Particle approximation of the doubly parabolic Keller-Segel equation in the plane

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Overview

Motivations

Particle approximation

Main Result

Strategy of proof

"Markovianization" argument

Overview

Keller-Segel model for chemotaxis (1970)

- **Chemotaxis**: directed movement of a cell population guided by chemical stimuli in their environment (that cells may emit) .
- Coupled non linear system on **population density** - $\rho(t,x)$ and **chemo-attractant concentration** - $c(t,x)$:

$$\begin{cases} \partial_t \rho(t,x) = \Delta \rho - \chi \nabla \cdot (\rho \nabla c), \quad t > 0,\ x \in \mathbb{R}^2, \\ \theta \partial_t c(t,x) = \triangle c - \lambda c + \rho, \quad t > 0,\ x \in \mathbb{R}^2. \\ \rho(0,x) = \rho_0(x), c(0,x) = c_0(x), \end{cases} \tag{1}$$

- Parameters:
  - $\chi > 0$: chemotactic sensitivity,
  - $\theta > 0$: ratio between the diffusion time scales of cells and chemical,
  - $\lambda \geq 0$: death rate of the chemo-attractant,
  - $\int \rho_0(dx) = 1$ total mass of cells rescaled.
- $\theta = 0$: parabolic-elliptic case (decoupled),
  $\theta > 0$: doubly parabolic (strongly coupled).

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Finite Time Blow Up VS Global Existence in $\mathbb{R}^2$

- FTBU: An **agglomeration of cells** emerges due to mutual cell attraction (some norm explodes in FT).

- Well known for parabolic-elliptic case:
  - $\chi < 8\pi$: GE (Blanchet-Dolbeault-Perthame, '06),
  - $\chi > 8\pi$: **FTBU** for $\int |x|^2 \rho_0(dx) < \infty$,
  - $\chi = 8\pi$: BU as $t \to \infty$.
    (See e.g. Perthame survey '05)

- Doubly parabolic:
  - $\chi < 8\pi$: GE (Calvez-Corrias '08) ,
  - $c_0 \equiv 0$, GE for any $\chi \leq \chi_\theta$ where $\chi_\theta \to \infty$ as $\theta \to \infty$
    (Biler-Guerra-Karch '15, Corrias-Escobedo-Matos '14)
  - **FTBU** open, recent result for $\chi > 8\pi$ and a class of (ρ_0, c_0).
    (Mizoguchi '21)

Our goal: derive the system (1) as a mean-field limit of an Interacting particle system.

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Overview

Mean field limit

Typical particle when $N = \infty$ with a 2 step approach:

1. Follows the potential c:

$$dX_t = \sqrt{2}dW_t + \chi\nabla c_t(X_t)dt.$$

   Denote $\rho_t := \mathcal{L}(X_t)$, for $t > 0$.

2. Feynman-Kac for c with ρ as source term:

$$c_t(x) = b_t^{c_0,\theta,\lambda}(x) + \int_0^t (K_{t-s}^{\theta,\lambda} * \rho_s)(x)\mathrm{d}s,$$

   where we denoted, for $(t,x) \in (0,\infty) \times \mathbb{R}^2$,

$$g_t^\theta(x) := \frac{\theta}{4\pi t}e^{-\frac{\theta}{4t}|x|^2}, \quad K_t^{\theta,\lambda}(x) := \frac{1}{\theta}e^{-\frac{\lambda}{\theta}t}g_t^\theta(x),$$

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Mean field limit

Putting everything together,

$$\begin{cases} X_t = X_0 + \sqrt{2}W_t + \chi \int_0^t \nabla b_s^{c_0,\theta,\lambda}(X_s)\mathrm{d}s + \chi \int_0^t \int_0^s (\nabla K_{s-u}^{\theta,\lambda} * \rho_u)(X_s)\mathrm{d}u\mathrm{d}s, \\ \rho_s = \mathrm{Law}(X_s), s \geq 0. \end{cases}$$

Notice

1. Past laws dependence,
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(well posedness with L^p spaces (T. '20) in $\mathbb{R}^2$)

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Particle system

For $N \geq 2$ it reads

$$X_t^{i,N} = X_0^{i,N} + \sqrt{2}W_t^i + \chi \int_0^t \nabla b_s^{c_0,\theta,\lambda}(s, X_s^{i,N})\mathrm{d}s$$
$$+ \frac{\chi}{N-1} \sum_{j \neq i} \int_0^t \int_0^s \nabla K_{s-u}^{\theta,\lambda}(X_s^{i,N} - X_u^{j,N})\mathrm{d}u\mathrm{d}s.$$

Notice

1. Non-Markovian system!
2. Singular interaction

$$\nabla K_t^{\theta,\lambda}(x) = -\frac{\theta}{8\pi t^2} e^{-\frac{\lambda}{\theta}t} e^{-\frac{\theta}{4t}|x|^2} x.$$

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How singular?

