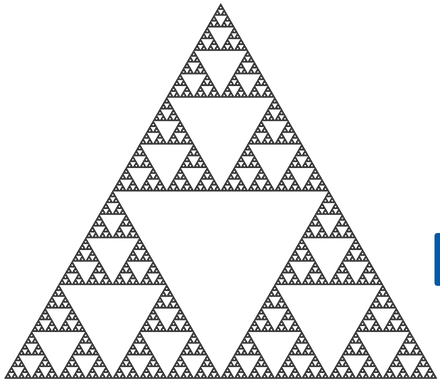

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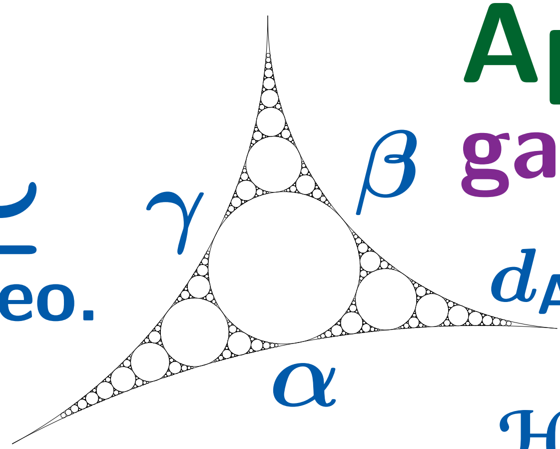

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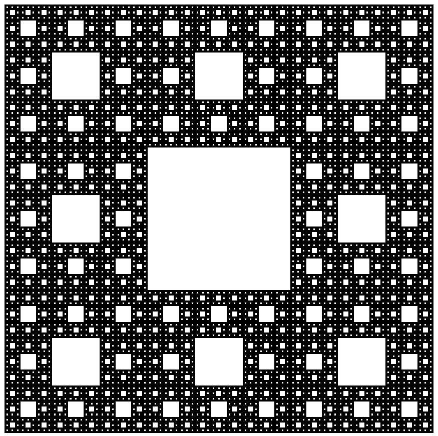

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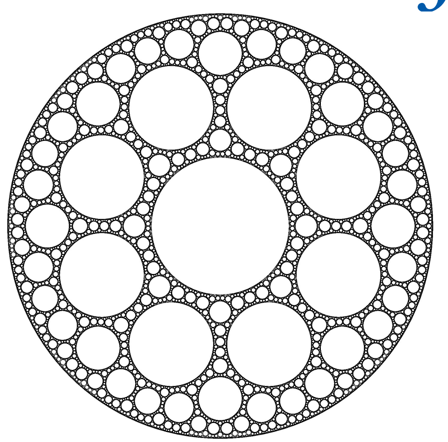

round SC

3. Quasi-conformal deformations of round SCs?

1 Fractals in self-similar & self-conformal geometries

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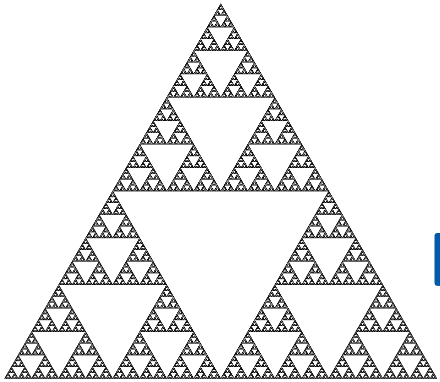

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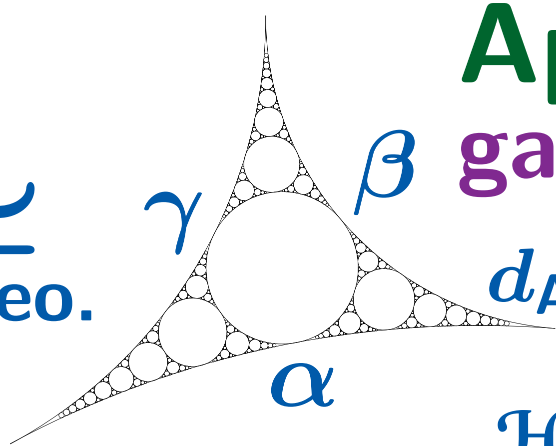

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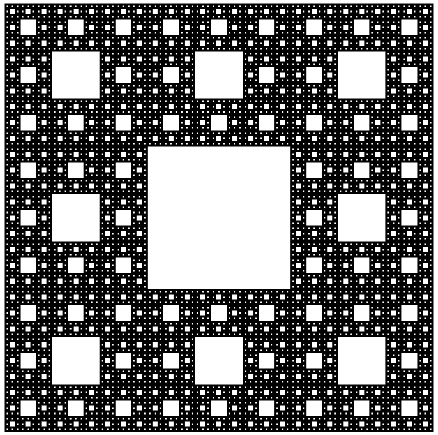

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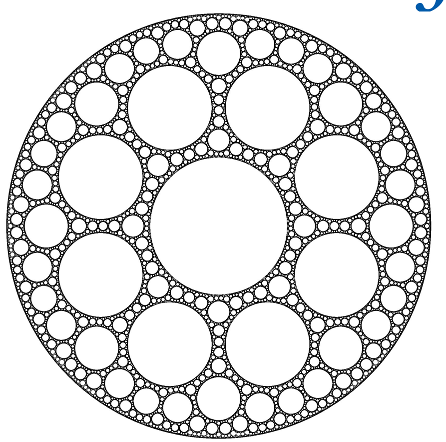

round SC

3. Quasi-conformal deformations of round SCs?
4. Self-conformal quasi-arcs $\partial \subset \partial_{\mathbb{C}} U$ (later this talk)

1 Fractals in self-similar & self-conformal geometries

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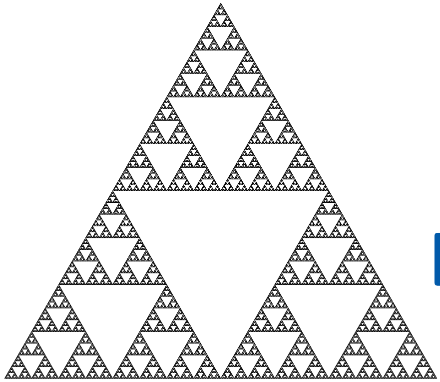
(Sierpiński Gasket)



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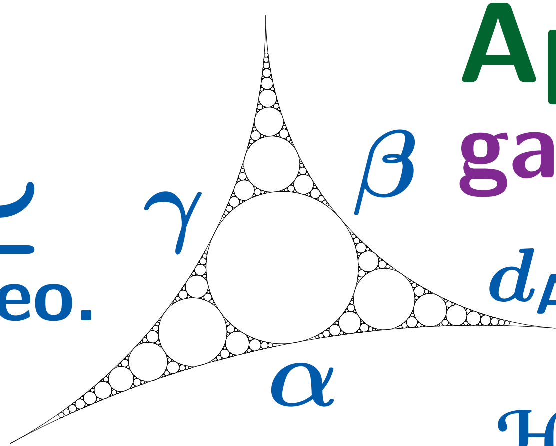

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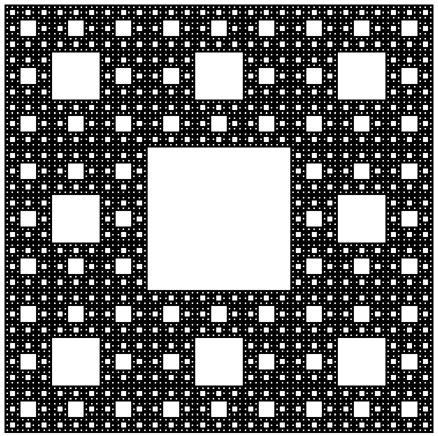
(Sierpiński Carpet)



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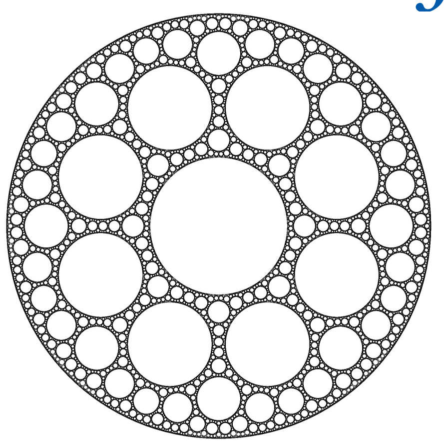

round SC

2. $\mathbf{SLE}_{\kappa}$-curve, $\kappa \in (0,4]$? (*cf.* Lawler–Rezaei '15)
3. Quasi-conformal deformations of round SCs?
4. Self-conformal quasi-arcs $\partial \subset \partial_{\mathbb{C}} U$ (later this talk)

1 Fractals in self-similar & self-conformal geometries

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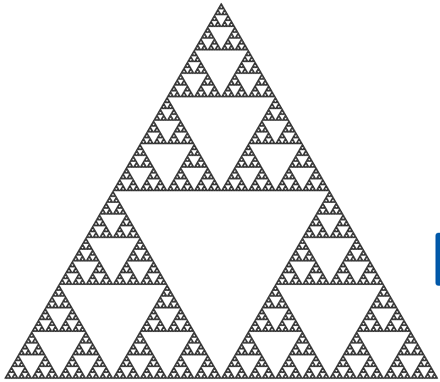
(Sierpiński Gasket)



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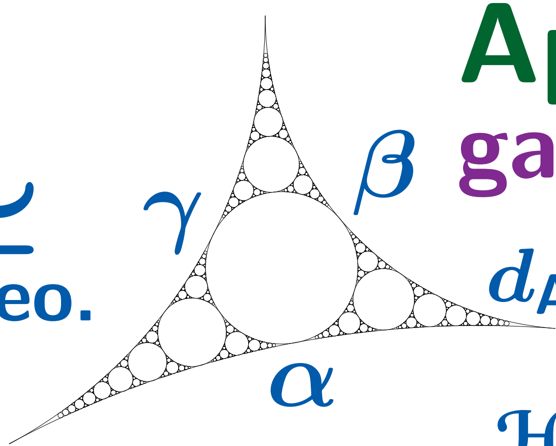

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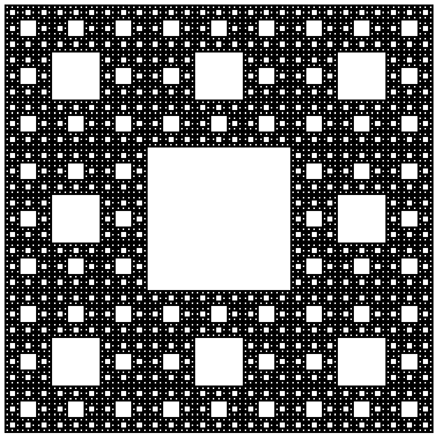
(self-similar) **SC**
(Sierpiński Carpet)



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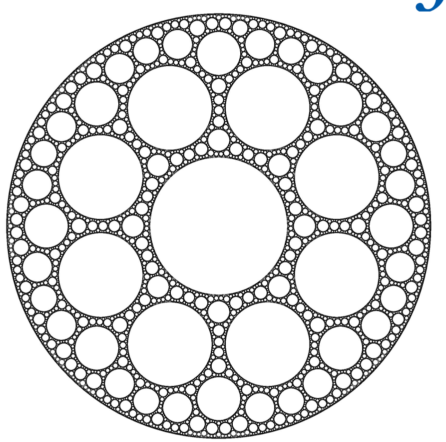
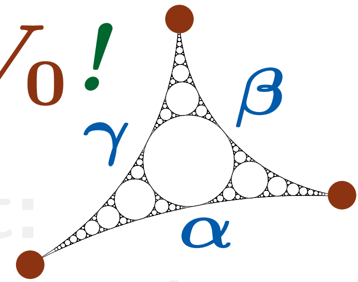


round SC

1. $\mathbf{CLE}_{\kappa}$-carpet, $\kappa \in (\frac{8}{3}, 4]$, of Sheffield–Werner '12?
2. $\mathbf{SLE}_{\kappa}$-curve, $\kappa \in (0, 4]$? (*cf.* Lawler–Rezaei '15)
3. Quasi-conformal deformations of round SCs?
4. Self-conformal quasi-arcs $\partial \subset \partial_{\mathbb{C}} U$ (later this talk)

2 Results for Apollonian gasket: $K_{\alpha,\beta,\gamma} \overset{\text{harmonic}}{\underset{\text{embedding}}{\hookrightarrow}} \mathbb{C}$

Thm (K., *cf.* Teplyaev '04). $^{\exists 1}(\mathcal{E}^{\alpha,\beta,\gamma}, \mathcal{F}_{\alpha,\beta,\gamma})$: non-zero, str. local, regular symmetric Dirichlet form over $K_{\alpha,\beta,\gamma}$, Re, Im are $\mathcal{E}^{\alpha,\beta,\gamma}$-*harmonic on* $K_{\alpha,\beta,\gamma} \setminus V_0$*!*



Rmk. Choice of a reference measure is irrelevant: $\mathcal{C}_{\alpha,\beta,\gamma} := \mathcal{F}_{\alpha,\beta,\gamma} \cap \mathcal{C}(K_{\alpha,\beta,\gamma})$ and $\mathcal{E}^{\alpha,\beta,\gamma}|_{\mathcal{C}_{\alpha,\beta,\gamma}}$ are unique.

Thm (K.). $\mathrm{LIP}|_{K_{\alpha,\beta,\gamma}}$ is a core of $(\mathcal{E}^{\alpha,\beta,\gamma}, \mathcal{F}_{\alpha,\beta,\gamma})$, and $^{\forall} u \in \mathrm{LIP}$,

$$\mathcal{E}^{\alpha,\beta,\gamma}(u,u) = \sum_{C \subset \mathrm{arc}\, K_{\alpha,\beta,\gamma}} \mathrm{rad}(C) \int_C |\nabla_C u|^2 d\mathrm{vol}_C.$$

2 Results for Apollonian gasket: $K_{\alpha,\beta,\gamma} \underset{\text{embedding}}{\overset{\text{harmonic}}{\hookrightarrow}} \mathbb{C}$

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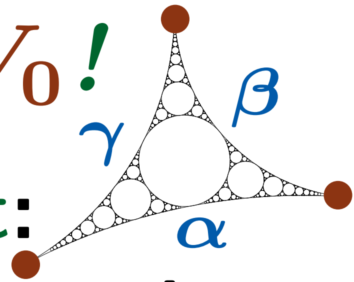

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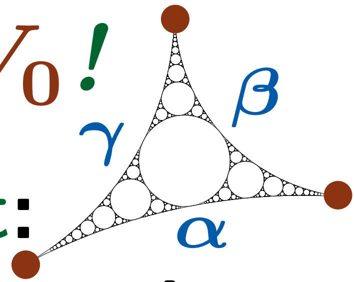
2 Results for Apollonian gasket: $K_{\alpha,\beta,\gamma} \overset{\text{harmonic}}{\underset{\text{embedding}}{\hookrightarrow}} \mathbb{C}$

Thm (K., *cf.* Teplyaev '04). $^{\exists 1}(\mathcal{E}^{\alpha,\beta,\gamma}, \mathcal{F}_{\alpha,\beta,\gamma})$: non-zero, str. local, regular symmetric Dirichlet form over $K_{\alpha,\beta,\gamma}$, Re, Im are $\mathcal{E}^{\alpha,\beta,\gamma}$-*harmonic on* $K_{\alpha,\beta,\gamma} \setminus V_0$*!*



Rmk. Choice of a reference measure is irrelevant: $\mathcal{C}_{\alpha,\beta,\gamma} := \mathcal{F}_{\alpha,\beta,\gamma} \cap \mathcal{C}(K_{\alpha,\beta,\gamma})$ and $\mathcal{E}^{\alpha,\beta,\gamma}|_{\mathcal{C}_{\alpha,\beta,\gamma}}$ are unique.

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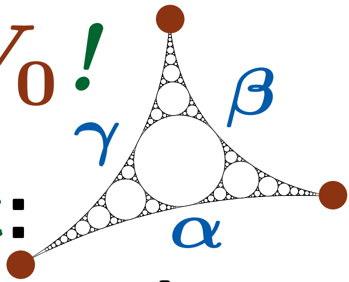

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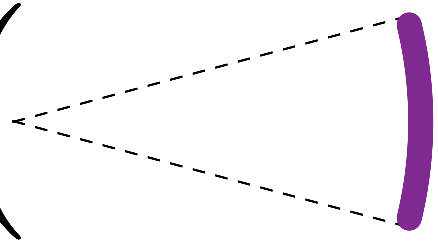
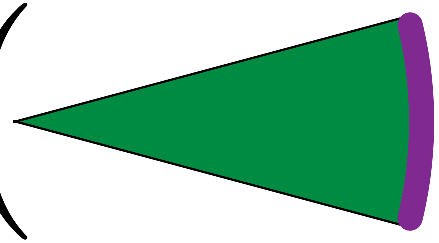
$\mu^{\alpha,\beta,\gamma}\Big($  $\Big) = 2\mathrm{Area}\Big($  $\Big)$!

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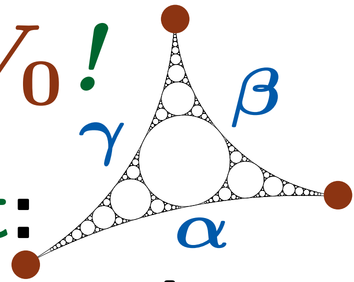

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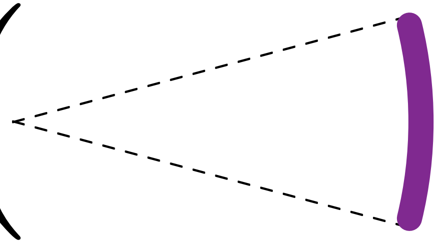
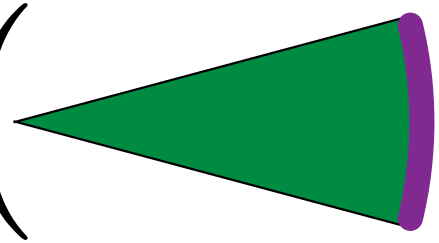
(NOT doubling!)

$\triangleright\ \mu^{\alpha,\beta,\gamma} := \mu^{\mathcal{E}^{\alpha,\beta,\gamma}}_{\langle \mathrm{Re} \rangle} + \mu^{\mathcal{E}^{\alpha,\beta,\gamma}}_{\langle \mathrm{Im} \rangle}$: volume meas.

$\mu^{\alpha,\beta,\gamma}\Big($  $\Big) = 2\,\mathrm{Area}\Big($  $\Big)$!

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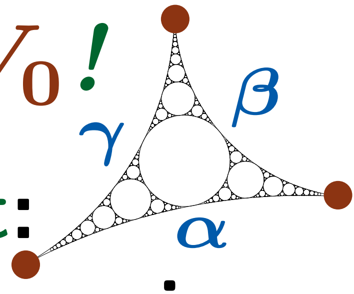

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(NOT doubling!)

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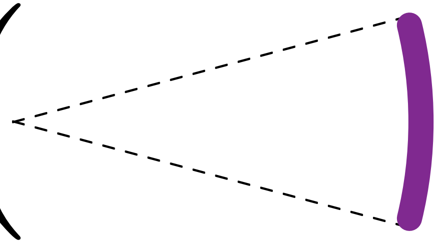
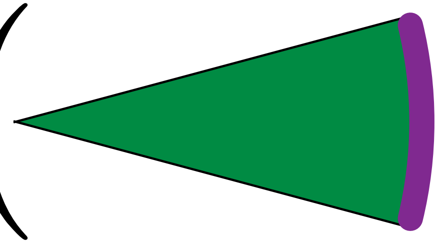
$\mu^{\alpha,\beta,\gamma}\Big($  $\Big) = 2\mathrm{Area}\Big($  $\Big)$!

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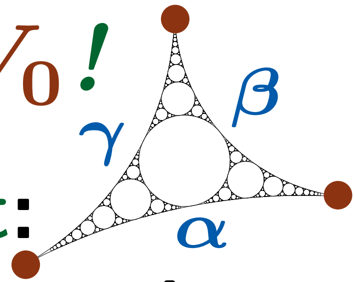

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(NOT doubling!)

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$\triangleright\ \mu^{\alpha,\beta,\gamma} := \sum_{C \subset_{\mathsf{arc}} K_{\alpha,\beta,\gamma}} \mathrm{rad}(C)\,\mathrm{vol}_C$: **volume meas.** (NOT doubling!)

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Remarks on Thm 1

- $\mu^{\alpha,\beta,\gamma} \perp \mathcal{H}^{d_{\mathsf{AG}}}|_{K_{\alpha,\beta,\gamma}}$ (for $\mathcal{H}^{d_{\mathsf{AG}}}(\mathsf{arcs}) = 0$ by $d_{\mathsf{AG}} > 1$).

- $\sum_n e^{-t\lambda_n^{\alpha,\beta,\gamma}} = \int p_t^{K_{\alpha,\beta,\gamma}}(x,x)\,d\mu^{\alpha,\beta,\gamma}(x) \overset{t\downarrow 0}{\sim} \dfrac{\mathcal{H}^{d_{\mathsf{AG}}}(K_{\alpha,\beta,\gamma})}{c t^{d_{\mathsf{AG}}/2}}$

$\Leftrightarrow$ Thm 1, BUT $p_t^{K_{\alpha,\beta,\gamma}}(x,x) \asymp_{c_x,t_x} t^{-1/2}$ for $\mu^{\alpha,\beta,\gamma}$-a.e. x!

