Random models on regularity-integrability structures

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BPHZ theorem

We consider singular SPDEs with renormalization, e.g., the dynamical Φ^4 model

$$(\partial_t - \Delta)\phi = \xi - (\phi^3 - \infty\phi) \quad \text{on } (0,\infty) \times \mathbb{R}^d.$$

How to prove the renormalizability? We use the local expansion

$$\phi(\cdot) \simeq \sum_{\tau} \phi_\tau(x)(\Pi_x \tau)(\cdot) \quad \text{near each } x \in (0,\infty) \times \mathbb{R}^d$$

by explicit random objects $\Pi_x \tau$.

Analytic & Algebraic steps of Regularity Structures

If all random objects $\Pi_x \tau$ and the products of up to 3 of them "make sense" (well-defined via renormalization), then ϕ^3 also makes sense.

The topic of this talk is

Probabilistic step of Regularity Structures

How to build the random objects?
(= BPHZ (Bogoliubov–Parasiuk–Hepp–Zimmerman) theorem)

Main result (rough)

We consider a class of SPDEs

$$(\partial_t - \Delta)u = F(u, \nabla u, \xi) \quad \text{on } (0,\infty)\times\mathbb{R}^d,$$

where ξ is the space or space-time white noise and F is a polynomial of $u, \nabla u$, and ξ but may contain the factor $f(u)$ for a sufficiently regular f if $u \in C^{0+}$.

Ingredients from Regularity Structures (defined later)

- *A family $\mathcal{F}$ of random objects (functionals of ξ) under consideration.*
- *An "expected regularity" $r : \mathcal{F} \to \mathbb{R}$.*
- *BPHZ model $\hat{\Pi}_x : \mathcal{F} \to \mathcal{S}'$ (uniquely defined from the law of ξ).*

BPHZ Theorem (Hairer–Steele '23+, Bailleul–H '23+)

If $\inf r(\mathcal{F} \setminus \{\xi\}) > -\frac{d+2}{2}$, then $\hat{\Pi}_x\tau$ makes sense for each $\tau \in \mathcal{F}$.

- The result holds for many SPDEs: Φ^4_d $(d < 4)$, 1d qgKPZ,...
- The same result holds for any stationary ξ satisfying Poincaré inequality.

Proofs of BPHZ theorem

(1) Feynman diagram approach. Stochastic estimates lead to the bounds of complicated iterated integrals $\to$ Graph theoretical arguments.

- Direct calculations (Hairer '14, H '16,...) have limitations on the size of $\sharp\mathcal{F}$.
- A general proof by Chandra–Hairer '16+ is tough reading (at least for me).

(2) Diagram-free approach. An inductive proof based on Poincaré inequality: There exists a constant $C > 0$ and it holds that

$$\mathbb{E}\big[|F(\xi) - \mathbb{E}[F(\xi)]|^2\big] \le C\mathbb{E}\big[\|\nabla F(\xi)\|_{L^2}^2\big]$$

for any bounded and smooth cylindrical functions $F : \mathcal{S}'(\mathbb{R}^{1+d}) \to \mathbb{R}$, where ∇ is the gradient operator along Cameron–Martin space $L^2(\mathbb{R}^{1+d})$.

- Linares–Otto–Tempelmayr–Tsatsoulis '21+ (Multiindex structure).
- Hairer–Steele '23+ (Pointed modelled distributions).
- Bailleul–H '23+ (**Regularity-Integrability Structure**).

Settings

Key ingredients $\boxed{\mathcal{F},\ r,\ \hat{\Pi}}$

- Duhamel form

$$
\begin{aligned}
&(\partial_t - \Delta)u = F(u, \nabla u, \xi) \\
\Leftrightarrow \quad & u = I(F(u, \nabla u, \xi)) + (\text{i.c.}) \qquad (I = (\partial_t - \Delta)^{-1})
\end{aligned}
$$

- Given $F(u, \nabla u, \xi)$, we define a family $\mathcal{F}$ of ξ-functionals sufficiently large to express the Duhamel equation. For example, if $F = -u^3 + \xi$, we can recursively define $\mathcal{F}$ by

  (1) $\xi \in \mathcal{F}$,
  (2) $\tau_1, \tau_2, \tau_3 \in \mathcal{F} \quad \Rightarrow \quad I(\tau_1),\ I(\tau_1)I(\tau_2),\ I(\tau_1)I(\tau_2)I(\tau_3) \in \mathcal{F}$,
  (3) (+“Polynomials”)

  Thus $\mathcal{F} = \{\xi, I(\xi), I(\xi)^2, I(\xi)^3, I(I(\xi)), I(I(\xi))I(\xi), \dots\}$.