Roughly speaking, if for some $R \in \mathbb{R}^2$ we have

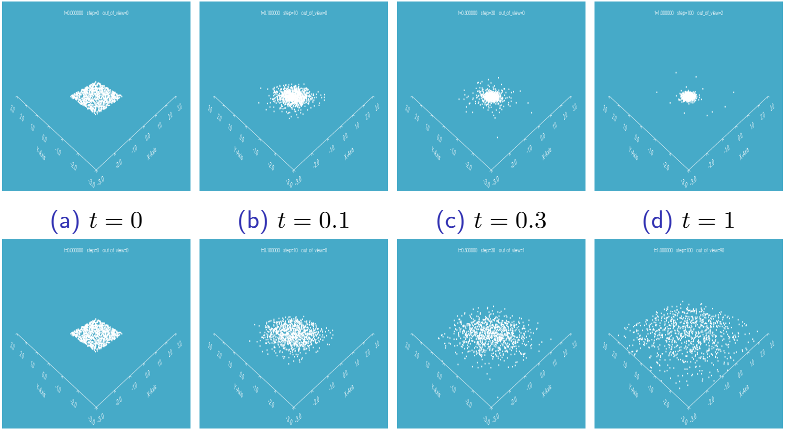
$$X_t^{1,N} = X_s^{2,N} + R, \quad s \in [t-1, t]$$

then the corresponding interaction (in the drift of $X^{1,N}$) looks like, e.g. when $\lambda = 0$,

$$\begin{aligned}\int_{t-1}^{t} \nabla K_{t-s}^{\theta,\lambda}(X_t^{1,N} - X_s^{2,N})\mathrm{d}s &= -\frac{\theta R}{8\pi}\int_{t-1}^{t} \frac{1}{(t-s)^2} e^{-\frac{\theta}{4(t-s)}|R|^2}\mathrm{d}s \\ &= -\frac{R}{2\pi|R|^2} e^{-\frac{\theta|R|^2}{4}} \overset{|R|\to 0}{\sim} -\frac{R}{2\pi|R|^2}.\end{aligned}$$

(Of course, this is an exaggerated situation.)

A simulation of the Particle system in $d = 2$



(a) $t = 0$ (b) $t = 0.1$ (c) $t = 0.3$ (d) $t = 1$

(e) $t = 0$ (f) $t = 0.1$ (g) $t = 0.3$ (h) $t = 1$

Figure: (a)-(d): χ large; (e)-(h): χ small.

Related works

- **Doubly parabolic case**
  - **in $1d$**: $\nabla K_t^{1d}(x) \sim \frac{x}{t^{3/2}} e^{-\frac{x^2}{4t}}$.
    Propagation of chaos in Jabir-Talay-T. ('18) using Girsanov transforms (impossible here as particles should collide, higher dimension $\to$ more singularity).
  - **in any d**: two particle system with mollified interaction by Stevens ('01).
- **Parabolic-elliptic case in $2d$**: $c_0 = 0$, $\lambda = 0$

$$dX_t^i = \sqrt{2} dW_t^i + \frac{\chi}{N} \nabla K(X_t^i - X_t^j) dt$$

where $\nabla K(x) = -\frac{x}{2\pi |x|^2}$.
Existence and convergence along subsequences $\chi \leq 2\pi$ in Fournier-Jourdain ('17) , $\chi \leq 8\pi$ Tardy ('21).

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Main result

We set $\mu^N := \frac{1}{N}\sum_{i=1}^{N}\delta_{(X_t^{i,N})_{t\geq 0}} \in \mathcal{P}(C([0,\infty),\mathbb{R}^2))$ a.s. and, for each $t \geq 0$, $\mu_t^N := \frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^{i,N}} \in \mathcal{P}(\mathbb{R}^2)$ a.s.

Theorem

*For each $\theta > 0$, there is $\chi_\theta > 0$ such that if $\chi \leq \chi_\theta$, then the system **has a solution** (for any exchangeable initial condition) for each $N \geq 2$ and, up to extraction of a subsequence, $(\mu_t^N)_{t\geq 0}$ **converges** to a solution $(\rho_t)_{t\geq 0}$ of (KS) if $\mu_0^N \xrightarrow{\mathbb{P}} \rho_0$.*

(of course we have as well the tightness of μ^N and convergence to a MP...)