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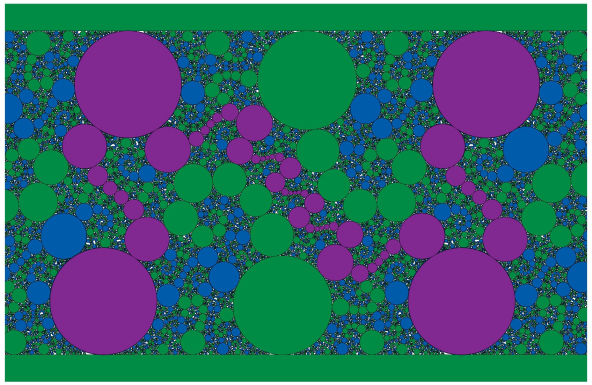
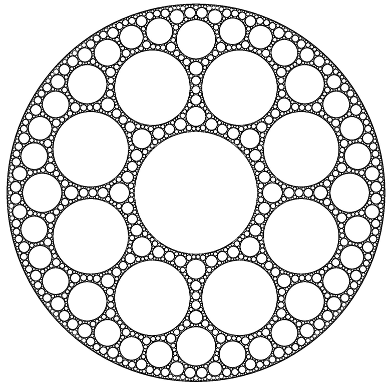
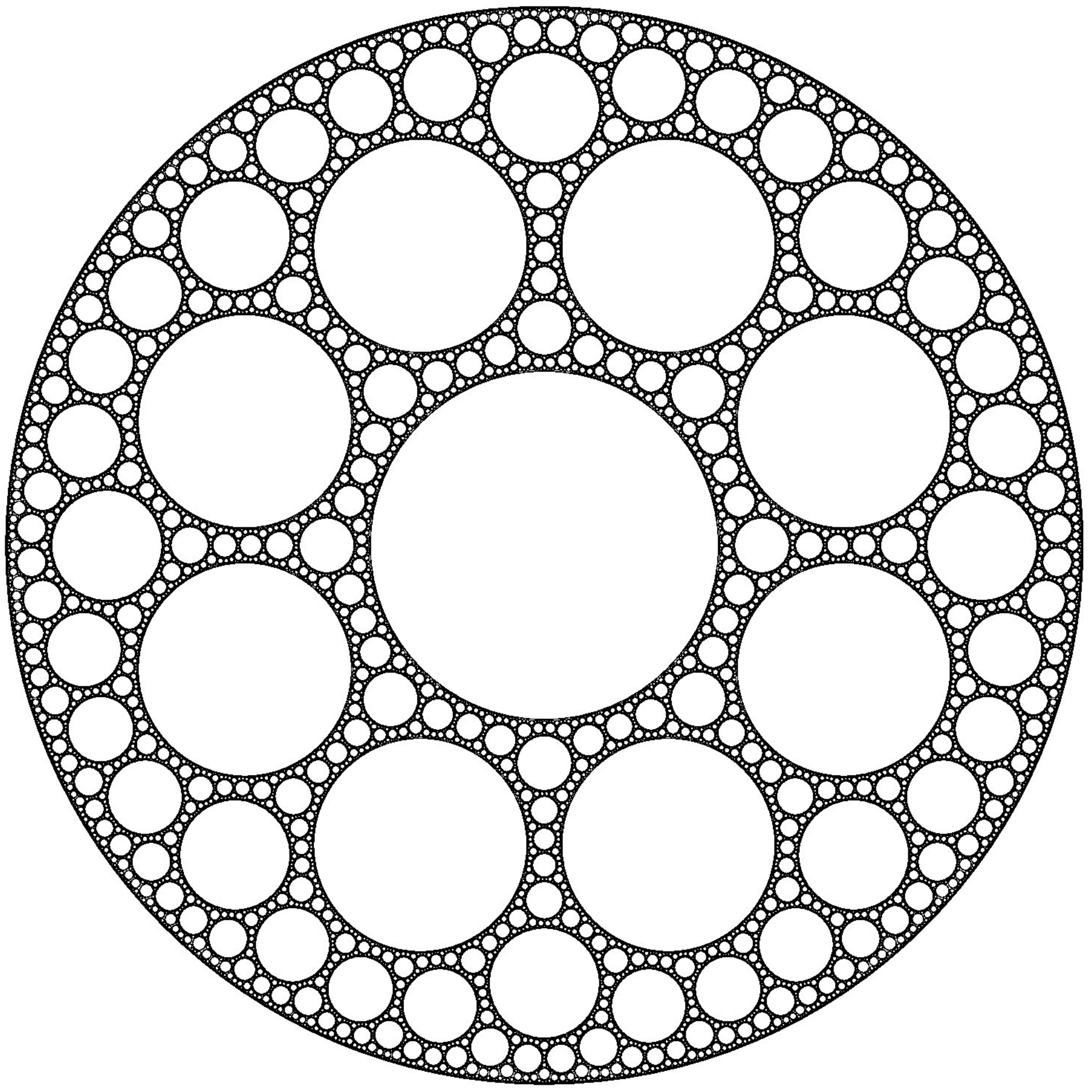
- (K.) Have extended to  & 

3 Some Kleinian groups G_m with $\partial_\infty G_m$ a RSC



▷ $m > 6$ $\left(\frac{\pi}{2}+\frac{\pi}{3}+\frac{\pi}{m}<\pi\right)$

▷ $\{\ell_k\}_{k=1}^{3}$: $\mathbb{B}^2$-geodesics, form $\triangle$, angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{m}$

▷ $\Gamma_m := \left\langle \{\mathrm{Inv}_{\ell_k}\}_{k=1}^{3} \right\rangle$
⇝ $\mathbb{B}^2 = \bigcup_{\tau\in\Gamma_m} \tau(\triangle_{\ell_1\ell_2\ell_3})$

● $S = S_m := \partial B_{\mathbb{B}^2}(0, \overset{\exists 1}{r_m})$: angle$(S, \ell_2) = \frac{\pi}{3}$.

▷ $G = G_m := \langle \Gamma_m, \mathrm{Inv}_S \rangle$
⇝ $\partial_\infty G_m$ is a *round* SC.

▷ $\partial_\infty G_m := \overline{\bigcup_{g\in G_m} g(\partial\mathbb{B}^2)}$: limit (i.e., min. cpt inv.) set of G_m

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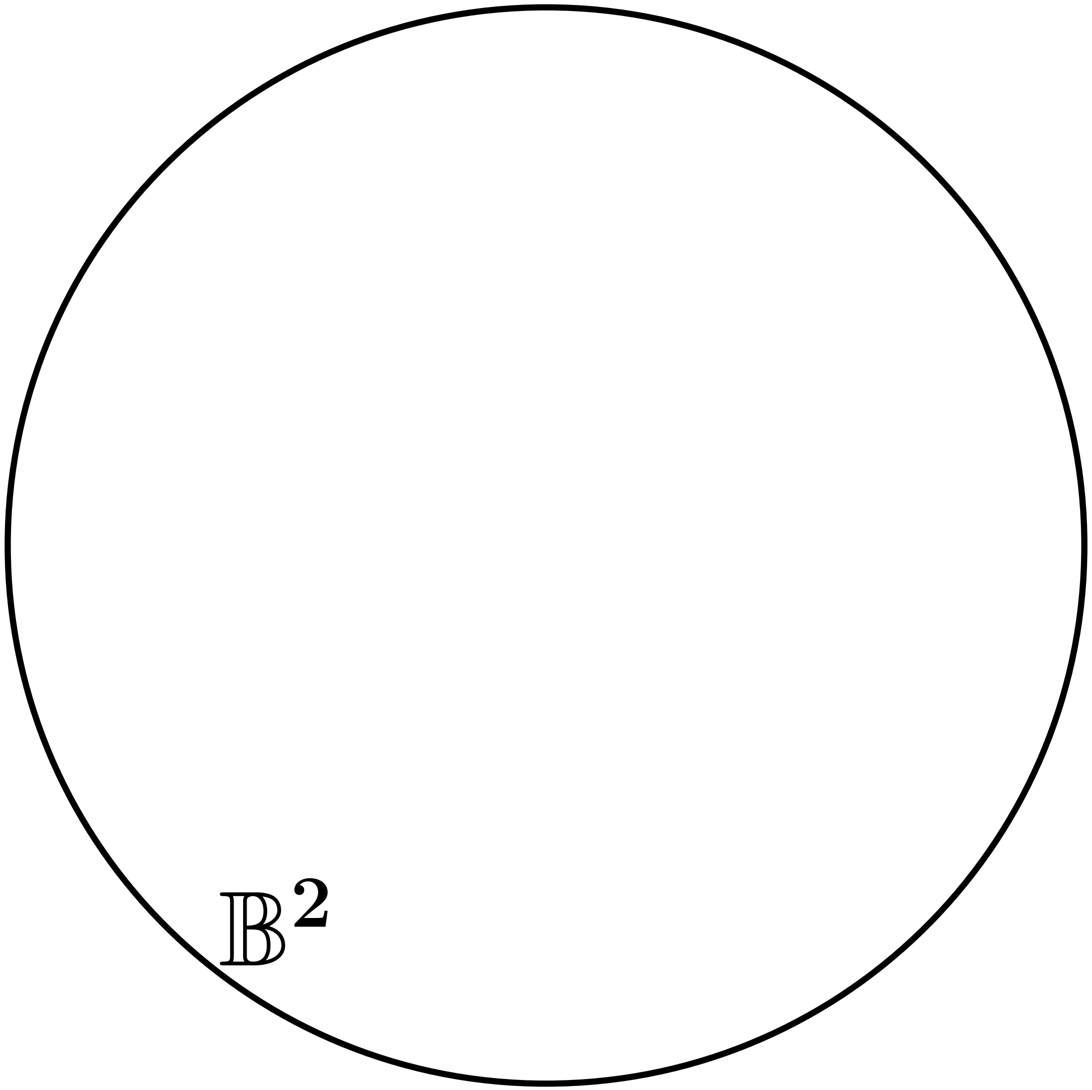

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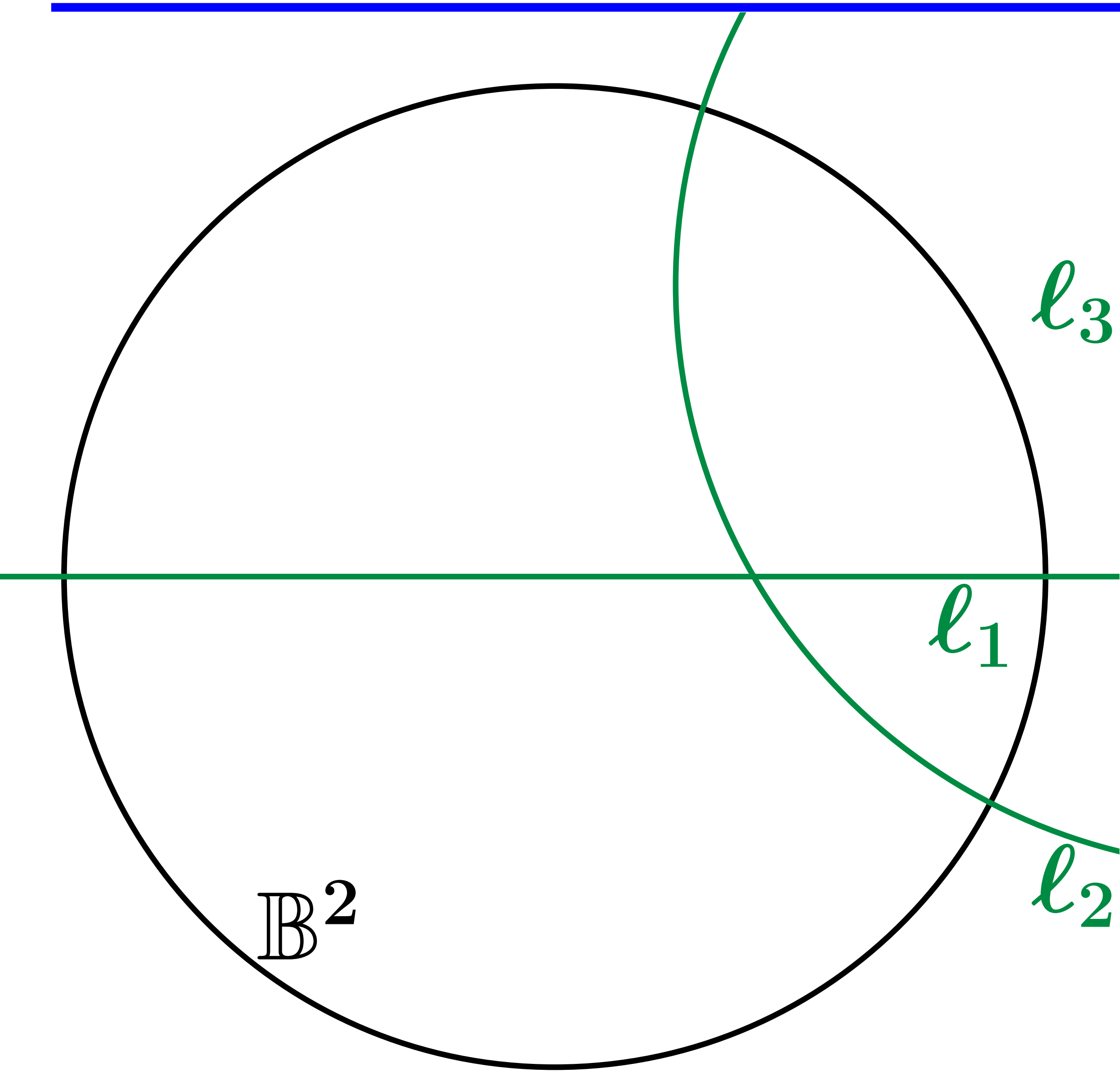

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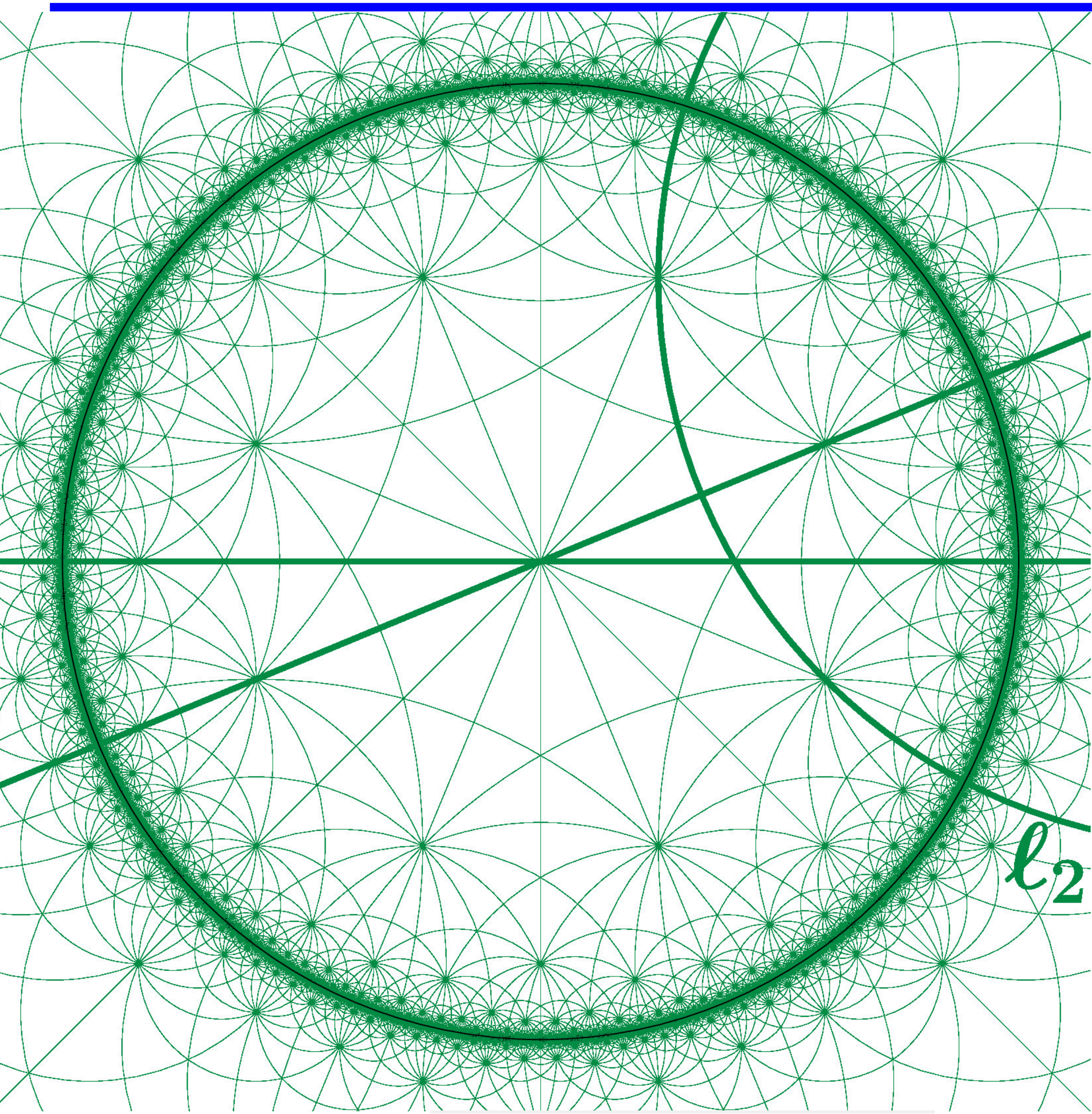

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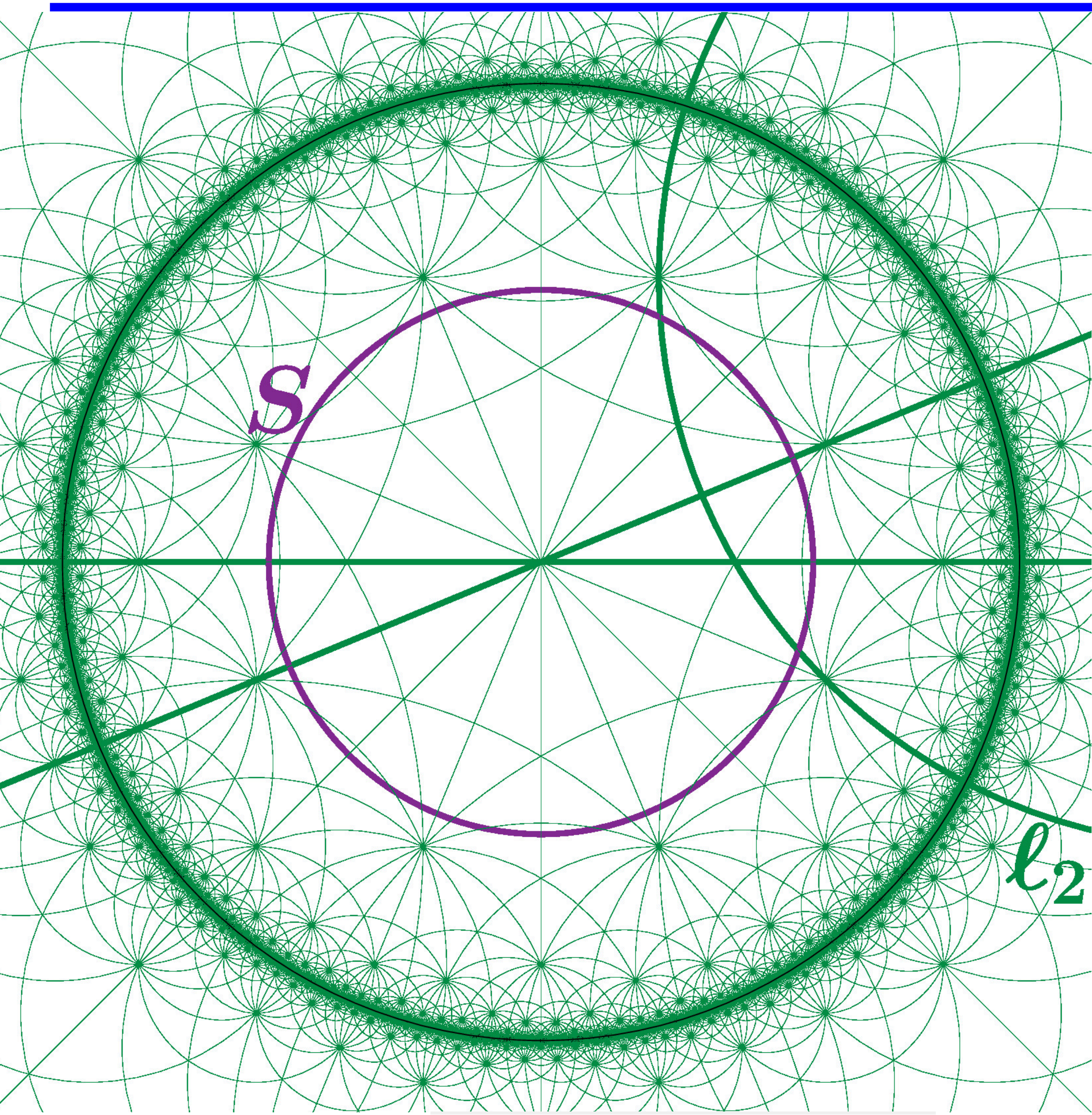

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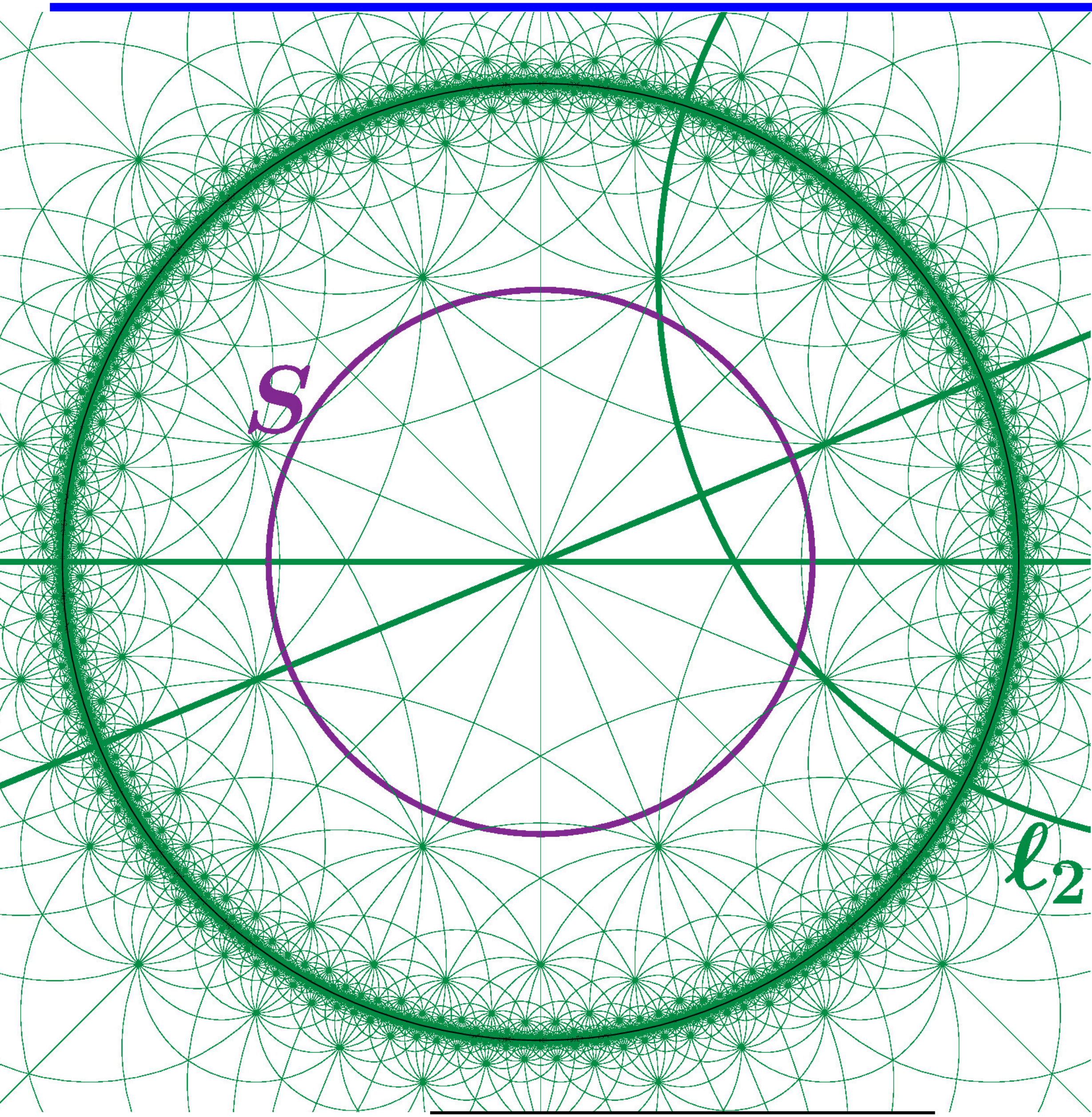