- We rather write $\Xi, \mathcal{I}(\Xi), \mathcal{I}(\Xi)^2, \dots$ instead of $\xi, I(\xi), I(\xi)^2, \dots$ to distinguish abstract symbols from real functions/distributions.

Key ingredients $\boxed{\mathcal{F},\ r,\ \hat{\Pi}}$

- We introduce another symbol $\dot{\Xi}$ expressing elements of CM space L^2.
- Note that each $\tau \in \mathcal{F}$ is multilinear with respect to Ξ. Define the family $\dot{\mathcal{F}}$ of first order derivatives of ξ-functionals by

$$\dot{\mathcal{F}} = \{\tau(\underset{1}{\Xi}, \dots, \underset{i}{\dot{\Xi}}, \dots, \underset{n}{\Xi})\,;\, \tau \in \mathcal{F},\ 1 \le i \le n\}.$$

- We also define the abstract derivative $D : \langle \mathcal{F} \rangle \to \langle \dot{\mathcal{F}} \rangle$ by

$$D\Xi = \dot{\Xi}, \quad D\mathcal{I}(\tau) = \mathcal{I}(D\tau), \quad D(\tau\sigma) = (D\tau)\sigma + \tau(D\sigma),$$

that is,

$$D\tau(\underset{1}{\Xi}, \dots, \underset{n}{\Xi}) = \sum_{i=1}^{n} \tau(\underset{1}{\Xi}, \dots, \underset{i}{\dot{\Xi}}, \dots, \underset{n}{\Xi}).$$

e.g., $D\Xi\mathcal{I}(\Xi) = \dot{\Xi}\mathcal{I}(\Xi) + \Xi\mathcal{I}(\dot{\Xi})$.

Key ingredients $\boxed{\mathcal{F},\ r,\ \hat{\Pi}}$

- Fix $p \in [2, \infty]$.
- The "expected regularity" $r_p : \mathcal{F} \cup \dot{\mathcal{F}} \to \mathbb{R}$ defined by

$$\begin{aligned} r_p(\Xi) &= -\tfrac{d+2}{2} - \varepsilon \quad \text{for fixed } \varepsilon > 0, \\ r_p(\dot{\Xi}) &= r_p(\Xi) + \tfrac{d+2}{p}, \\ r_p(\tau\sigma) &= r_p(\tau) + r_p(\sigma), \\ r_p(\mathcal{I}(\tau)) &= r_p(\tau) + 2. \end{aligned}$$

cf. The number $r_p(\dot{\Xi})$ comes from the interpolation

$$\underset{\text{CM sp.}}{L^2} \subset B^{-\varepsilon}_{2,\infty} \subset B^{-(d+2)(1/2-1/p)-\varepsilon}_{p,\infty} \subset \underset{\text{Noise sp.}}{B^{-(d+2)/2-\varepsilon}_{\infty,\infty}}$$

- The "expected integrability" $i_p : \mathcal{F} \cup \dot{\mathcal{F}} \to \mathbb{R}$ defined by

$$i_p(\tau) = \begin{cases} \infty & (\tau \in \mathcal{F}), \\ p & (\tau \in \dot{\mathcal{F}}). \end{cases}$$

- A grading by (r_p, i_p) leads the concept of **regularity-integrability structures**.