About the threshold

The particular form is quite complicated (but **explicit**) and **independent** of ρ_0, c_0. Optimizing the condition numerically we have:

- $\chi_{\theta=1} = 1.39$,
- $\chi_{\theta=0.00001} = 3.28$,
- $\chi_\theta \overset{\theta\to\infty}{\sim} \frac{1.65}{\sqrt{\theta}}$.

(The last point is troubling, as at least when $c_0 \equiv 0$ one can find for any χ a θ such that the limit is well posed.)

Some comments

- The only information about the limit for all $t \geq 0$,

$$\int_0^t \int_{\mathbb{R}^2} \int_0^s \int_{\mathbb{R}^2} (K_{s-u}^{\theta,\lambda}(x-y) + |\nabla K_{s-u}^{\theta,\lambda}(x-y)|)\rho_u(\mathrm{d}y)\mathrm{d}u\rho_s(\mathrm{d}x)\mathrm{d}s < \infty,$$

  $\rightarrow$ very weak, **measure valued solution to (KS)** (slightly different then the ones in Biler et al and Corrias et al).

- Difficult to show **uniqueness** to (KS) of such solutions (or propagation of regularity) $\rightarrow$ **not a propagation of chaos result** (does not coincide with the MP in more regular spaces, see T. (2020)).
- Initial condition only exchangeable particles (can be a **dirac**); initial concentration c_0 only in $L^{2^+}(\mathbb{R}^2)$

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Overview

Strategy ($\theta = 1, \lambda = 0, c_0 \equiv 0$)

- Remember that $\nabla K_t(x) = \nabla g_t(x) \sim -\frac{x}{t^2} e^{\frac{-|x|^2}{4t}}$.
- Control a priori the 2-by-2 interaction. Set

$$D_s^{1,2,N} := \int_0^s \nabla K_{s-u}(X_s^{1,N} - X_u^{2,N}) \mathrm{d}u,$$

we prove there exists $\gamma \in (\frac{3}{2}, 2)$ s.t.

$$\sup_{N \geq 2} \mathbb{E}\Big[\int_0^t |D_s^{1,2,N}|^{2(\gamma-1)} \mathrm{d}s \Big] < \infty \qquad \text{for all } t > 0.$$

Then, you can do this on a ε-regularised PS and get tightness, pass to the limit....

Key idea

- We want to perform a "*Markovianization*" of the interaction. Informally

$$|D_t^{1,2,N}| \sim \frac{1}{|X_t^{1,N} - X_t^{2,N}|}$$

- Rigorously, we will prove that for χ small, there exists $\gamma \in (\frac{3}{2}, 2)$ and C (independent of N) such that

$$\mathbb{E}\Big[\int_0^t |D_s^{1,2,N}|^{2(\gamma-1)} \mathrm{d}s\Big] \leq C\mathbb{E}\Big[\int_0^t |X_s^{1,N} - X_s^{2,N}|^{-2(\gamma-1)} ds\Big].$$

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We bound the **path dependent** interaction by a **current time dependent** one of **elliptic order**.

Now, as the interaction is of order $\frac{1}{|x|}$, we can proceed as in the elliptic case to treat it (Fournier-Jourdain).

For $\alpha \in (0,1)$ applying Ito and using exchangeability :

$$\frac{d}{dt}\mathbb{E}|X_t^1 - X_t^2|^\alpha \geq C_\alpha \mathbb{E}|X_t^1 - X_t^2|^{\alpha-2} - \frac{\chi}{N-1} C_\alpha \sum_{j=2}^{N} \mathbb{E}[|X_t^1 - X_t^2|^{\alpha-1} |D_t^{1,j}|]$$

Using Holder, exchangeability and the Markovianization

$$\frac{d}{dt}\mathbb{E}|X_t^1 - X_t^2|^\alpha \geq (C_\alpha - C\chi C_\alpha)\mathbb{E}|X_t^1 - X_t^2|^{\alpha-2}$$

Choose $\alpha = 4 - 2\gamma \in (0,1)$, suppose χ small and rearrange

$$\int_0^T \mathbb{E}|X_t^1 - X_t^2|^{2(1-\gamma)} \, dt \leq A_T.$$

Overview

3 ingredients

- A **suitable Itô formula** for the path dependent interaction,
- Apply it to a **convenient function**,
- A key **functional inequality**.

Time-space Itô

Denote $R^{i,j}_{t,s} := X^i_t - X^j_s$

Let $F : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ be of class $C^{1,2}_b(\mathbb{R}_+ \times \mathbb{R}^2)$. For all $t > 0$,

$$\mathbb{E}\Big[\int_0^t F(t-s, R^{1,2}_{t,s})ds\Big] = \mathbb{E}\Big[\int_0^t F(0, R^{1,2}_{s,s})ds\Big] + \mathbb{E}\Big[\int_0^t \int_0^u (\partial_t F + \Delta F)(u-s, R^{1,2}_{u,s})ds\ du\Big] + \frac{\chi}{N-1}\sum_{j=2}^N \mathbb{E}\Big[\int_0^t \Big(\int_0^u \nabla F(u-s, R^{1,2}_{u,s})ds\Big) \cdot D^{1,j}_u du\Big].$$

(Ito between s and t on X^1 with X^2_s fixed + integrate in s + Fubini.)