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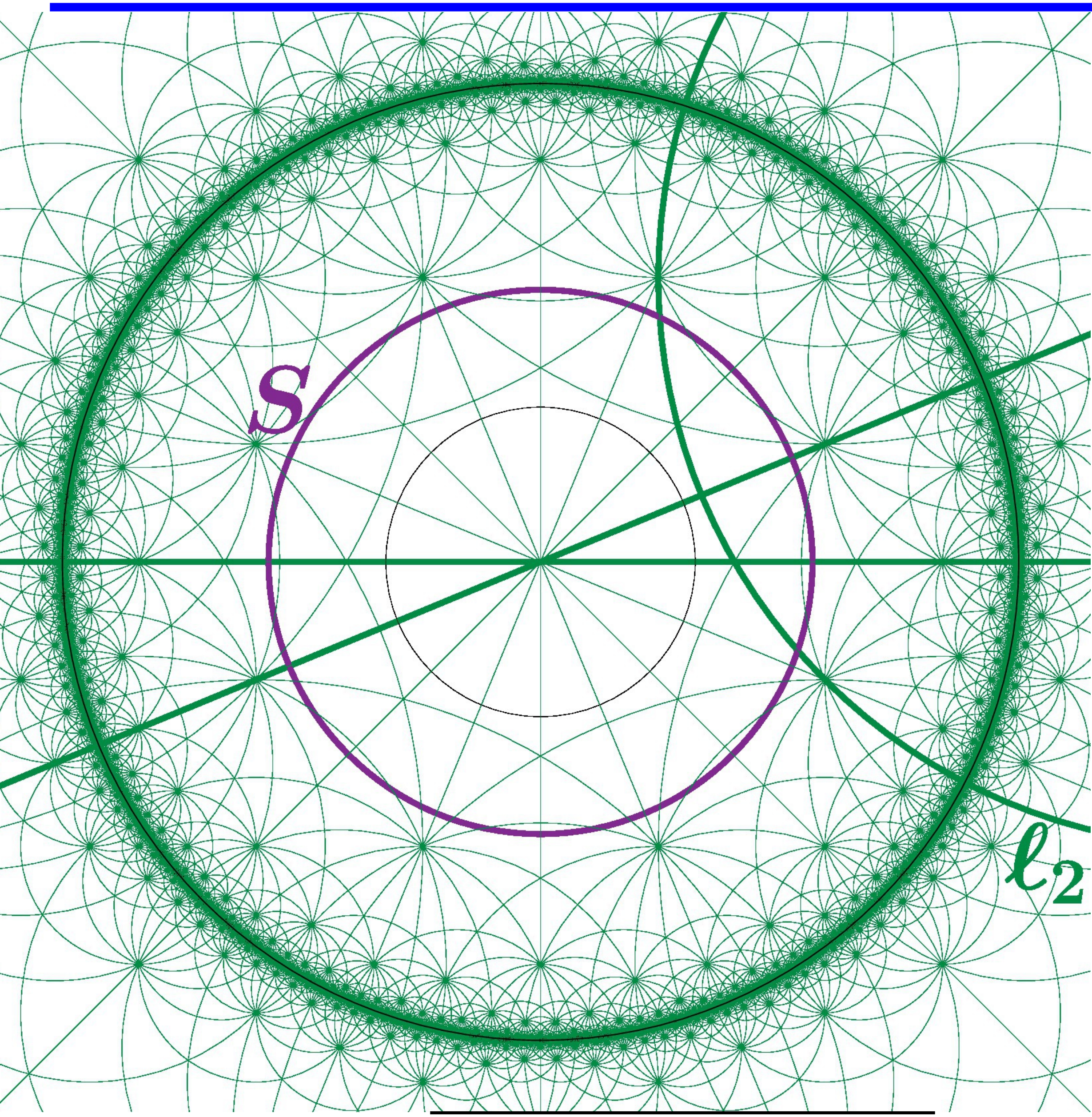

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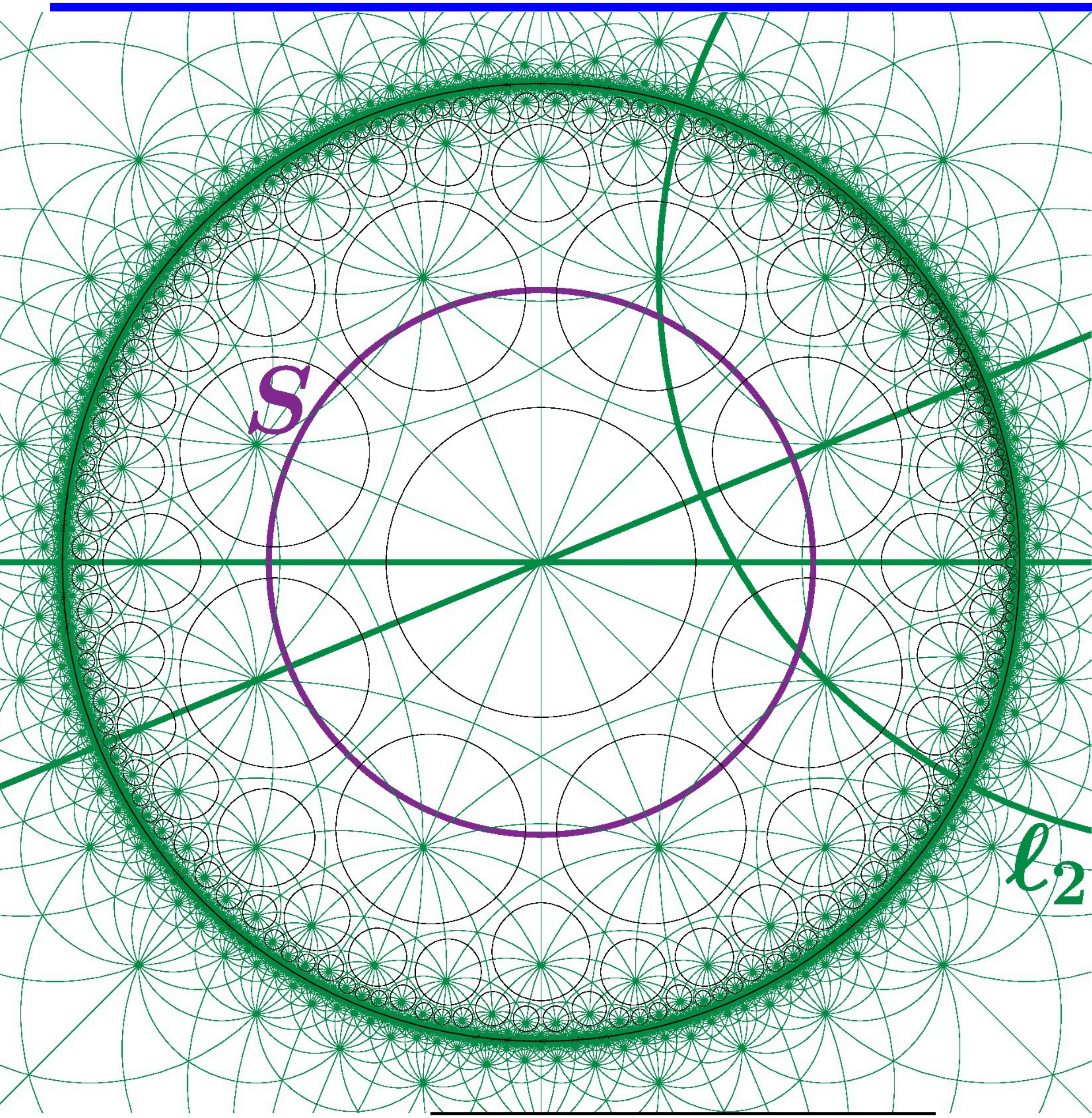

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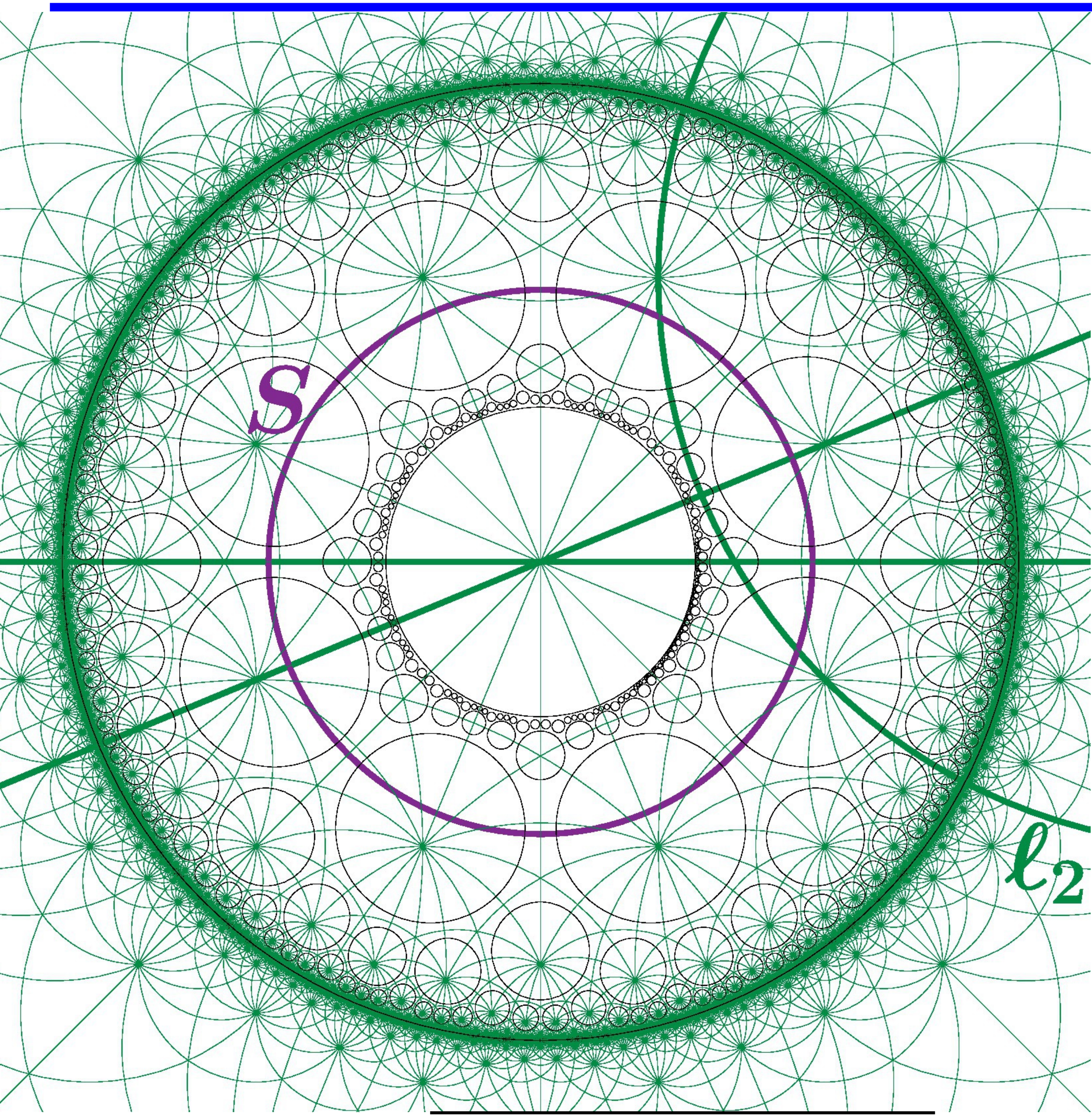

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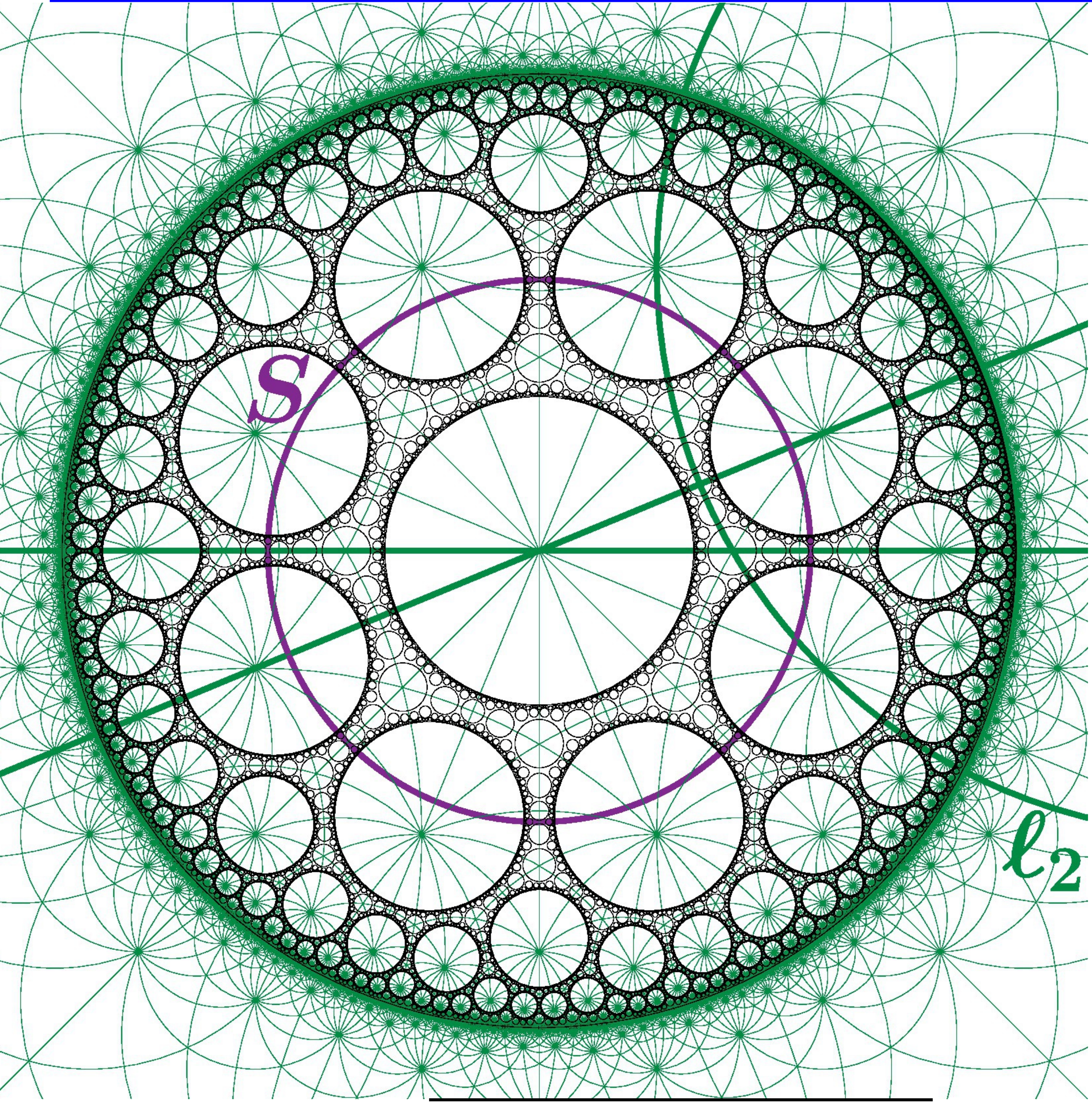

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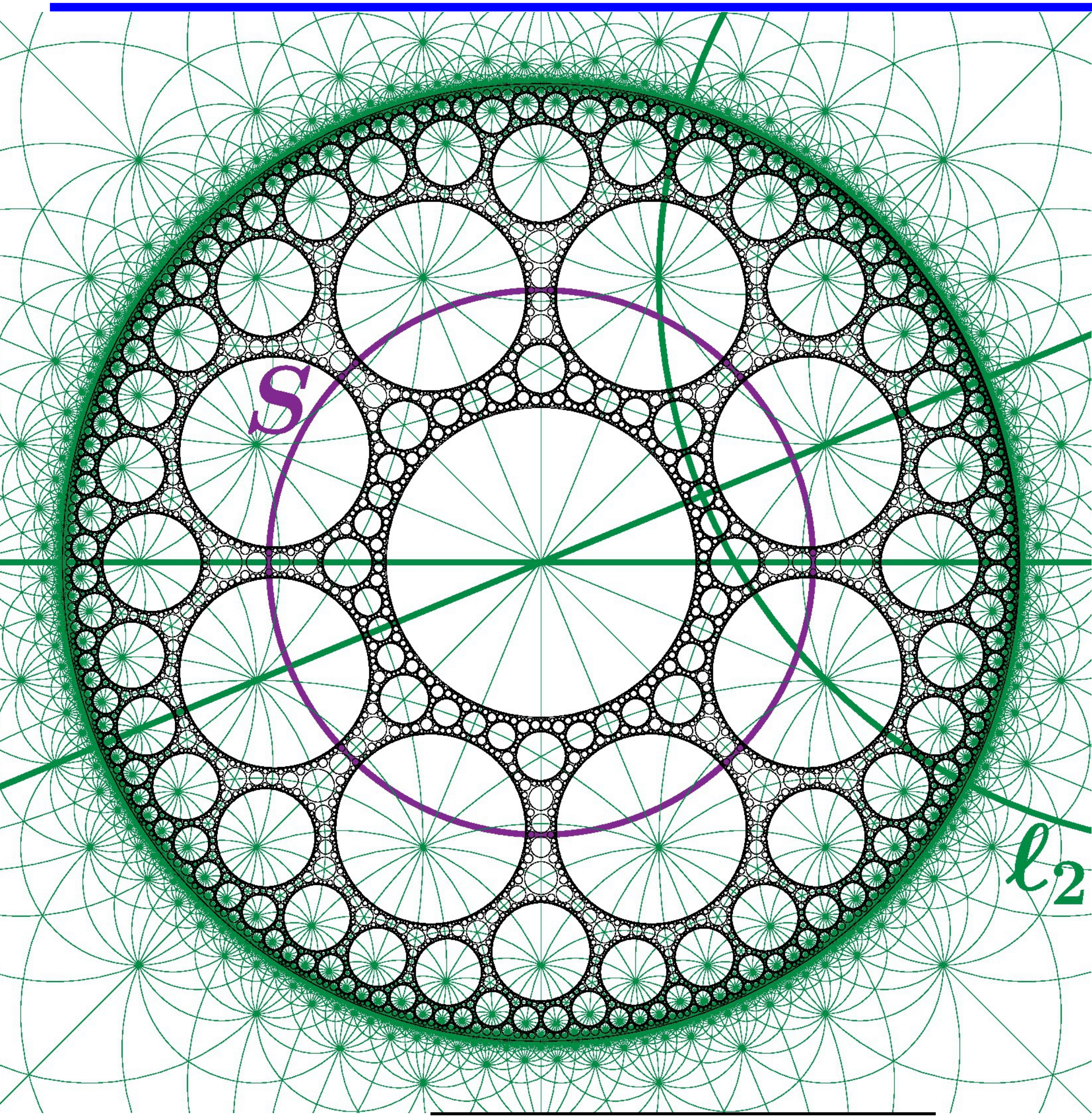

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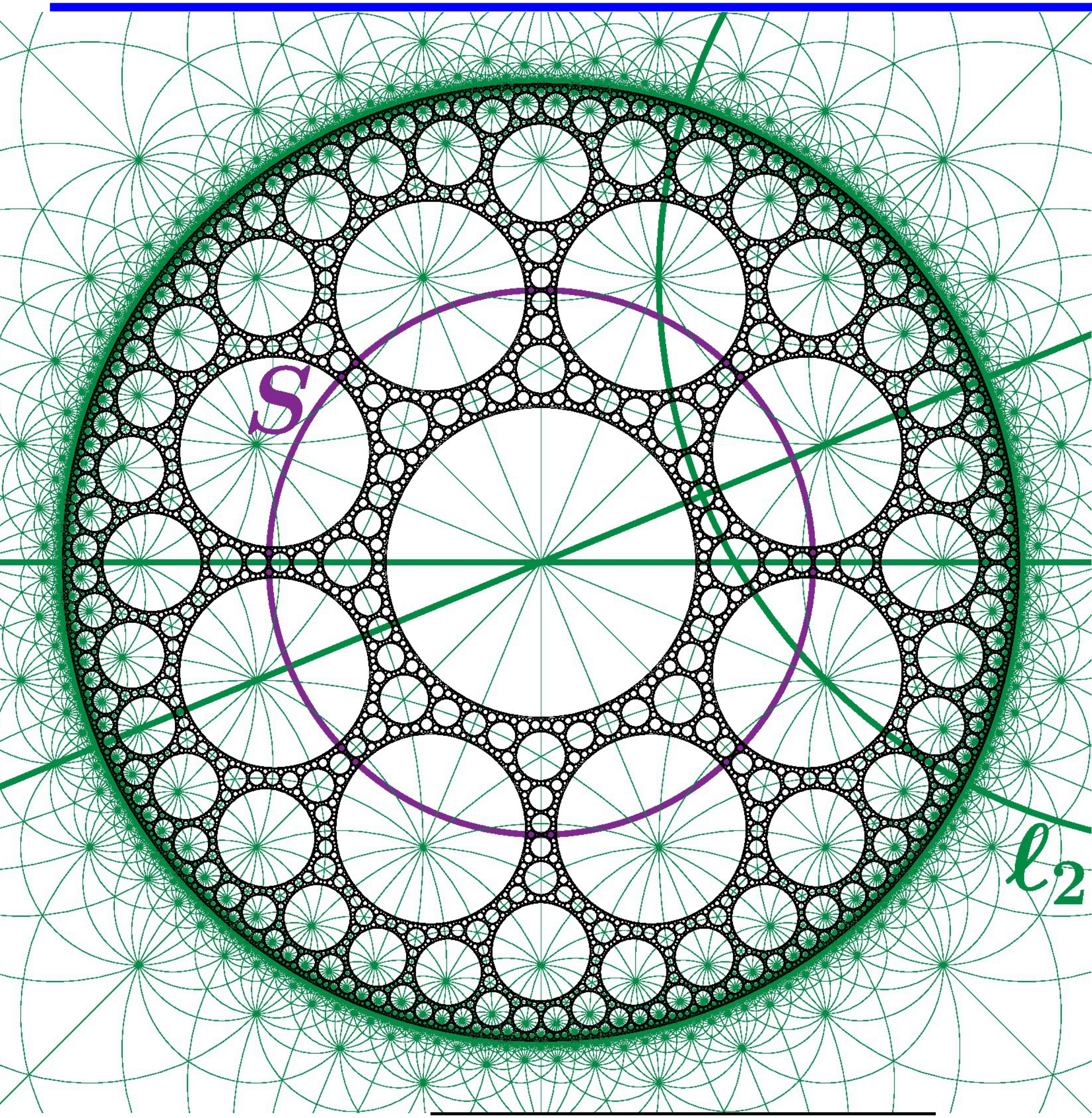