Key ingredients $\boxed{\mathcal{F},\ r,\ \hat{\Pi}}$

- First we define the canonical model.
- Fix a sequence $\{\varrho_n\}_{n=1}^{\infty} \subset C_0^\infty(\mathbb{R}^{1+d})$ converging to δ_0 as $n \to \infty$.
- For any $\xi \in B_{\infty,\infty}^{-(d+2)/2-\varepsilon}$, $h \in L^2$, $n \in \mathbb{N}$, $p \in [2,\infty]$, and $x \in \mathbb{R}^{1+d}$, we define the map
$$\Pi_x = \Pi_x^{\xi,h,n,p} : \mathcal{F} \cup \dot{\mathcal{F}} \to C^\infty(\mathbb{R}^{1+d})$$
as follows.
$$\begin{aligned}
(\Pi_x \Xi)(\cdot) &= \xi_n(\cdot) := (\xi * \varrho_n)(\cdot),\\
(\Pi_x \dot{\Xi})(\cdot) &= h_n(\cdot) := (h * \varrho_n)(\cdot),\\
(\Pi_x(\tau\sigma))(\cdot) &= (\Pi_x\tau)(\cdot)(\Pi_x\sigma)(\cdot),\\
(\Pi_x \mathcal{I}(\tau))(\cdot) &= I(\Pi_x\tau)(\cdot) - \sum_{k\in\mathbb{N}^{1+d},\, |k|_s < r_p(\mathcal{I}(\tau))} \frac{(\cdot - x)^k}{k!} \partial^k I(\Pi_x\tau)(x),
\end{aligned}$$
where $|k|_s := 2k_0 + \sum_{i=1}^d k_i$ is the parabolic scale of $k = (k_i)_{i=0}^d \in \mathbb{N}^{1+d}$.
Point. The order of the Taylor expansion varies depending on p.
- (+ Operators Γ_{yx} describing the consistency of $\{\Pi_x\}$.)

Key ingredients $\boxed{\mathcal{F},\ r,\ \hat{\Pi}}$

- **Q**. Does the canonical model $\{\Pi_x^{\xi,h,n,p}\}$ converge as $n \to \infty$?
- **A**. False in general. We need a transformation

$$\Pi_x^{\xi,h,n,p} \mapsto \hat{\Pi}_x^{\xi,h,n,p}$$

  to obtain a convergence result.
- Following Bruned–Hairer–Zambotti '18, we can define the unique BPHZ model $\hat{\Pi}_x = \hat{\Pi}_x^{\xi,h,n,p}$ by canceling the expectations.

  e.g., $\hat{\Pi}_x(\mathcal{I}(\Xi))^2 = \Pi_x(\mathcal{I}(\Xi))^2 - C_n$.
- Our proof is also valid for transformations via preparation maps (introduced by Bruned '18).

Main result

Theorem (Bailleul–H '23+)

Assume $\inf r_\infty(\mathcal{F}\setminus\{\Xi\}) > -\frac{d+2}{2}$. *For any* $h\in L^2$ *and* $p\in[2,\infty]$, *the BPHZ model* $\{\hat\Pi_x^n = \hat\Pi_x^{\xi,h,n,p}\}$ *converges as* $n\to\infty$ *in the sense that, there exists a family* $\{\hat\Pi_x\tau\}_{\tau\in\mathcal{F}\cup\dot{\mathcal{F}}}$ *of random Schwartz distributions such that*

$$\sup_{\theta\in(0,1]} \theta^{-r_p(\tau)/4}\|e^{-a|x|}P_\theta(\hat\Pi_x^n\tau - \hat\Pi_x\tau)(x)\|_{L_x^{i_p(\tau)}} \xrightarrow{n\to\infty} 0$$

in probability, for any $\tau\in\mathcal{F}\cup\dot{\mathcal{F}}$ *and* $a>0$. *The convergence is uniform over* h *such that* $\|h\|_{L^2}\le 1$.
(+ Convergence results for Γ *operators.)*

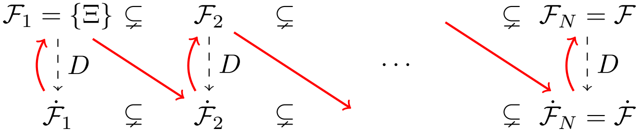
- $P_\theta := e^{\theta(\partial_t-\Delta)(\partial_t+\Delta)}$ $(\theta>0)$.
- Besov norm for $\alpha<0$ and $p\in[1,\infty]$:

$$\|\xi\|_{B^\alpha_{p,\infty}} \asymp \sup_{\theta\in(0,1]} \theta^{-\alpha/4}\|P_\theta\xi\|_{L^p}. \qquad (\xi\in\mathcal{S}'(\mathbb{R}^{1+d}))$$

Strategy

- We may assume that $\sharp\mathcal{F} < \infty$. (For each $\tau \in \mathcal{F}$, we consider only the minimum sector containing τ.)
- We define the order $\mathcal{F} = \{\tau_1, \dots, \tau_N\}$ based on the "complicatedness" and define $\mathcal{F}_i = \{\tau_1, \dots, \tau_i\}$.
- $\dot{\mathcal{F}}_i := \left\{\tau(\underset{1}{\Xi}, \dots, \underset{k}{\dot{\Xi}}, \dots, \underset{n}{\Xi})\,;\, \tau \in \mathcal{F}_i,\ 1 \le k \le n\right\}$.