A good F

Notice that

$$|\nabla g_t(x)| \leq \frac{C_\beta}{(t+\beta|x|^2)^{\frac{3}{2}}}, \quad \beta > 0.$$

Choose

$$F(t,x) = -(t+\beta|x|^2)^{1-\gamma}, \quad \gamma \in (\frac{3}{2}, 2).$$

So that

$$(\partial_t F + \Delta F)(t,x) \geq C_\beta (t+\beta|x|^2)^{-\gamma}, \quad \text{for } \beta \text{ small},$$

and

$$|\nabla F| \leq C(t+\beta|x|^2)^{\frac{1}{2}-\gamma}.$$

Plugging this F

Using exchangeability

$$
\begin{aligned}
\text{(negative)} = &-\mathbb{E}\Big[\int_0^t |X_s^1 - X_s^2|^{2(1-\gamma)} ds\Big] \\
&+ \text{(geq)}\mathbb{E}\Big[\int_0^t \int_0^u (u - s + \beta|X_u^1 - X_s^2|^2)^{-\gamma} ds\ du\Big] \\
&- \text{(leq)}C_\chi\mathbb{E}\Big[\int_0^t \Big(\int_0^u (u - s + \beta|X_u^1 - X_s^2|^2)^{\frac{1}{2}-\gamma} ds\Big)|D_u^{1,3}| du\Big]
\end{aligned}
$$

Remember $|D_u^{1,3}| \leq C_\beta \int_0^u (u - s + \beta|X_u^1 - X_s^3|^2)^{-\frac{3}{2}} ds$

Now, it we would have some kind of Holder inequality to compare the terms on RHS we would be very happy.

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Key functional inequality

Let $b > a > 0$ and $t > 0$. For any measurable function $f : [0,t] \to \mathbb{R}_+$, we have

$$\int_0^t \frac{1}{(s+f(s))^{1+a}} ds \le \kappa(a,b) \Big(\int_0^t \frac{1}{(s+f(s))^{1+b}} ds \Big)^{\frac{a}{b}},$$

where

$$\kappa(a,b) = \frac{a+1}{a} \Big[\frac{b}{b+1} \Big]^{\frac{a}{b}}.$$

(The constant $\kappa(a,b)$ is optimal (for any value of $t > 0$))

Everything magically comes into place as applying FI

- for $a = 1/2, b = \gamma - 1$ we have

$$|D_u^{1,3}| \leq C_\beta \int_0^u (u - s + \beta |X_u^1 - X_s^3|^2)^{-\frac{3}{2}} ds$$

$$\leq \tilde{C}(\gamma, \beta) \Big(\int_0^u (u - s + \beta |X_u^1 - X_s^3|^2)^{-\gamma} ds \Big)^{\frac{1}{2(\gamma - 1)}}$$

- for $a = \gamma - \frac{3}{2}, b = \gamma - 1$ we have

$$\int_0^u (u - s + \beta |X_u^1 - X_s^2|^2)^{\frac{1}{2} - \gamma} ds$$

$$\leq \bar{C}(\gamma, \beta) \Big(\int_0^u (u - s + \beta |X_u^1 - X_s^2|^2)^{-\gamma} ds \Big)^{\frac{2\gamma - 3}{2(\gamma - 1)}}$$

and most importantly $\frac{1}{2(\gamma-1)} + \frac{2\gamma-3}{2(\gamma-1)} = 1.$

A classical Holder in both $\mathbb{E}$ and $\int_0^t$ separates the two terms and after exchangeability and rearranging lead to, provided χ **small**,

$$\mathbb{E}\Big[\int_0^t \int_0^u (u-s+|X_u^1-X_s^2|^2)^{-\gamma} ds\ du\Big]$$

$$\leq C(\chi,\gamma,\beta)\mathbb{E}\Big[\int_0^t |X_s^1-X_s^2|^{2(1-\gamma)} ds\Big].$$

Combine the **drift bound** from the previous slide and the **above** to finally get the Markovianization

$$\mathbb{E}\Big[\int_0^t |D_s^{1,2,N}|^{2(\gamma-1)} \mathrm{d}s\Big] \leq C\mathbb{E}\Big[\int_0^t |X_s^{1,N}-X_s^{2,N}|^{-2(\gamma-1)} ds\Big].$$

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