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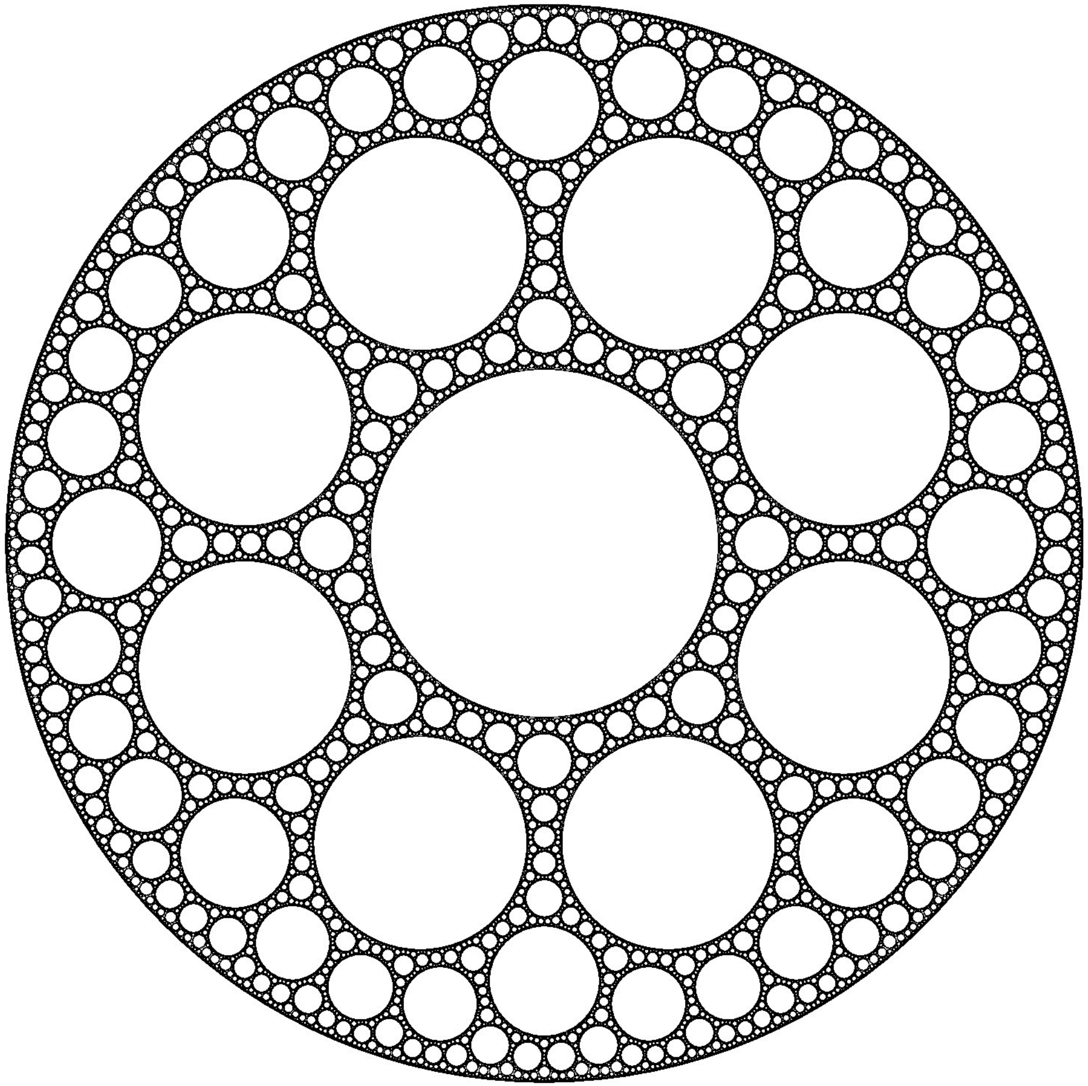

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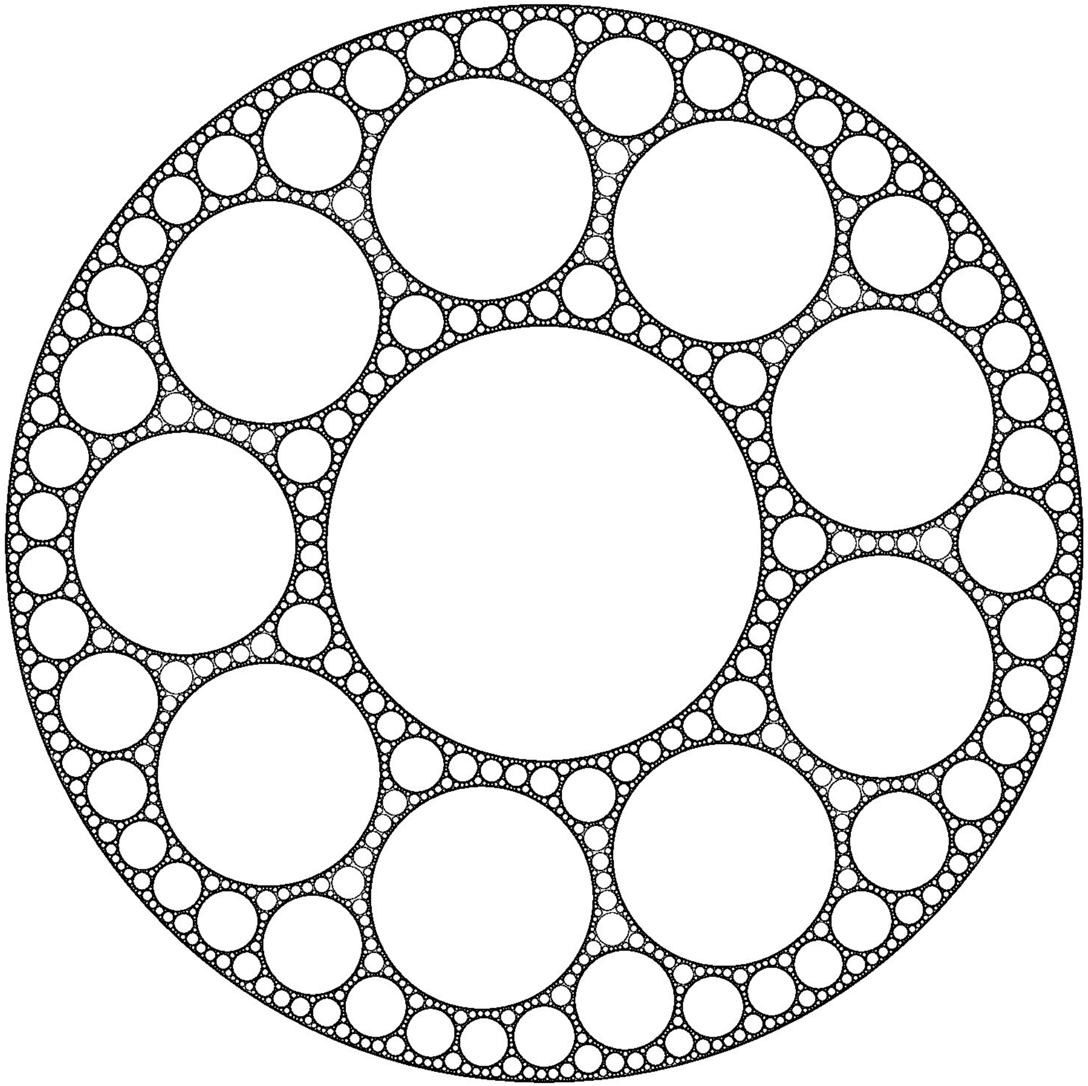

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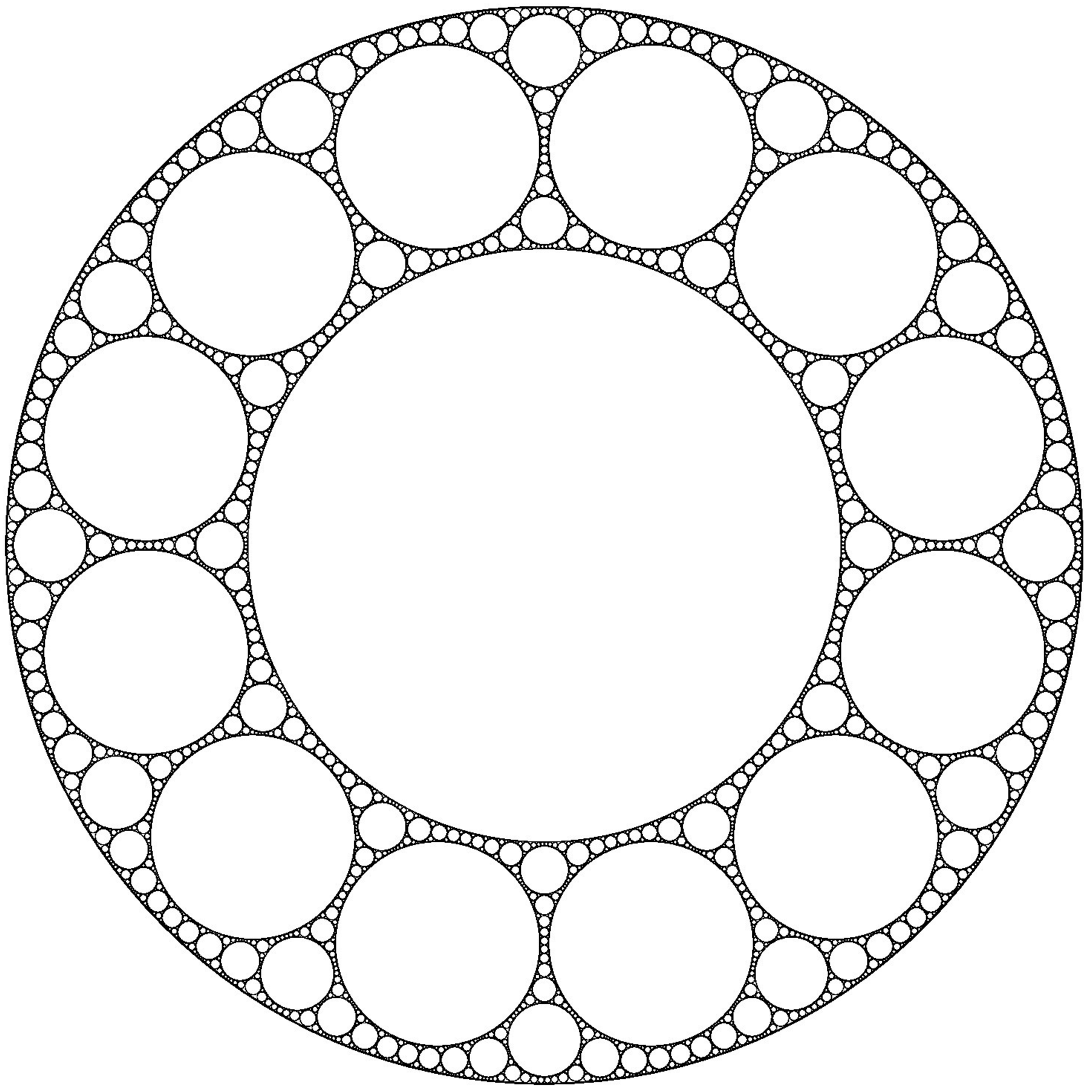

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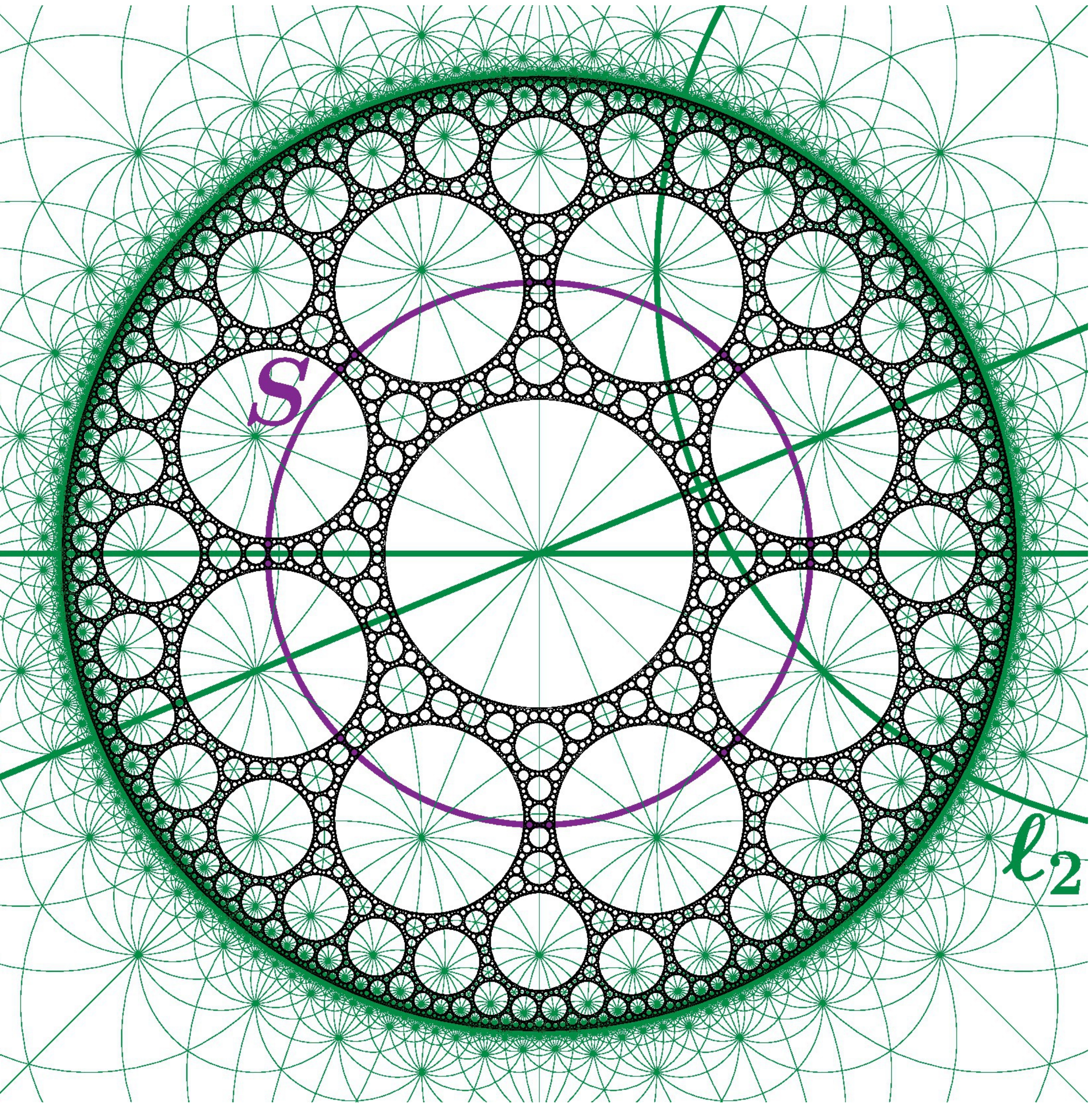

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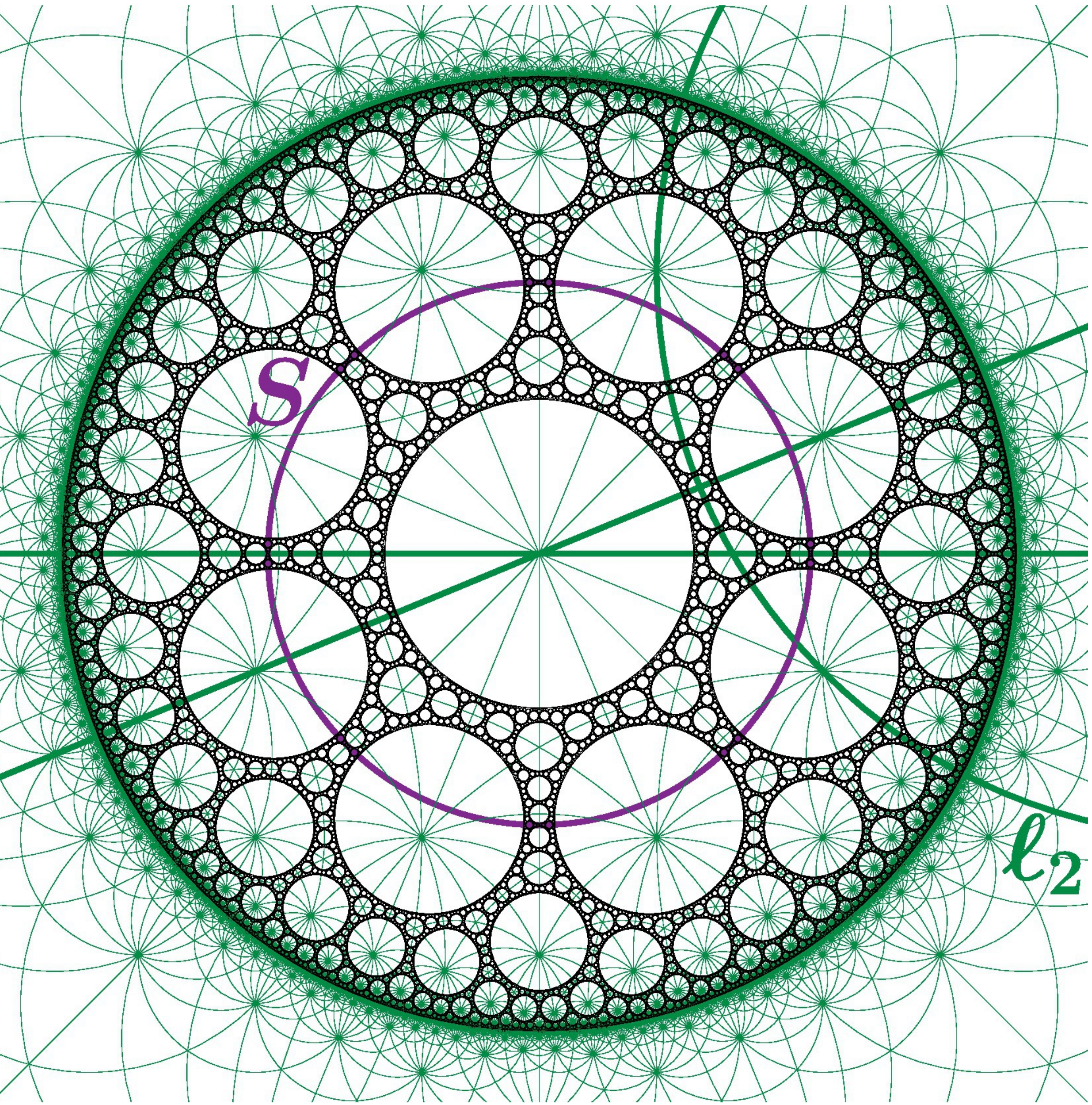

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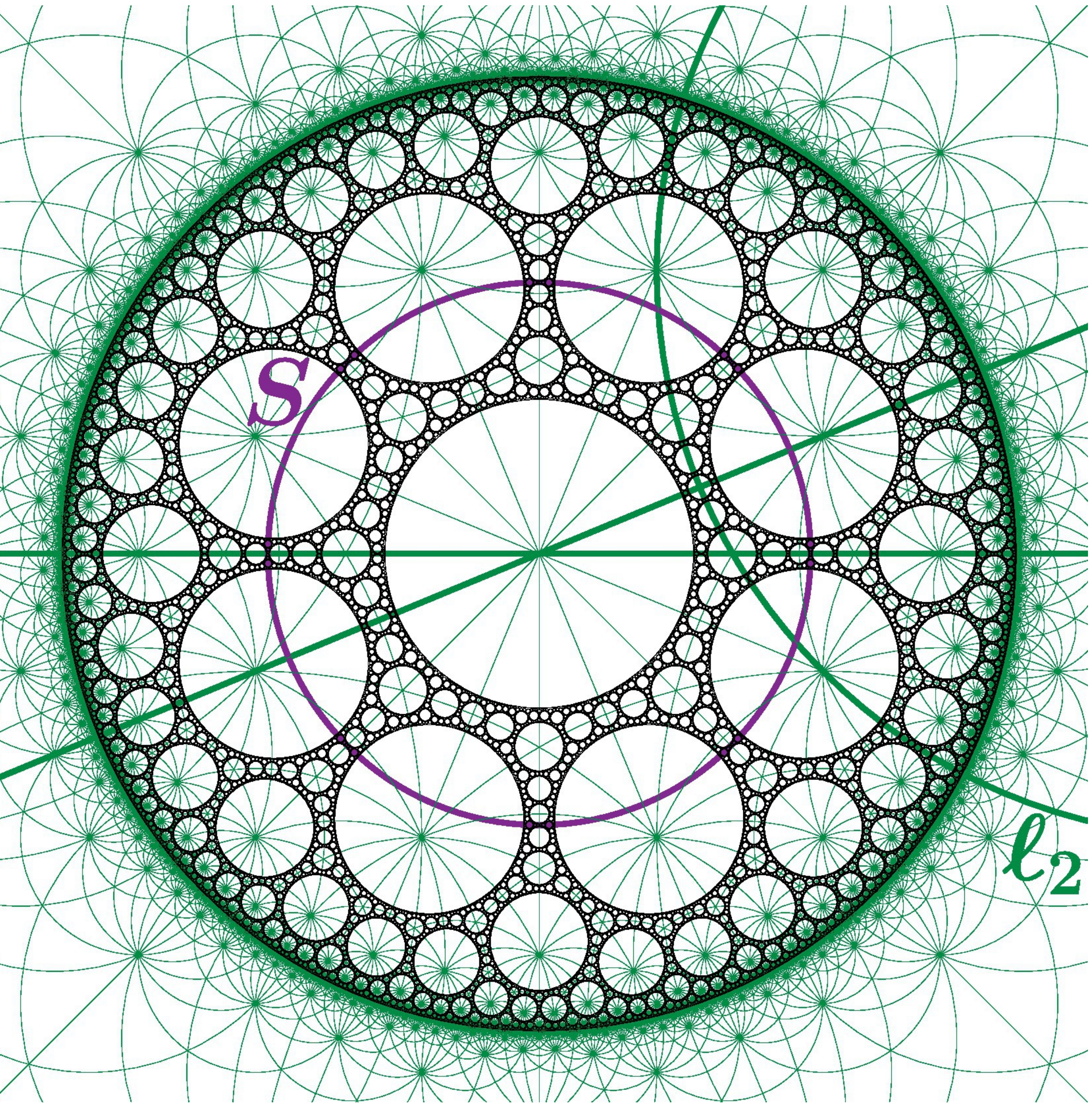