Induction



- The result for $\dot{\mathcal{F}}_1$ ($\Leftrightarrow h_n \to h$ in $B^{-\varepsilon}_{2,\infty}$) is trivial.
- Induction steps: $\begin{cases} (1)\ \mathcal{F}_{i-1} \cup \dot{\mathcal{F}}_i \to \mathcal{F}_i, & \text{Poincaré ineq.} \\ (2)\ \mathcal{F}_i \cup \dot{\mathcal{F}}_i \to \dot{\mathcal{F}}_{i+1} \text{ for } p = 2, & \text{Reconstruction} \\ (3)\ \mathcal{F}_i \cup \dot{\mathcal{F}}_i \to \dot{\mathcal{F}}_{i+1} \text{ for any } p \in [2, \infty]. & \text{Taylor exp.} \end{cases}$

Sketch of the proof

(1) Probabilistic step ($\mathcal{F}_{i-1} \cup \dot{\mathcal{F}}_i \to \mathcal{F}_i$)

For any $\tau \in \mathcal{F}_i$, stochastic moments of $\hat{\Pi}_x \tau$ are controlled by those of

$$\nabla_h \hat{\Pi}_x^{\xi,h,n,\infty} \tau = \hat{\Pi}_x^{\xi,h,n,\infty} D\tau.$$

The result for $D\tau \in \dot{\mathcal{F}}_i$ is already known by induction.

(2) Analytic step ($\mathcal{F}_i \cup \dot{\mathcal{F}}_i \to \dot{\mathcal{F}}_{i+1}$ for $p = 2$)

For any $\tau \in \dot{\mathcal{F}}_{i+1} \setminus \{\dot{\Xi}\}$, the assumption implies

$$r_\infty(\tau) > -\frac{d+2}{2} \quad \text{hence} \quad r_2(\tau) > 0.$$

In this case, the result for $\hat{\Pi}_x^{\xi,h,n,2} \tau$ is automatically obtained by the extended reconstruction theorem for the RIS with $p = 2$.

cf. $B^\alpha_{\infty,\infty} \times B^\beta_{2,\infty}$ is well-defined iff. $\alpha + \beta > 0$.

(3) Algebraic step ($\mathcal{F}_i \cup \dot{\mathcal{F}}_i \to \dot{\mathcal{F}}_{i+1}$ for any $p \in [2,\infty]$)

Point. BPHZ model $\hat{\Pi}_x^p = \hat{\Pi}_x^{\xi,h,n,p}$ discontinuously varies with respect to p. e.g.,

$$\hat{\Pi}_x^p \mathcal{I}(\dot{\Xi}) = \begin{cases} I(h_n)(\cdot) - I(h_n)(x) - \sum_{i=1}^d \partial_i I(h_n)(x)((\cdot)_i - x_i) & (p < \frac{6}{1+2\varepsilon}), \\ I(h_n)(\cdot) - I(h_n)(x) & (p \geq \frac{6}{1+2\varepsilon}). \end{cases}$$

Proposition

For any $\tau \in \dot{\mathcal{F}}_{i+1}$, we have the formula

$$\hat{\Pi}_x^p \tau = \hat{\Pi}_x^2 \tau + \sum_{\tau_1 \in \dot{\mathcal{F}}_i, \tau_2 \in \mathcal{F}_i} \mathsf{h}_x^p(\tau_1) \hat{\Pi}_x \tau_2,$$

where $\mathsf{h}_x^p(\tau_1) \in L_x^{p(\tau_1)-}$ with $p(\tau_1) = \frac{d+2}{\frac{d+2}{2} - r_2(\tau_1)}$ (discontinuous points of $p \mapsto \hat{\Pi}_x^p \tau$).

The result for $p = 2$ is extended to all $p \in (2,\infty]$.

Merci!