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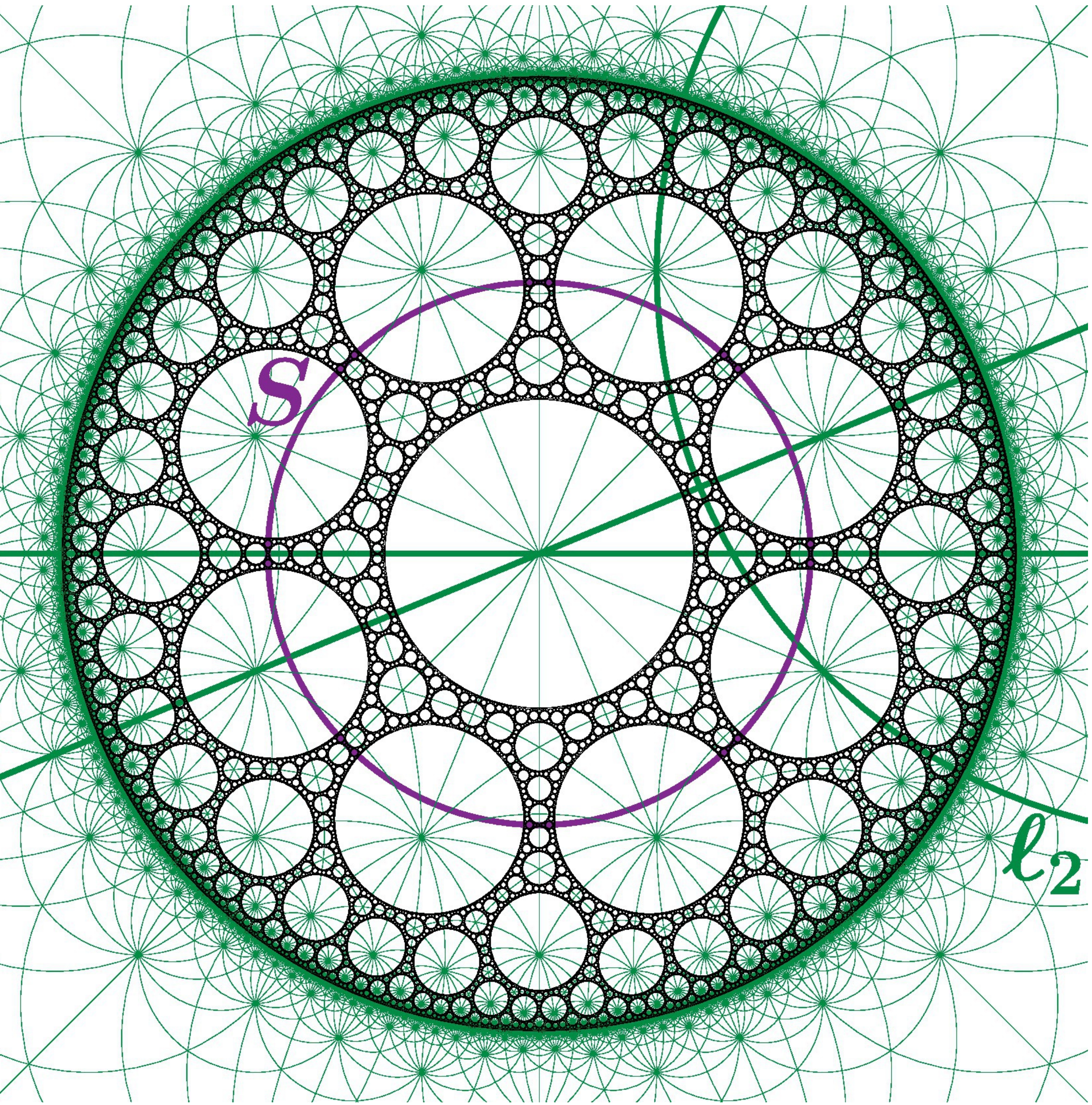

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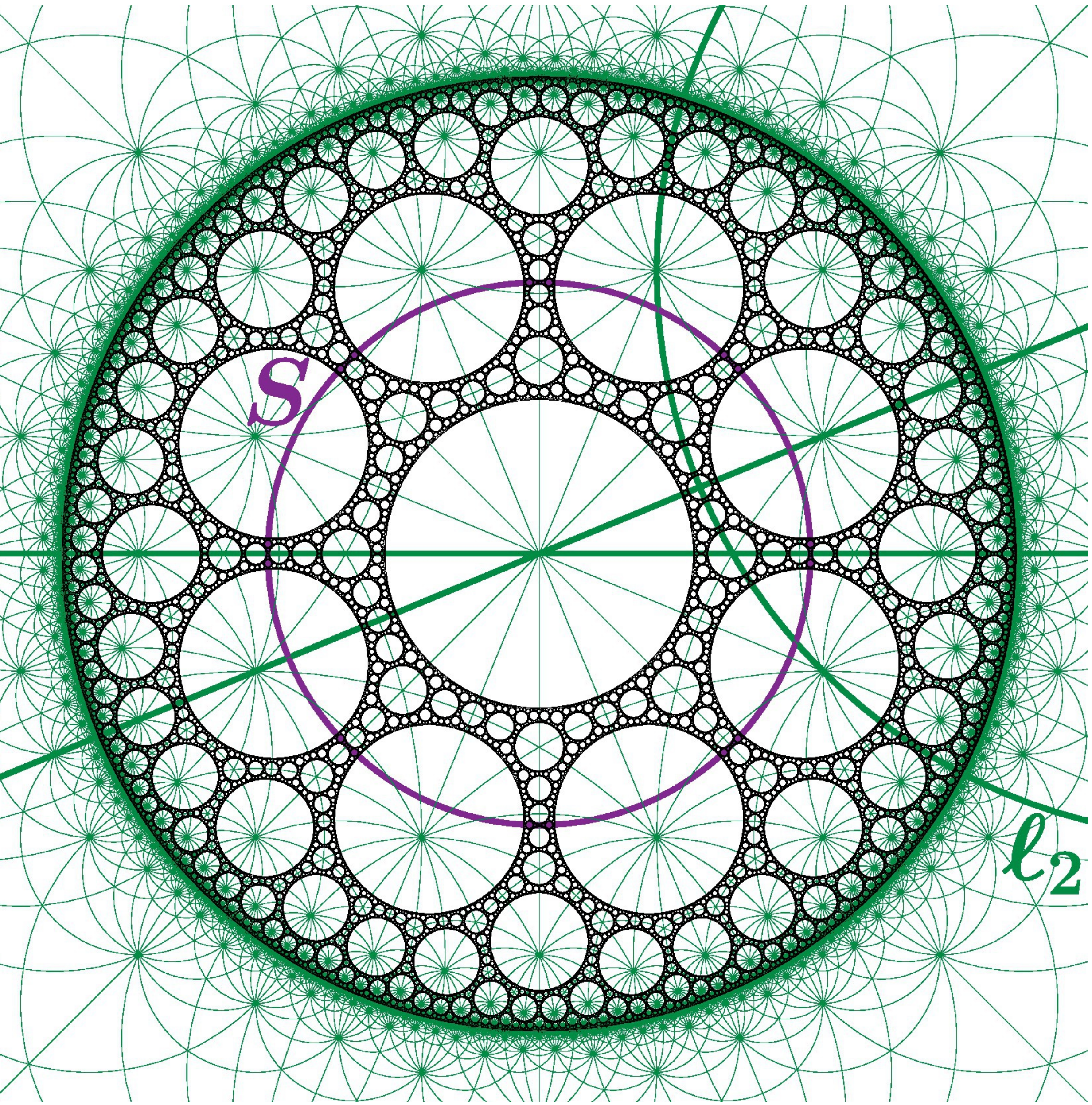

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Thm 2 (K.). $\lim_{\lambda \to \infty} \#\{n \in \mathbb{N} \mid \lambda_n^U \leq \lambda\} / \lambda^{d/2} = c_m \mathcal{H}^d(U)$.

4 Case of non-circle-packing self-conformal fractals?

3.Quasi-conformal deformations of round SCs?

4 Case of non-circle-packing self-conformal fractals?

3. $f(\partial_\infty \tilde{G})$ (∂_∞ of qc deform. $f\tilde{G}f^{-1}$ of $\tilde{G} \underset{\text{fin}}{\subset} G_m$)?

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4. Self-conformal quasi-arcs $\partial \subset \partial_{\mathbb{C}} U$ (in progress):

4 Case of non-circle-packing self-conformal fractals?

2. SLE_κ-curve, $\kappa \in (0,4]$? (*cf.* Lawler–Rezaei '15)
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1. CLE_κ-carpet, $\kappa \in (\frac{8}{3}, 4]$, of Sheffield–Werner '12?
2. SLE_κ-curve, $\kappa \in (0, 4]$? (*cf.* Lawler–Rezaei '15)
3. $f(\partial_\infty \tilde{G})$ (∂_∞ of qc deform. $f\tilde{G}f^{-1}$ of $\tilde{G} \underset{\text{fin}}{\subset} G_m$)?
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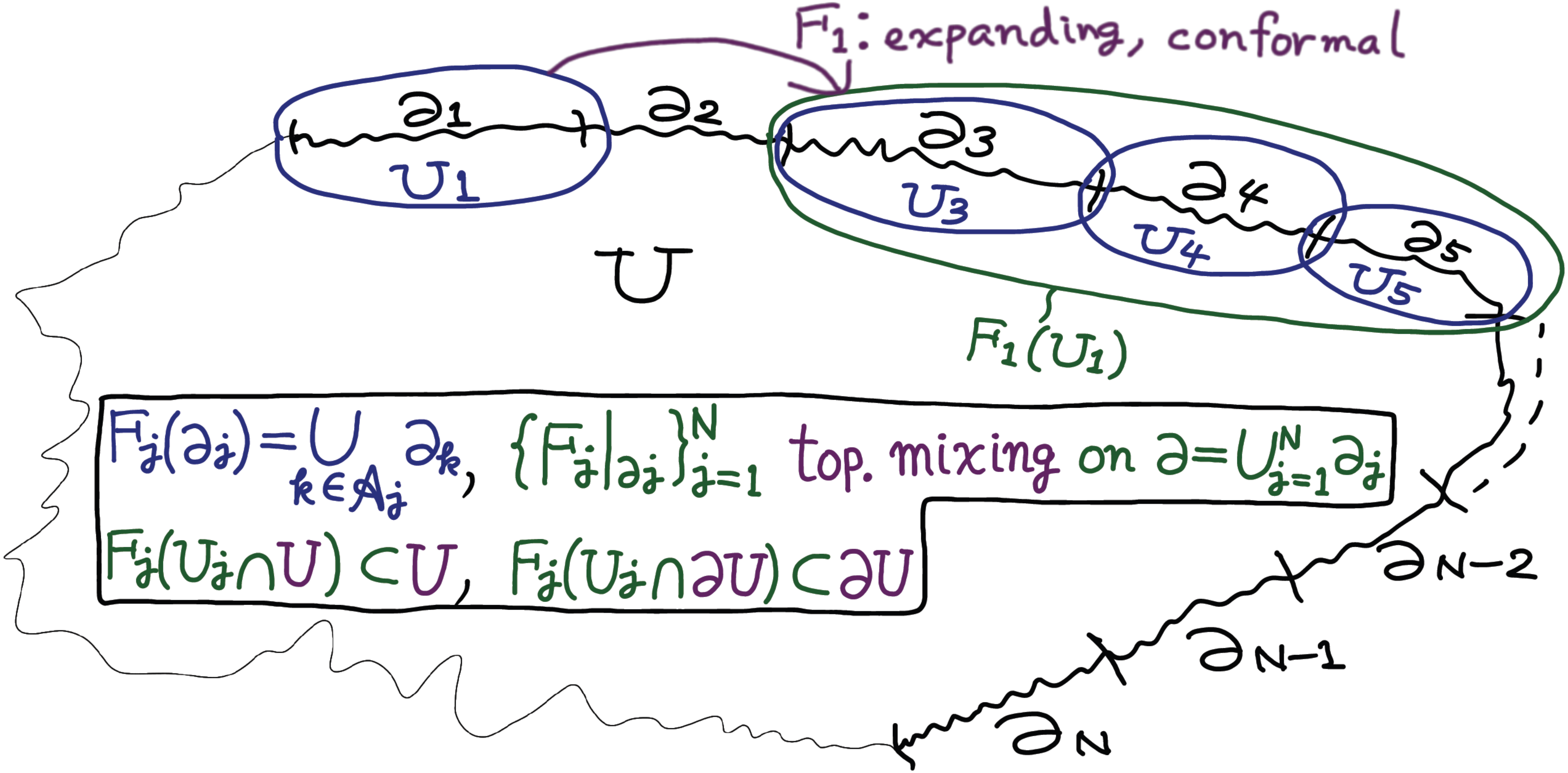

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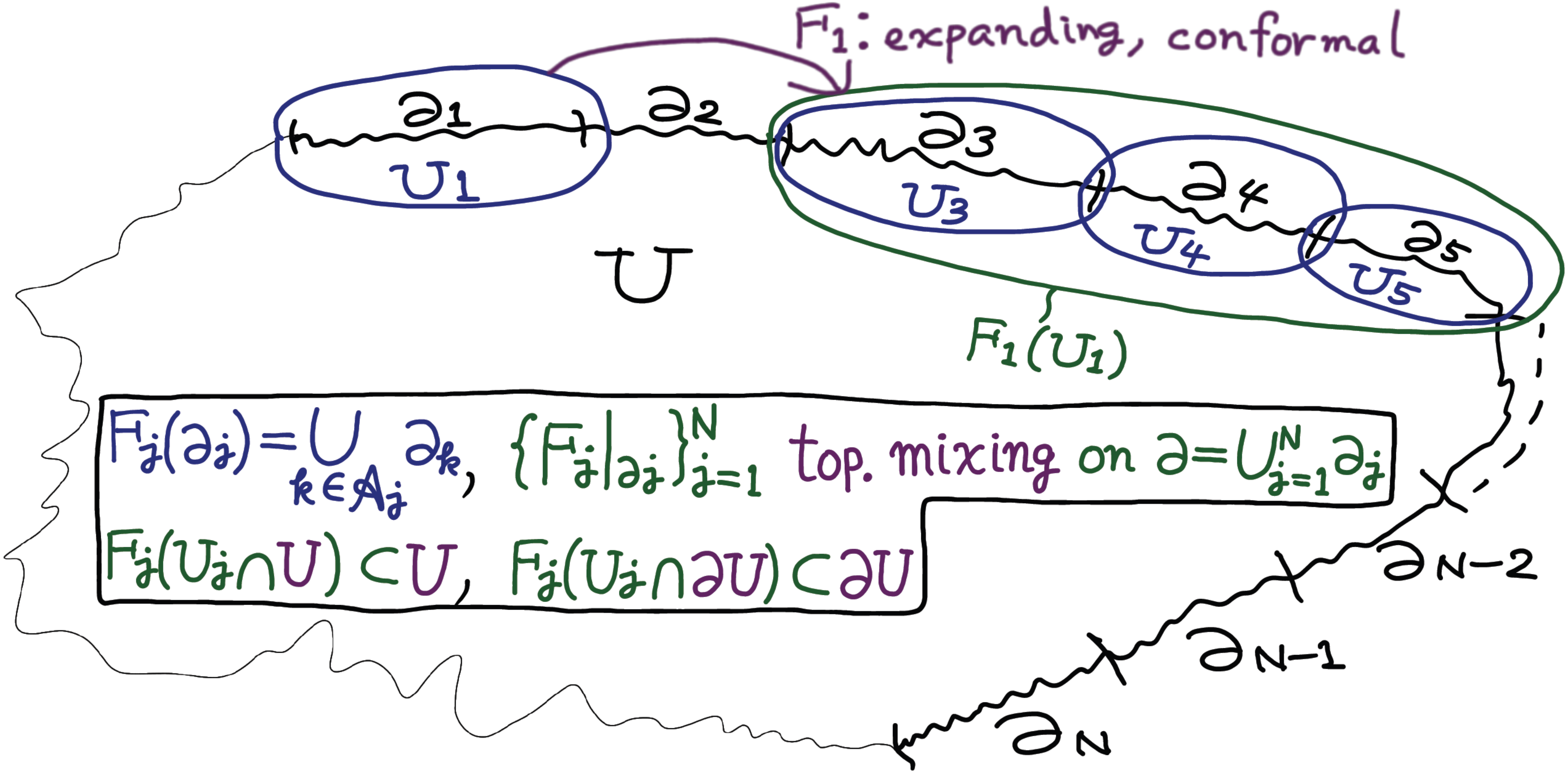

$F_j(\partial_j) = \bigcup_{k \in A_j} \partial_k$, $\{F_j|_{\partial_j}\}_{j=1}^N$ top. mixing on $\partial = \bigcup_{j=1}^N \partial_j$

$F_j(U_j \cap U) \subset U$, $F_j(U_j \cap \partial U) \subset \partial U$

Exmp. ● (Bowen '79) ∂_∞ of quasi-Fuchsian groups without parabolics

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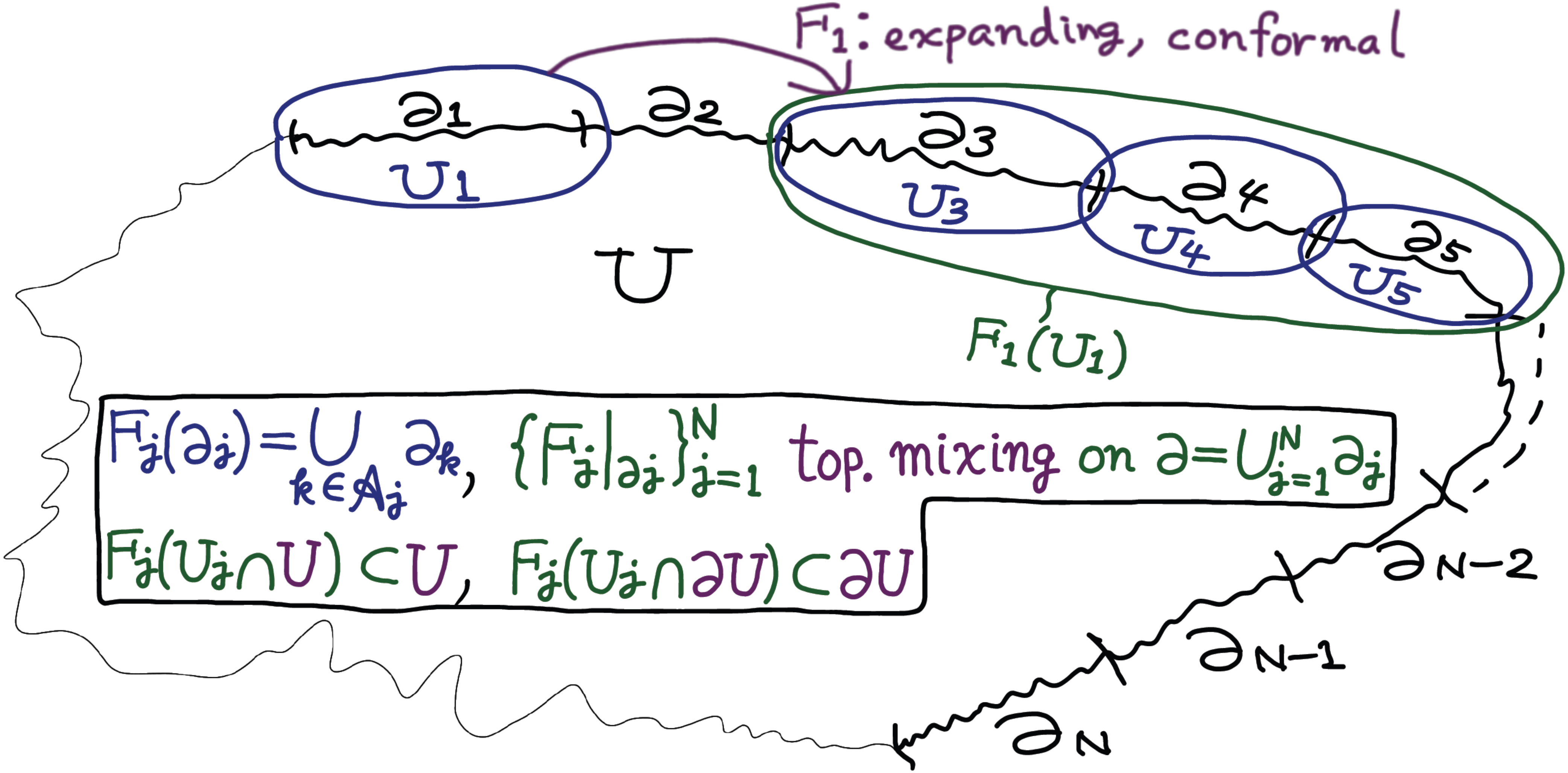

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● (*cf.* Makarov '90) $\mathrm{Julia}(z^2 + c)$ for $c \in \mathbb{C}$ with $|c| > 0$ small

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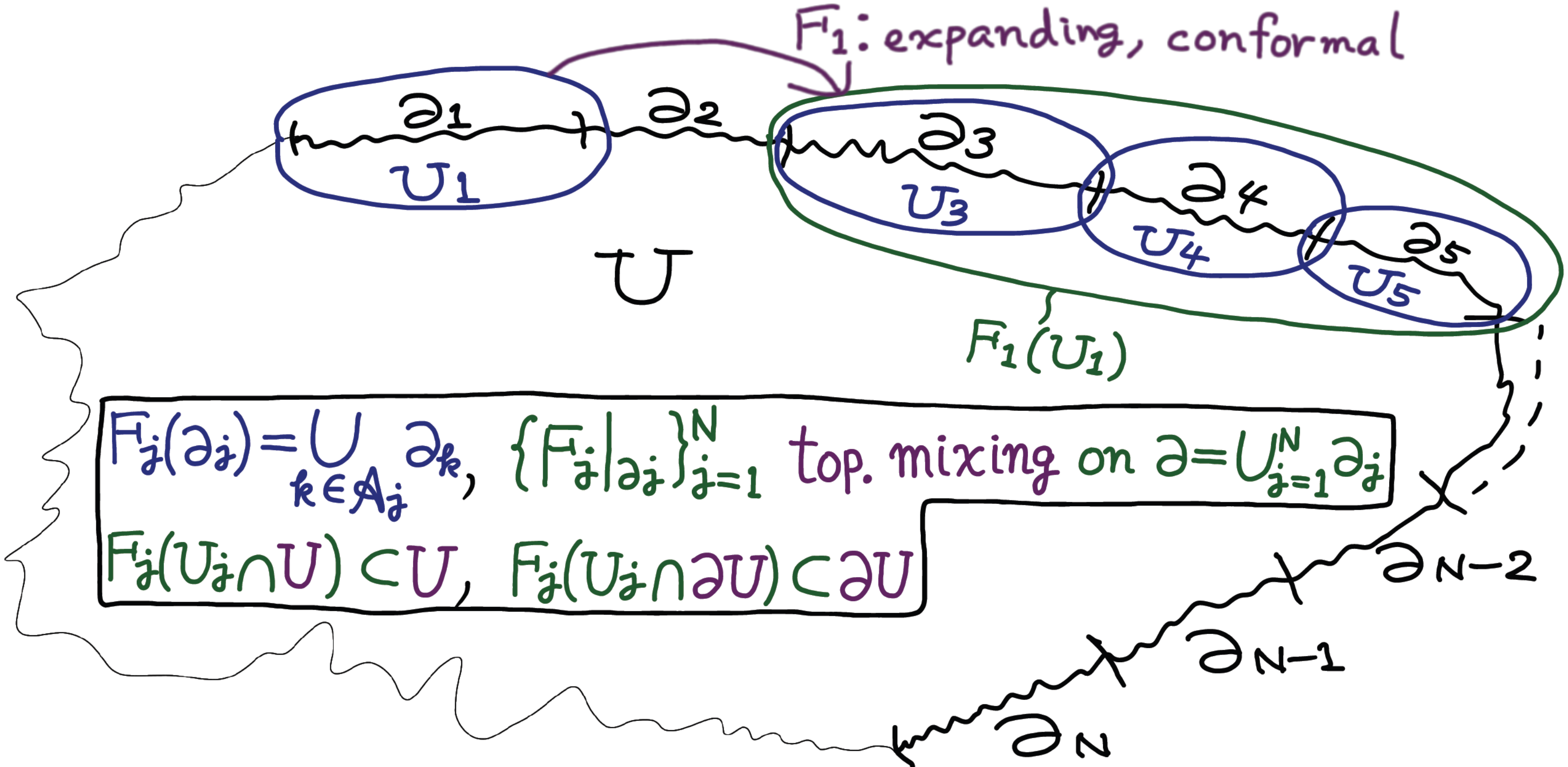
▷ $\mathcal{E}^{\partial}(u, v) := \int_{\partial} (du/d\omega_{U,q_0})(dv/d\omega_{U,q_0})\, d\omega_{U,q_0}$



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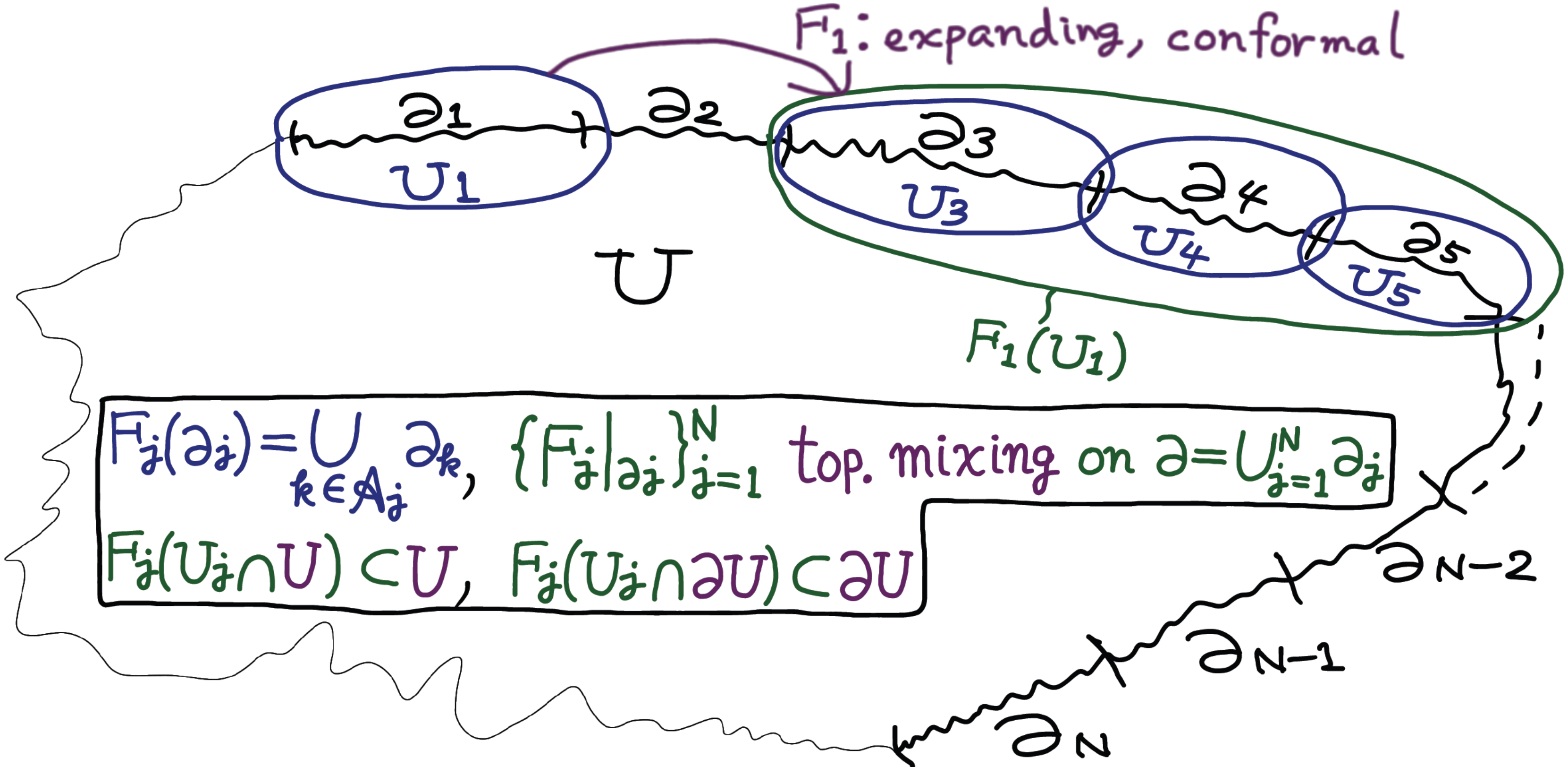
● $^{\exists 1} d_{\mathrm{w}} \geq 2$, with $d_{\mathrm{f}} := \dim_{\mathrm{H}} \partial$, $^{\exists}\mu_{\partial}$: meas. on ∂,

$\mu_{\partial}([x,y]_{\partial}) \asymp |x-y|^{d_{\mathrm{f}} d_{\mathrm{w}}} / \omega_{U,q_0}([x,y]_{\partial})$. (*cf.* Makarov '90)



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Exmp. ● (Bowen '79) ∂_{∞} of quasi-Fuchsian groups without parabolics
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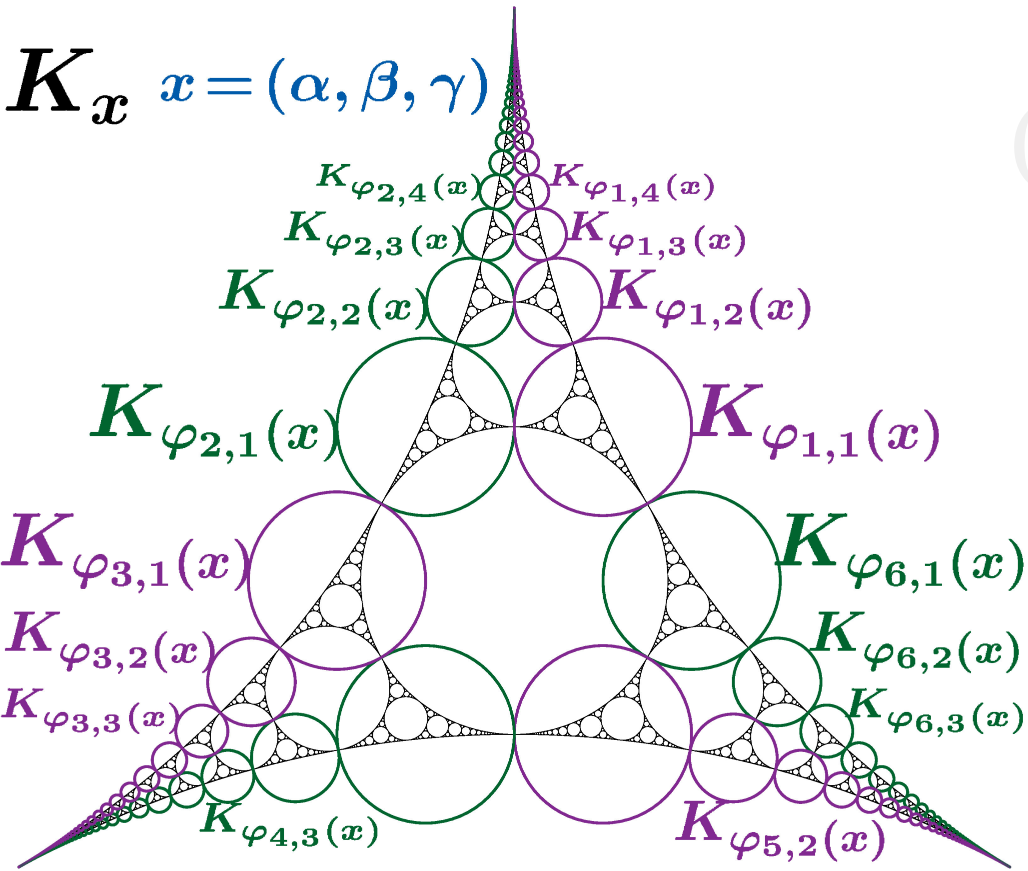

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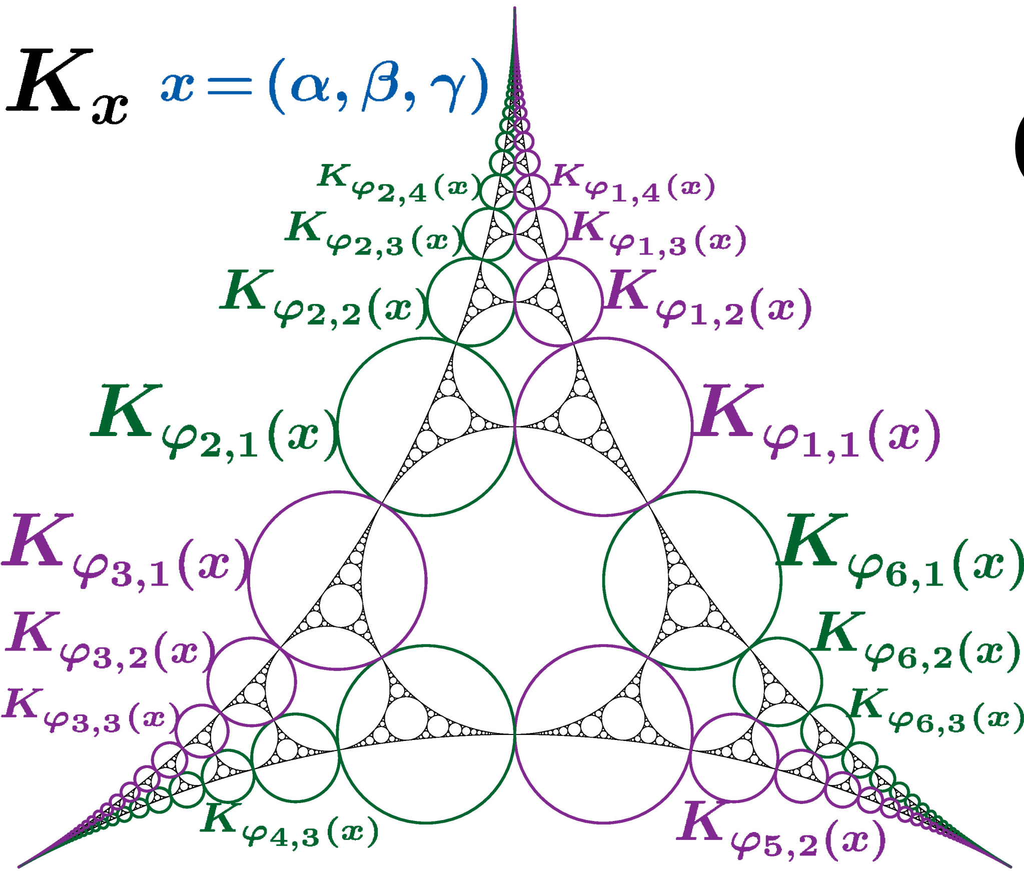

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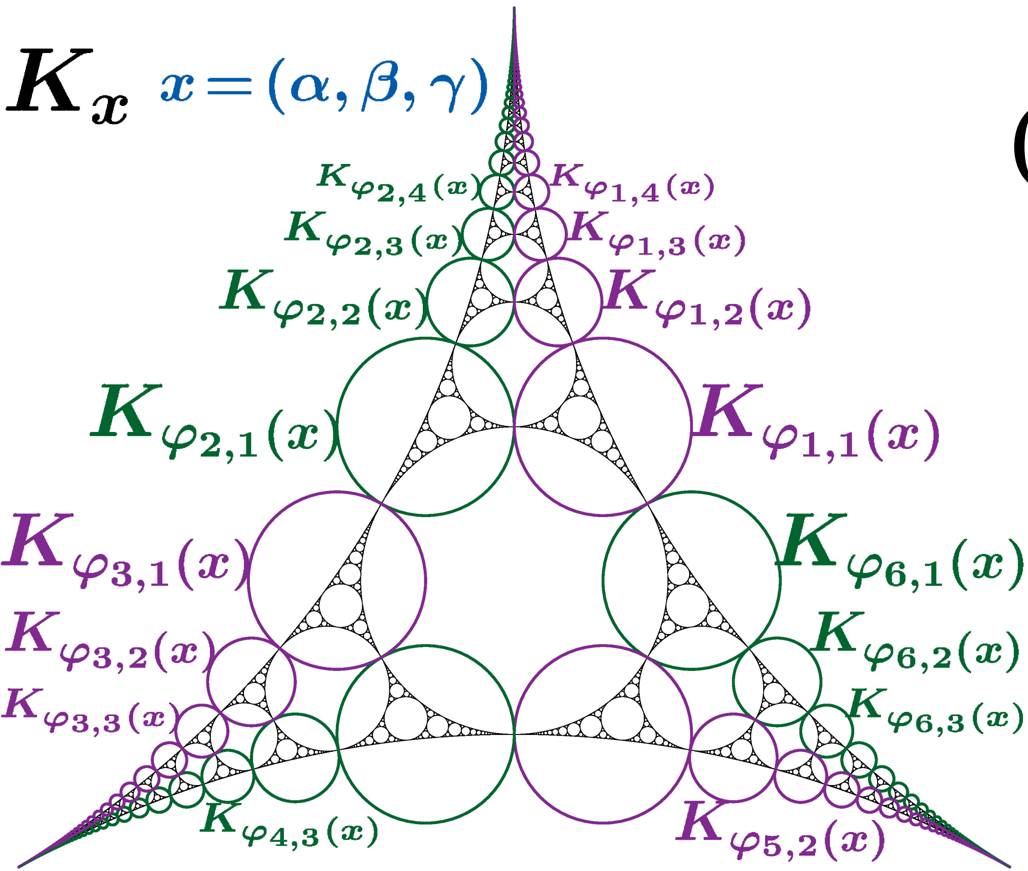

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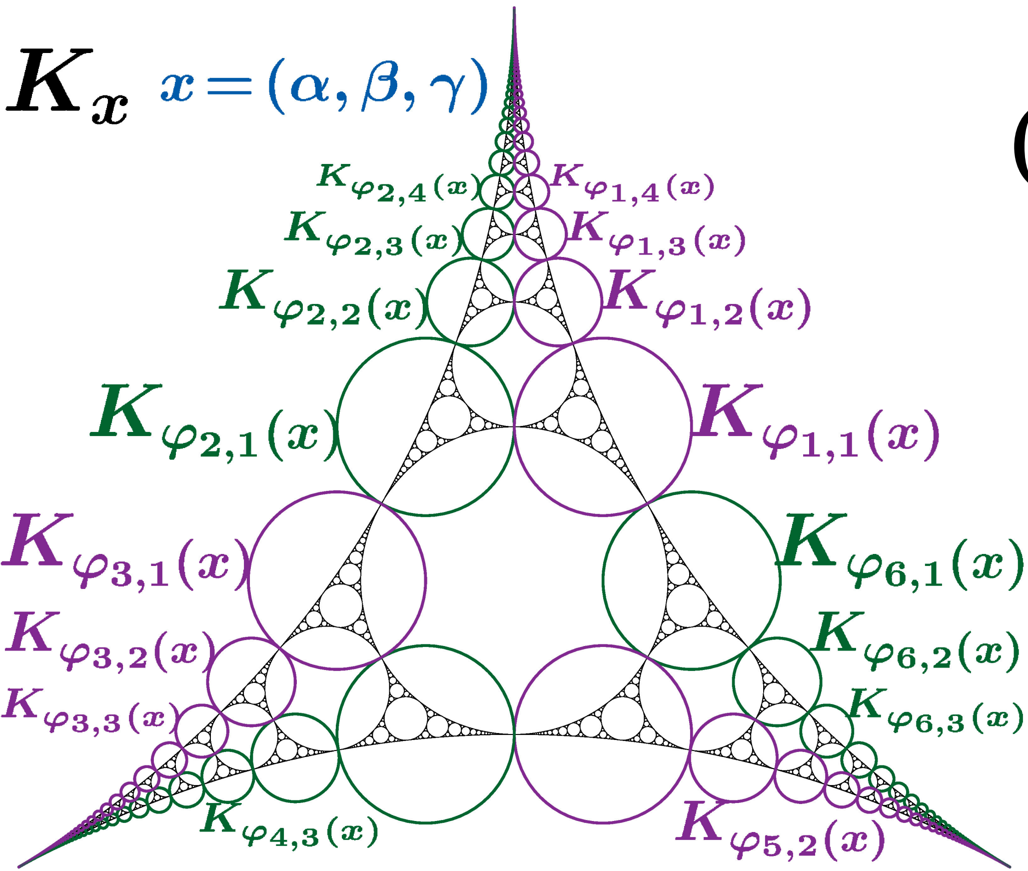

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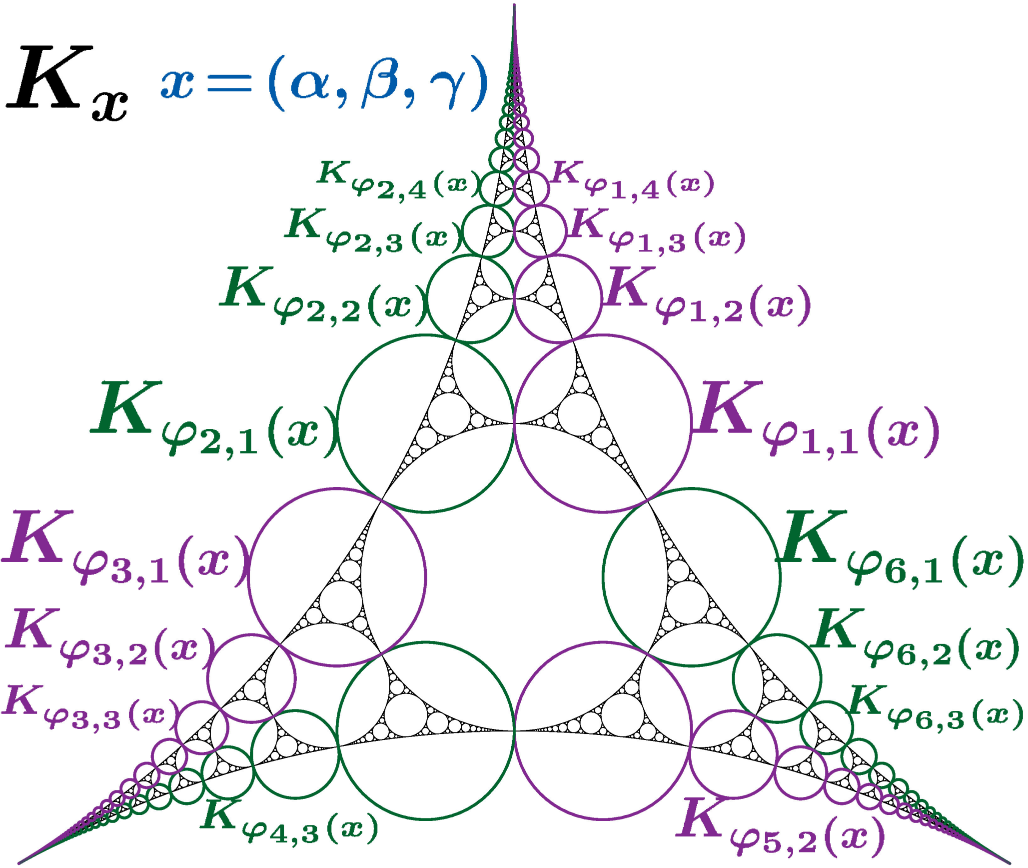

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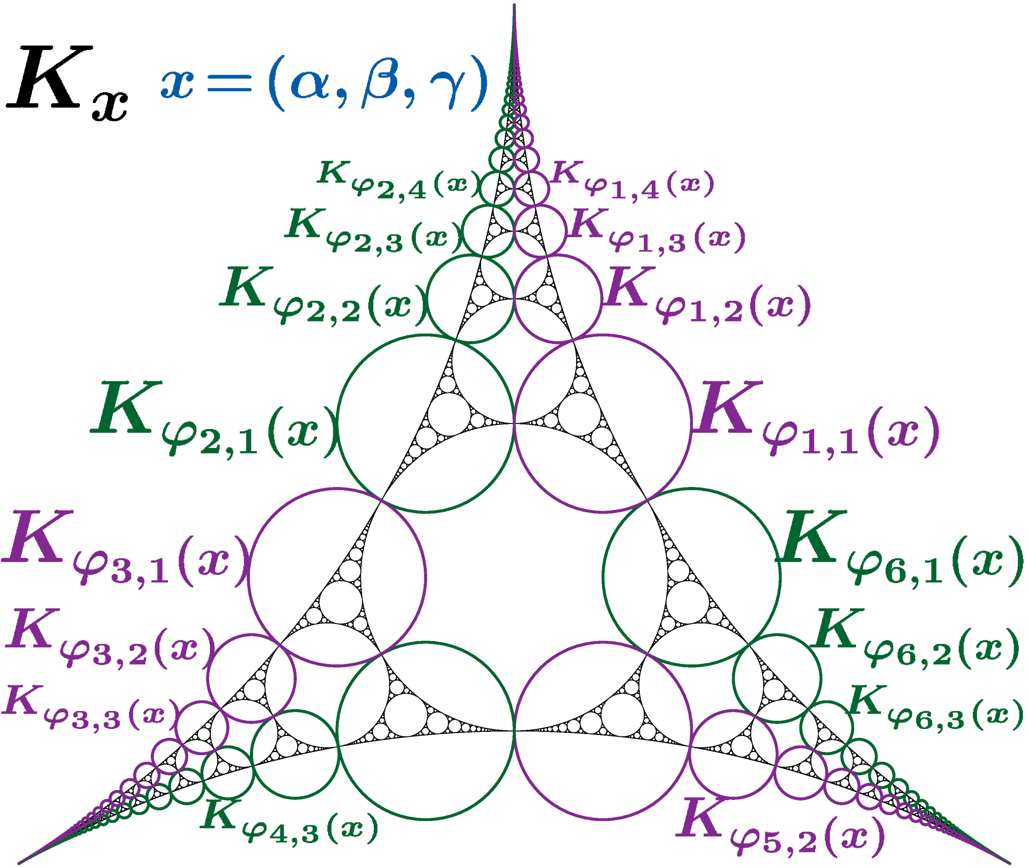

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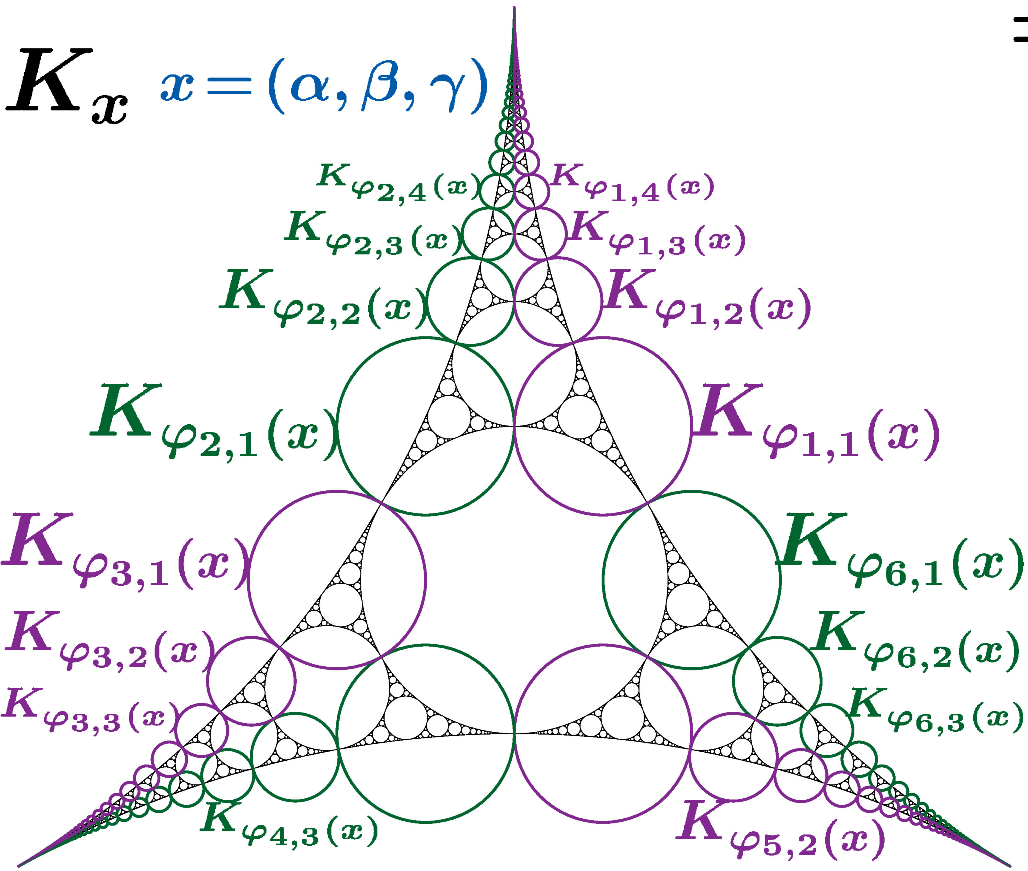

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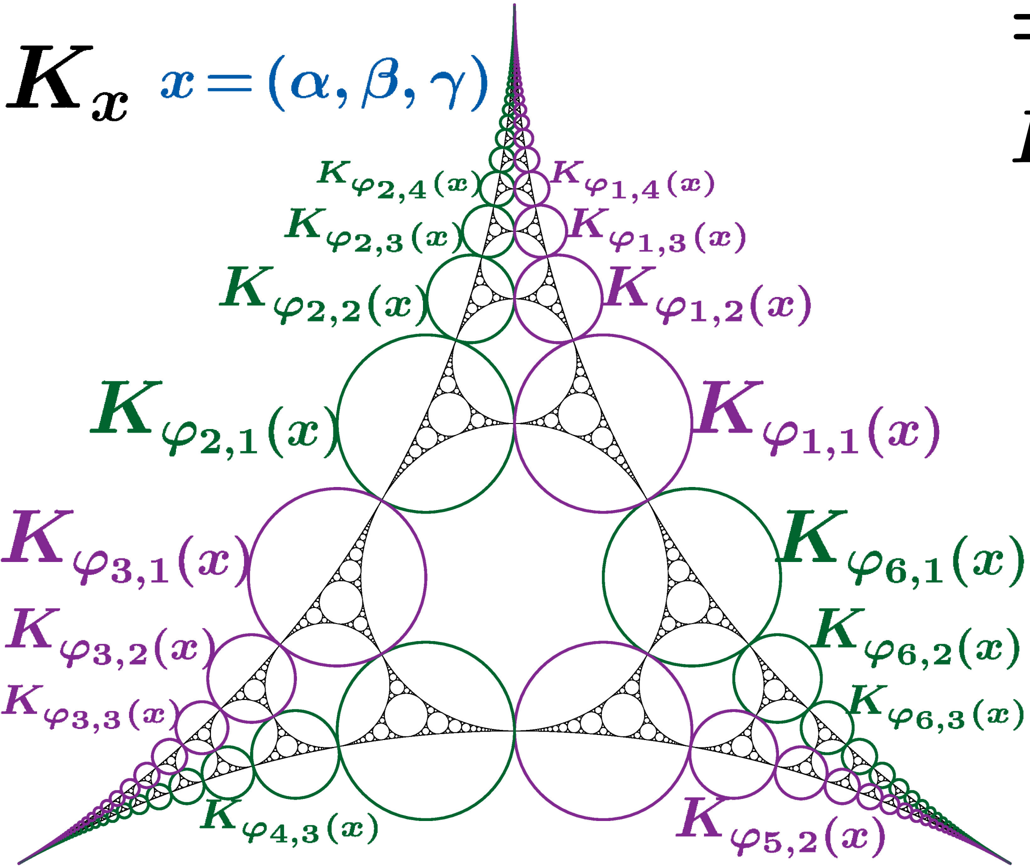

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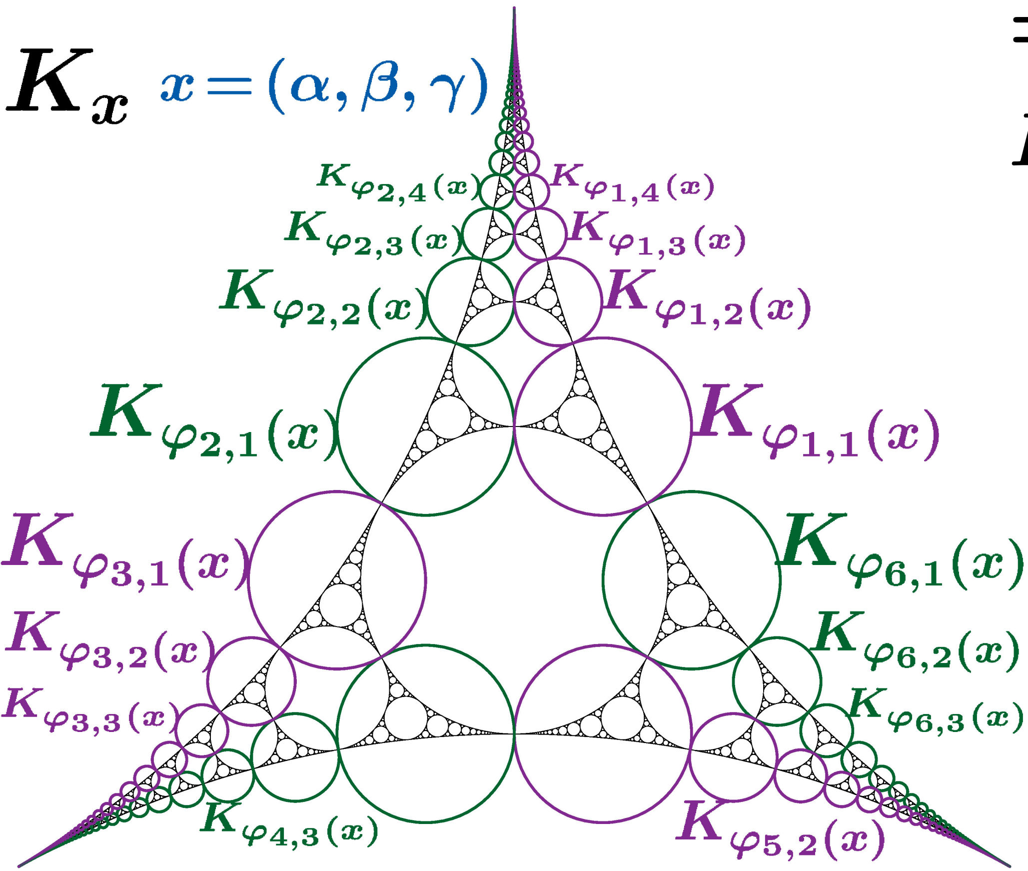

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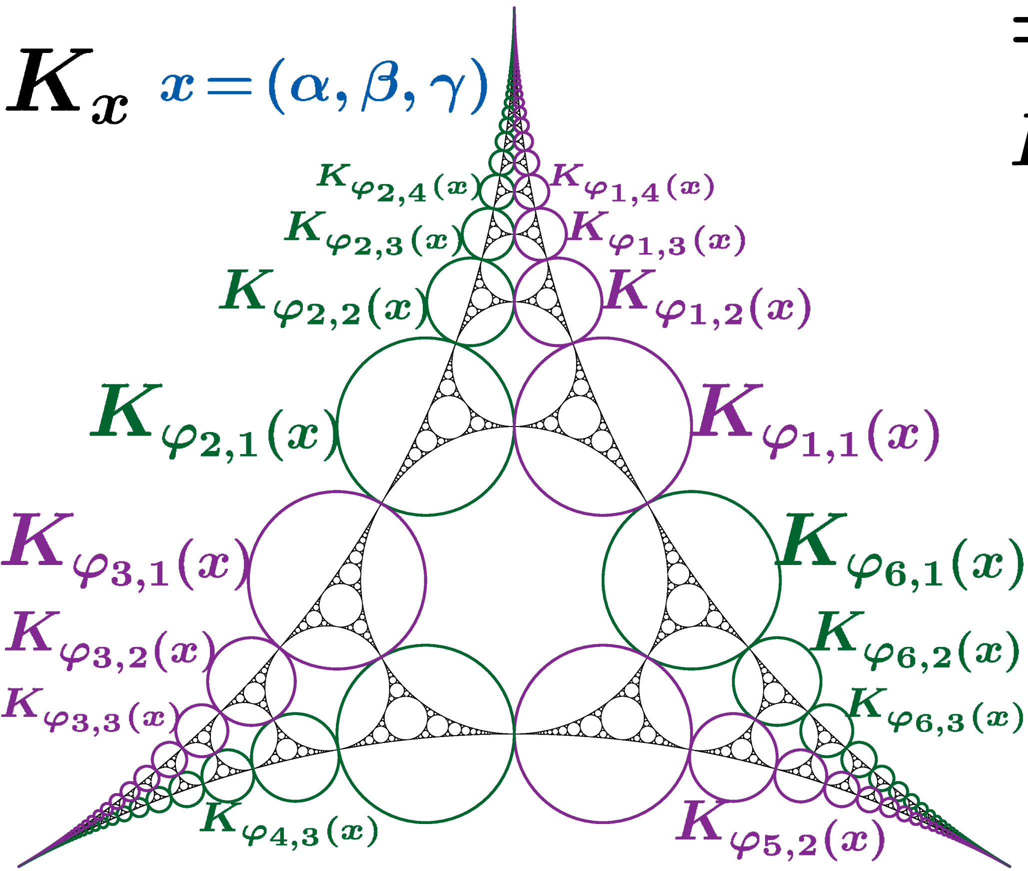

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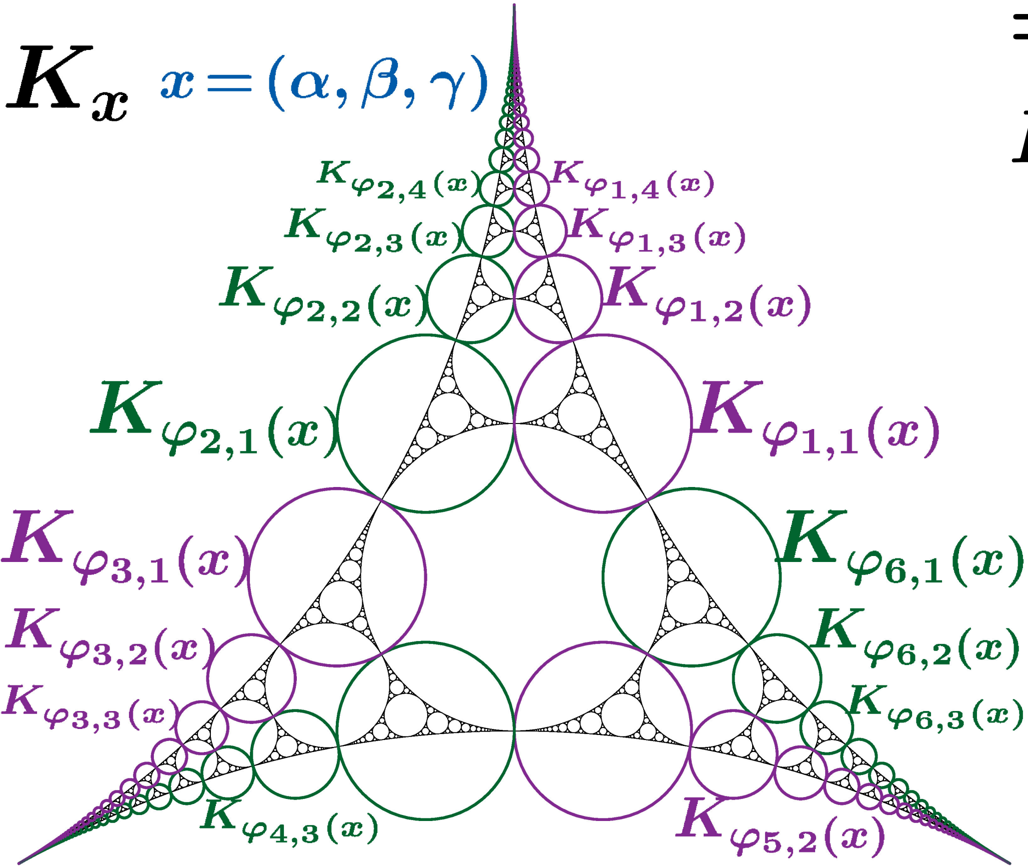

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$$= \mathbb{E}_x\Big[\sum_{n=0}^{\infty} \mathcal{R}([X_n], s - V_n)\Big]$$

$$\xrightarrow[\text{Kesten '74}]{s\,\to\,\infty} \int_{\Gamma\times\mathbb{R}} \mathcal{R}(x,s)\,d\nu(x)\,ds!$$

Need: ● $|\mathcal{R}(x,s)| \leq c' e^{-c|s|^{\alpha}}$.

●$\{(X_n, V_n)\}_{n=0}^{\infty}$ unique. ergodic

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cf. p. 30, Figure 3 of R. D. Mauldin & M. Urbański, Adv. Math. 136 (1998), 26–38

Thm 1 (K.). $^{\exists}c_0 \in (0,\infty)$, $^{\forall}\alpha,\beta,\gamma \in (0,\infty)$, $(d := d_{\mathsf{AG}})$

$$\lim_{\lambda\to\infty} \#\{n \in \mathbb{N} \mid \lambda_n^{\alpha,\beta,\gamma} \le \lambda\}/\lambda^{d_{\mathsf{AG}}/2} = c_0 \mathcal{H}^{d_{\mathsf{AG}}}(K_{\alpha,\beta,\gamma}).$$

▷ $\{[X_n]\}_{n=0}^{\infty}$: MC on $\Gamma_{\text{(shapes)}}$, $x \rightsquigarrow [\varphi_{k,l}(x)]$ w. prob. $\mathcal{H}^d(K_{\varphi_{k,l}(x)})$

▷ $V_n := -\frac{1}{d}\log \mathcal{H}^d(K_{X_n})$ (the changes in size along $\{[X_n]\}_{n=0}^{\infty}$)

▷ $K_x \setminus V_0 = \bigcup_{k,l} K_{\varphi_{k,l}(x)}$

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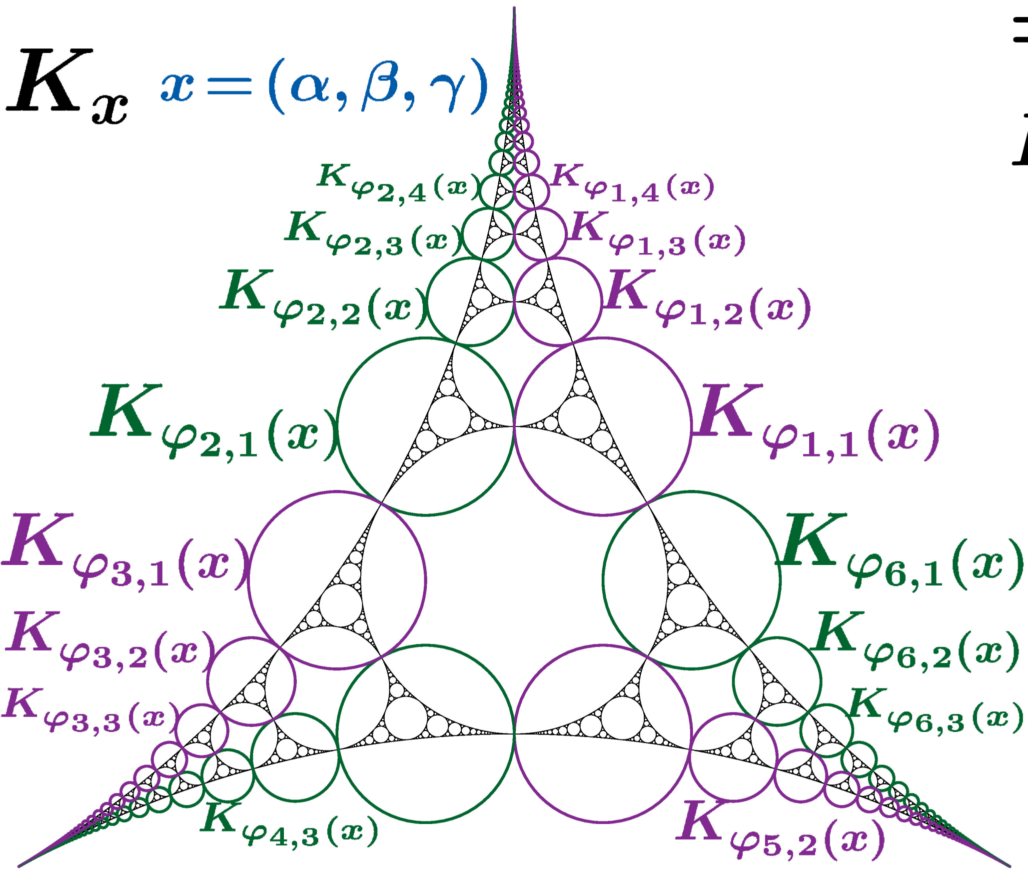

Need: ● $|\mathcal{R}(x,s)| \le c' e^{-c|s|^{\alpha}}$.

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cf. p. 30, Figure 3 of R. D. Mauldin & M. Urbański, Adv. Math. 136 (1998), 26–38

Cor (Special case of Oh–Shah '12). $\exists c > 0$, $\forall \alpha, \beta, \gamma \in (0, \infty)$,

$$\#\{C \subset_{\text{circle}} K_{\alpha,\beta,\gamma} \mid \mathrm{ra}(C)^{-1} \leq \lambda\} \overset{\lambda\to\infty}{\sim} c \cdot \mathcal{H}^d(K_{\alpha,\beta,\gamma})\lambda^d.$$

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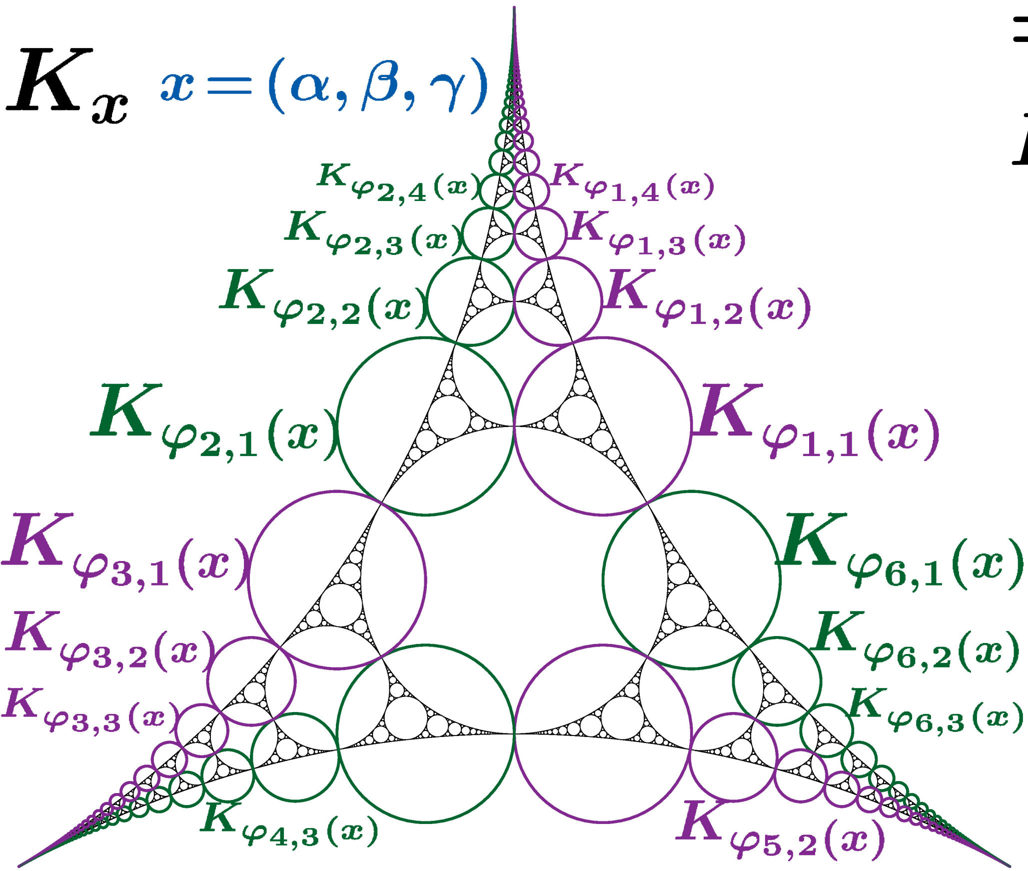

▷ $F(x, s) := e^{-ds}\mathcal{N}_x^{\mathrm{rad}}(e^s)$

$$= \mathcal{R}(x, s) + \sum_{k,l} \mathcal{H}^d(K_{\varphi_{k,l}(x)}) \cdot F\left([\varphi_{k,l}(x)], s + \tfrac{1}{d}\log \mathcal{H}^d(K_{\varphi_{k,l}(x)})\right)$$

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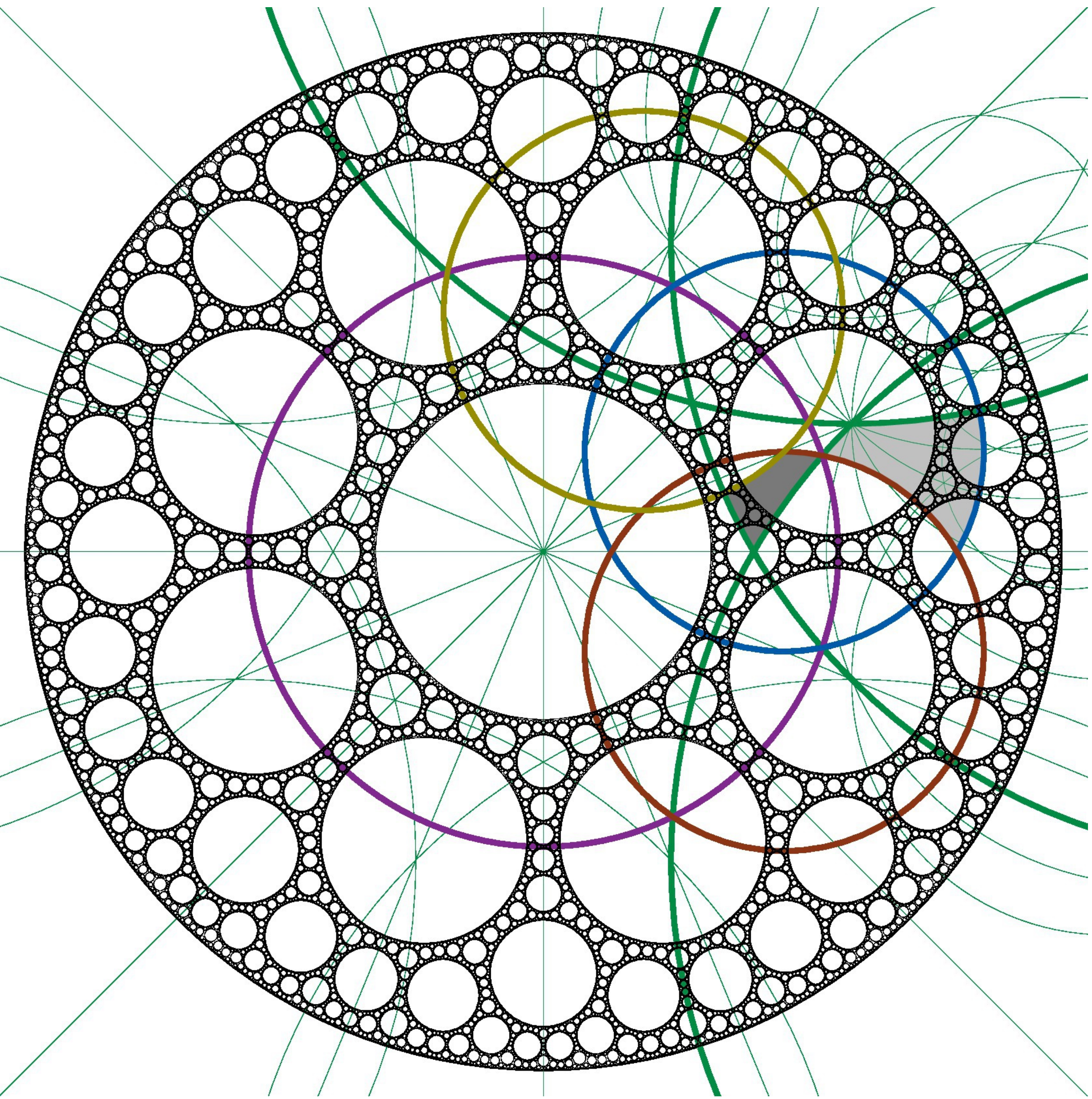
5 Proof ingredients for $G = G_m$ with $\partial_\infty G$ a RSC



● A "self-similar" decomp. ("fundamental domain" for the action $G \curvearrowright \partial_\infty G$)

▷ $\{\ell_k\}_{k=1}^3$: $\mathbb{B}^2$-geodesics, form $\triangle$, angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{m}$

▷ $\Gamma_m := \langle \{\mathbf{Inv}_{\ell_k}\}_{k=1}^3 \rangle$

$\rightsquigarrow \mathbb{B}^2 = \bigcup_{\tau \in \Gamma_m} \tau(\triangle_{\ell_1 \ell_2 \ell_3})$

● $S = S_m := \partial B_{\mathbb{B}^2}(0, {}^{\exists 1} r_m)$: $\mathrm{angle}(S, \ell_2) = \frac{\pi}{3}$.

▷ $G = G_m := \langle \Gamma_m, \mathbf{Inv}_S \rangle$, $\partial_\infty G_m := \overline{\bigcup_{g \in G_m} g(\partial \mathbb{B}^2)}$

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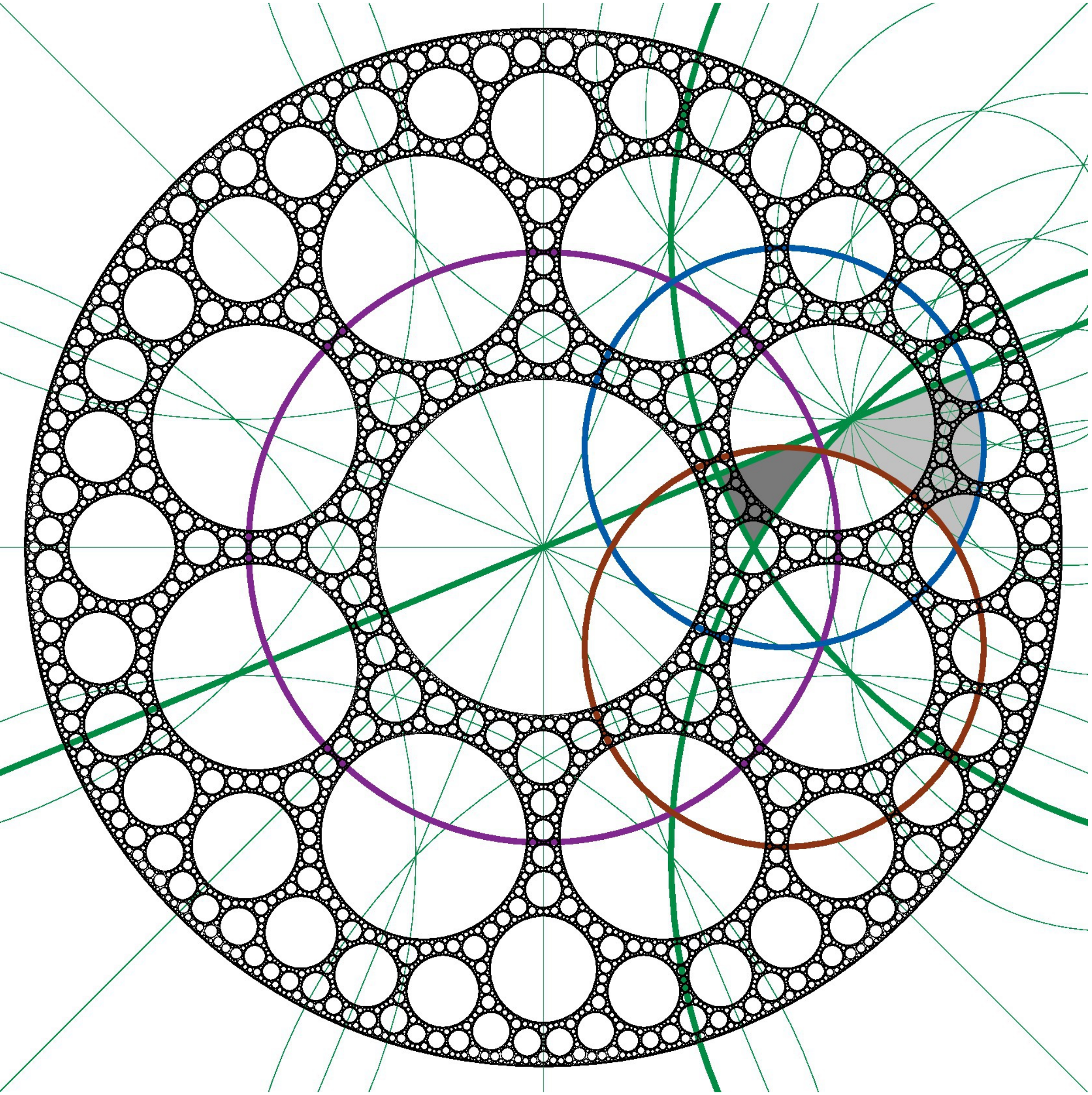

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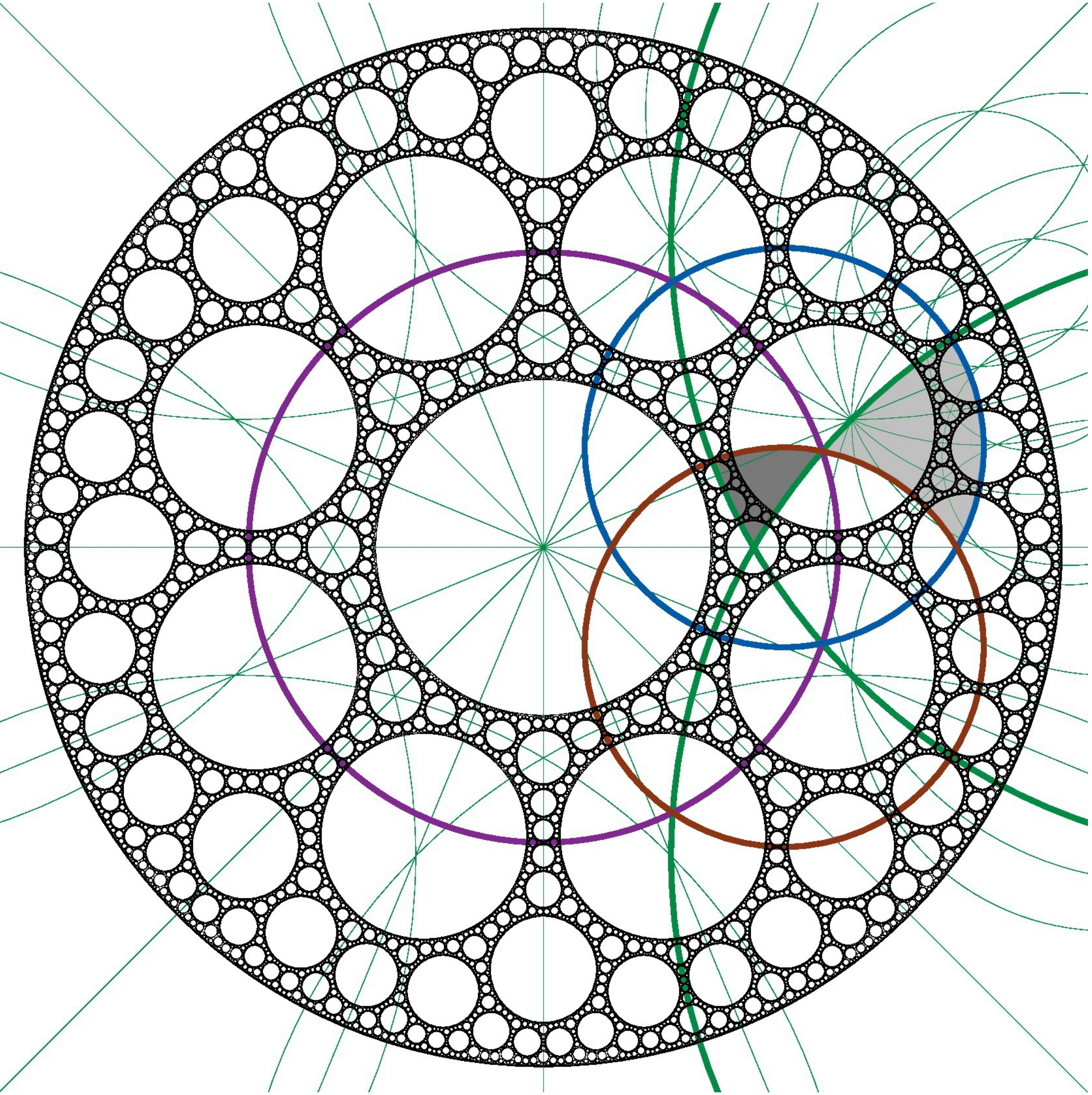
5 Proof ingredients for $G = G_m$ with $\partial_\infty G$ a RSC



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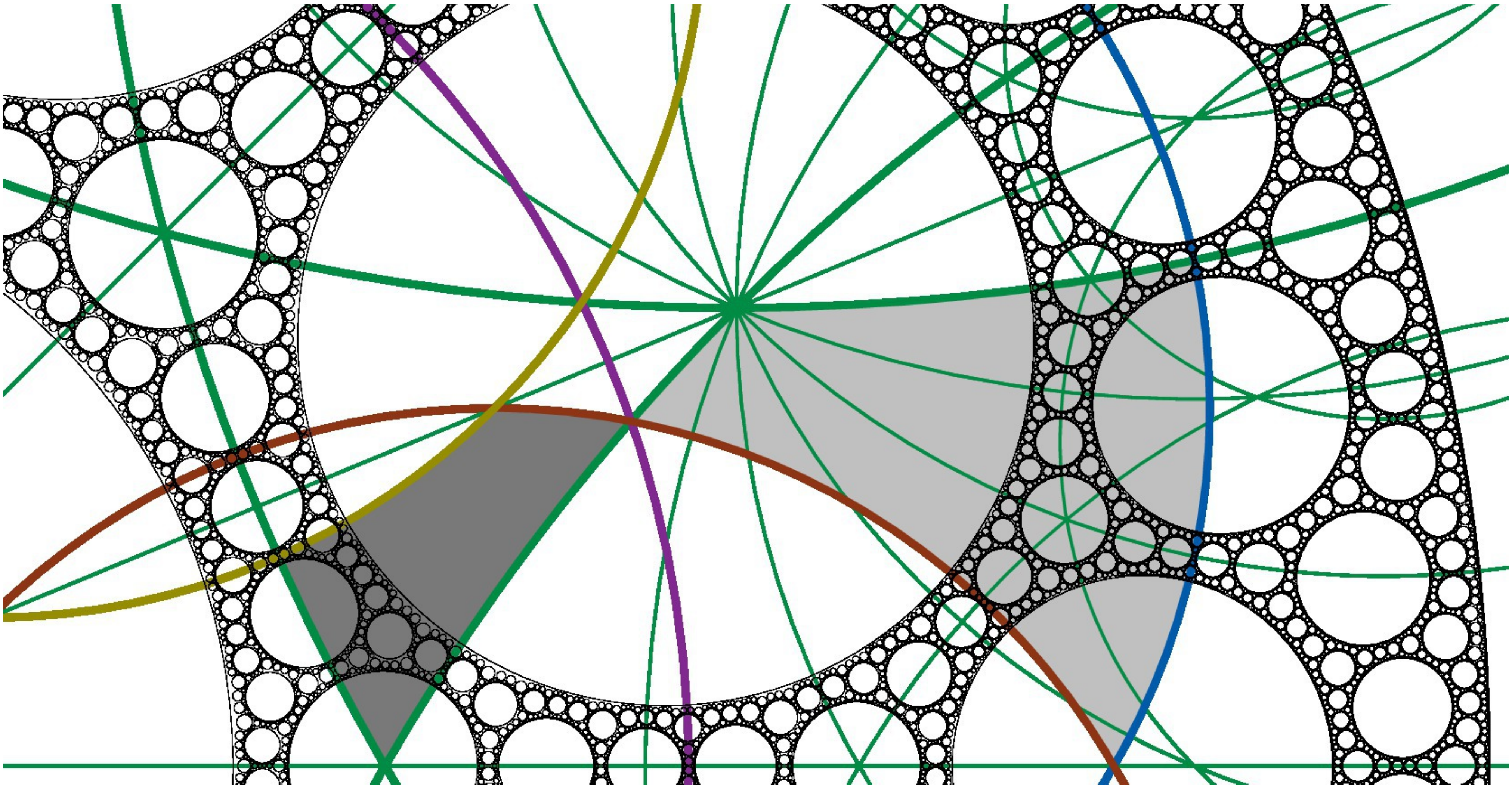

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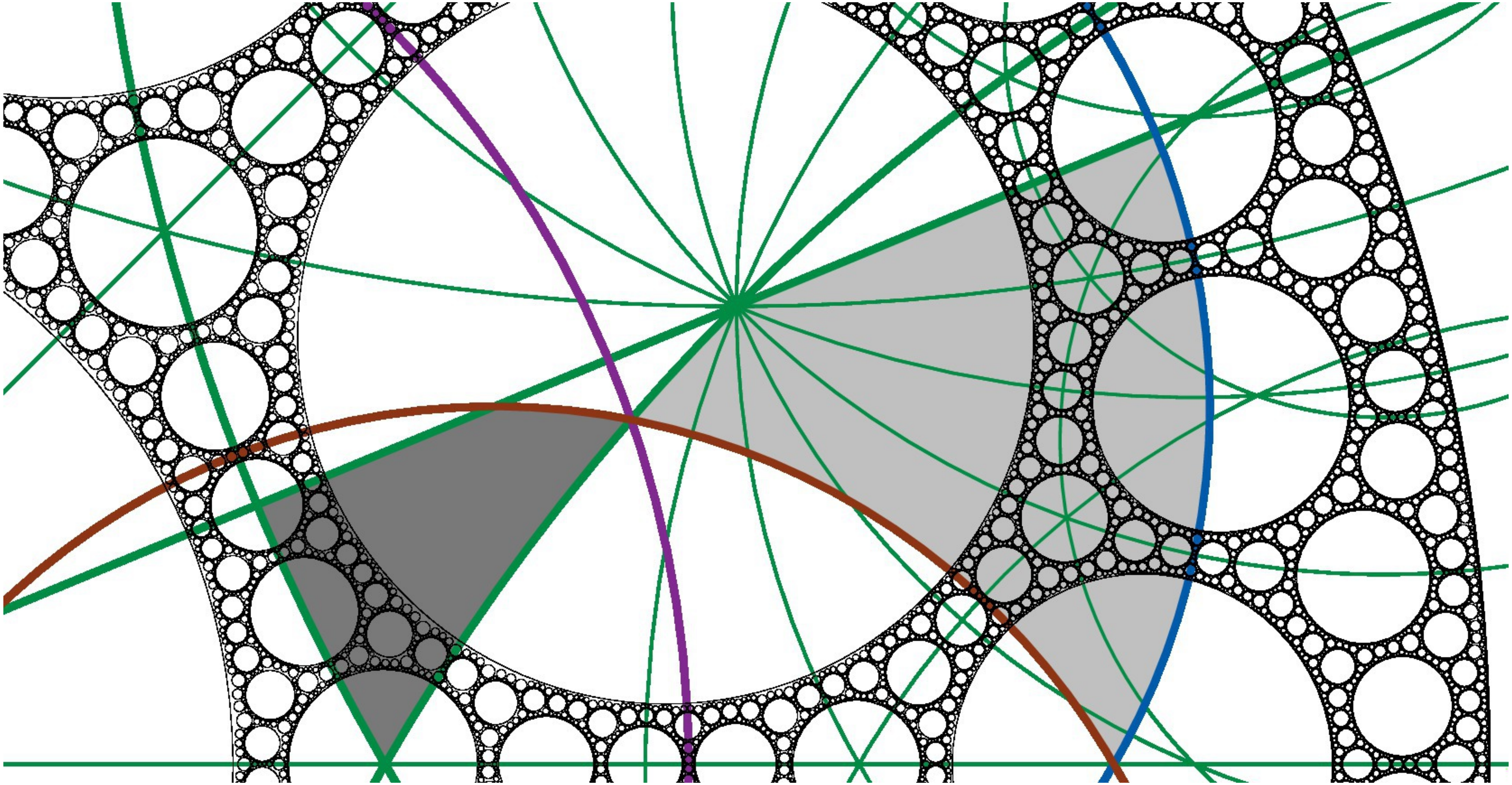

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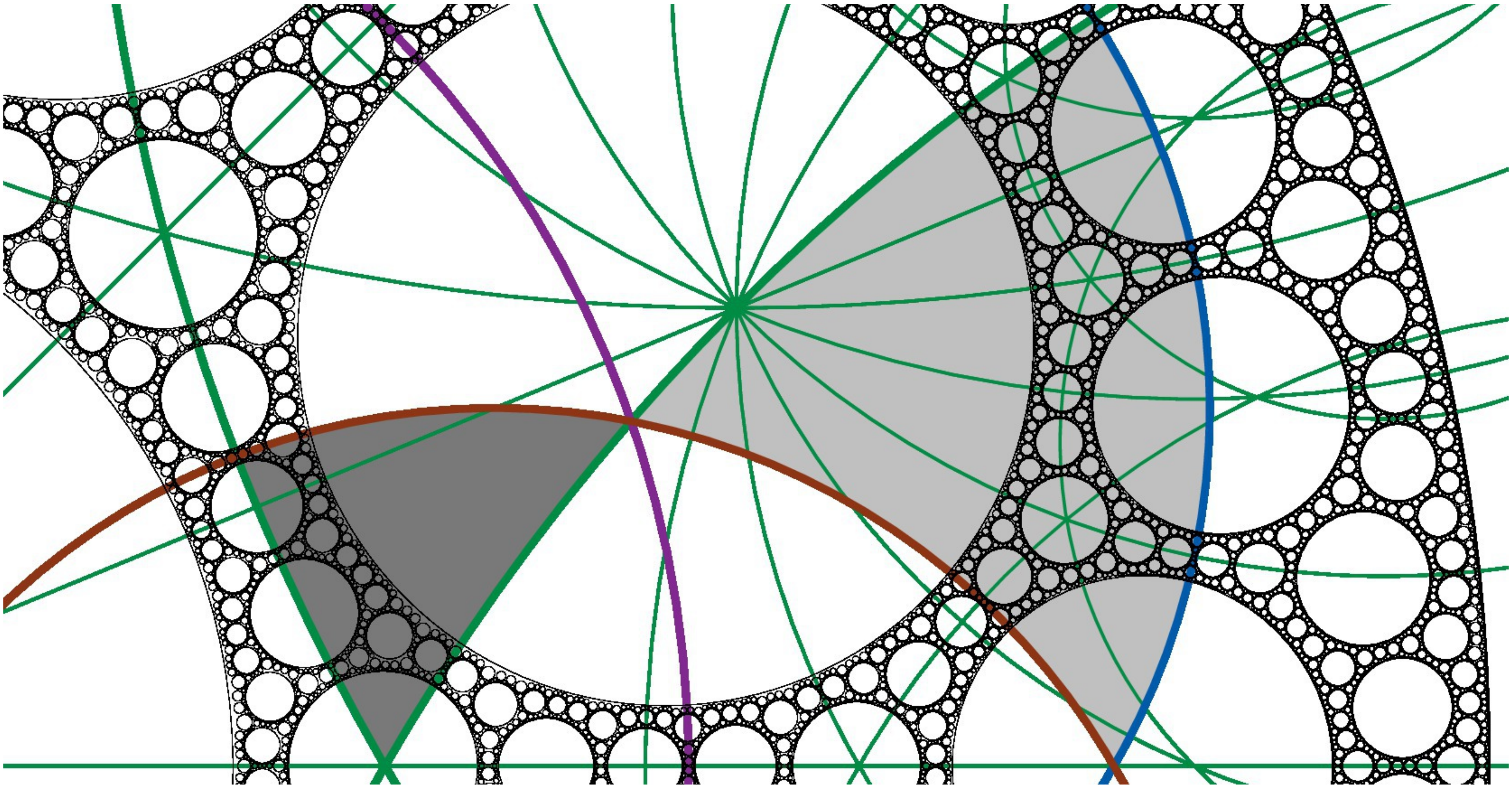

- **A "self-similar" decomp. ("fund. dom." for $G \curvearrowright \partial_\infty G$)**

(requires concrete knowledge of $G \curvearrowright \partial_\infty G$; NOT extend easily)

- **A version of the "2-dimensional" Nash inequality**

($\rightsquigarrow$ the spectrum of Δ_{K_g} is discrete & ${}^{\exists}p_t^{K_g}(x,y) \le c_g t^{-1}$)

- $\iota : K_g \hookrightarrow \mathbb{R}^2$ **is** $\mathcal{E}^g$*-harmonic* **&** $\Gamma_{\mathcal{E}^g}(\iota,\iota) = \nu^g$

($\rightsquigarrow \{\langle X_t^g, z\rangle\}_t$ slower than $\{B_t^{\mathbb{R}}\}_t \rightsquigarrow \mathbb{P}_x[\tau_{B(x,r)} \le t]$ small!)

$\rightsquigarrow$ **can now run a Markov chain on the** ***Euclidean shapes of the cells*** **and apply Kesten's renewal thm [AOP '74].**

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▷ $\nu^g := \sum_{C \subset_{\text{arc}} K_g} \mathrm{rad}(C)\mathrm{vol}_C$ (**NOT doubling!**)

▷ ${}^\forall u \in \mathrm{LIP}|_{K_g}$, $\mathcal{E}^g(u,u) := \sum_{C \subset_{\text{arc}} K_g} \mathrm{rad}(C) \int_C |\nabla_C u|^2 \, d\mathrm{vol}_C$ (*cf.* **Osada '07**)

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- (Grigor'yan–Hu–Lau '10) Bound for $p_t^{K_g}(x, x) - p_t^U(x, x)$
in terms of $\sup_{s\in[t/2,t]} p_s^{K_g}(x, \cdot)|_{\partial_{K_g} U}$ & $\mathbb{P}_x[\tau_U \le t]$

$\rightsquigarrow$ can now run a Markov chain on the *Euclidean shapes of the cells* and apply Kesten's renewal thm [AOP '74].

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- (*cf.* Bonk '11) The circles in $\partial_\infty G$ are unif. rel. separated: ${}^\forall j \neq k$, $\mathrm{dist}(C_j, C_k) \geq \delta_m \min\{\mathrm{rad}(C_j), \mathrm{rad}(C_k)\}$.

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