Regularity structures for singular SPDEs

Joint work with M. Hoshino & S. Kusuoka

1 What is singular?

- **Theorem** – *The product map*

$$(a, b) \in C^\alpha \times C^\beta \mapsto ab \in C^{\alpha \wedge \beta}$$

is well-defined and continuous iff $\alpha + \beta > 0$.

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  - Heat equation with white noise potential

$$(\partial_t - \Delta)u = \zeta u,$$

    for a space white noise $\zeta = \zeta(x)$ on a 2-dimensional torus: ζ is $(-1)^-$ regular.

  - The 2-dimensional Φ^4 **equation**

$$(\partial_t - \Delta)u = \xi - u^3,$$

    for a spacetime white noise $\xi = \xi(t,x)$ on 2-dimensional space torus that is $(-2)^-$ (parabolic) regular.

  - The 1-dimensional **(gKPZ) equation**

$$(\partial_t - \partial_x^2)u = f(u)\xi + g(u)(\partial_x u)^2,$$

    for a spacetime white noise $\xi = \xi(t,x)$ on 1-dimensional space torus that is $(-3/2)^-$ (parabolic) regular.

2 The mantra of singular SPDEs

Adopt a langage in which we disentangle the task of solving the PDE from the task of making sense of some ill-defined products.

Mantra – *For a given equation, if one can make sense of the a priori ill-defined products when* u *is replaced by some reference random objects then one can make sense of these products when* u *"looks like" the reference objects.*

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► **Theorem** – *There is a large class of equations where one can build some random reference objects in a meaningful way – this is what renormalization is about* $\rightsquigarrow$ *Masato's talk.*

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We use **reference objects** to describe f locally

$$f(\cdot) \simeq \sum_{\tau \in \mathcal{B}} f_\tau(x)\,(\Pi_x \tau)(\cdot), \quad \text{near each } x,$$

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A **model** = an a priori definition of a number of functionals of the "noise".

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Is $u = R^M(\mathbf{u}^M)$ the solution of an equation?

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Renormalized equation – An automated approach in a semilinear setting: From Bruned, Chandra, Chevyrev & Hairer (Renormalising SPDEs in regularity structures) and Bailleul & Bruned (Locality for singular stochastic PDEs), for a large class of models M^ϵ built from a regularized noise ξ^ϵ, there is an explicit function $C^\epsilon(\cdot)$ such that the function $u^\epsilon = \mathcal{R}^{M^\epsilon}(\mathbf{u}^{M^\epsilon})$ is a solution to the renormalized equation

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If M^ϵ converges then $\mathbf{u}^{M^\epsilon}$ and u^ϵ converge.

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▶ **Theorem** (Local in time well-posedness) – *Take* $u_0 \in C^{0^+}(\mathbf{T})$ *and a function* $a \in C_b^3$ *with values in a compact subset of* $(0, \infty)$*. In the full subcritical regime, one can construct a regularity structure, containing infinitely many trees* $\tau^{\mathbf{p}}$ *of any fixed degree, within which, for any admissible model, the quasilinear equation* (1) *is well-posed locally in time.*

Here τ runs over the finitely many trees of the BHZ regularity structure of the semilinear (gKPZ) equation and $\mathbf{p}$ is an integer-valued decoration on the edges of τ.

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– Denote by $\ell^{\lambda,\epsilon}(\tau)$ the BPHZ counterterms of the equation

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▶ **Theorem** (Renormalized equation) – *There is an explicit polynomial function $\chi_a(\tau)$ of a and its derivatives such that the solution u^ϵ to*

$$\partial_t u^\epsilon - a(u^\epsilon)\partial_x^2 u^\epsilon = f(u^\epsilon)\xi^\epsilon + g(u^\epsilon)(\partial_x u^\epsilon)^2 + \sum_\tau \frac{\ell^{a(u^\epsilon(\cdot)),\epsilon}(\tau)}{S(\tau)}\chi_a(\tau)(u^\epsilon)\,\mathcal{F}(\tau)(u^\epsilon),$$

with initial condition $u_0 \in C^{0^+}(\mathbf{T})$, converges (in law) on a random time interval, as $\epsilon \downarrow 0$.

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Local in time well-posedness

- Modelled distributions built with *series instead of finite sums*: Needed in Picard iteration

$$u \simeq K^M\Big(F(u)\zeta + G(u)(Du)^2 + \big(A(u) - 1\big)D^2 u\Big).$$

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- Here the map $\Phi : u \mapsto K^M\big((A(u)-1)D^2u\big)$ is *not contracting* but it has a decomposition

$$\Phi = \Phi_1 + \Phi_2$$

where Φ_2 is a contraction for a small enough time horizon and an *iterate of Φ_1 is a contraction*. (Details are being written.)

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The solution of the coherence relation has here a particular structure

$$\mathcal{F}^a(\tau^{\mathbf{p}})(\mathsf{c}_0, \mathsf{c}_0') = \chi_a(\tau)(\mathsf{c}_0)\,\big(a(\mathsf{c}_0) - 1\big)^{|\mathbf{p}|}\,\mathcal{F}(\tau)(\mathsf{c}_0)$$

for some functions $\chi_a(\tau)$ and $\mathcal{F}(\tau)$ defined inductively, with $\chi_a(\tau)$ a polynomial function of a and its derivatives.

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▶ **Lemma** – *For any τ with null* **p***-decoration the function*

$$\lambda \mapsto \ell^{\lambda,\epsilon}(\tau)$$

is analytic in any given bounded interval $(a, b) \subset (0, +\infty)$ with $a > 0$ and

$$\frac{1}{n!}\,\partial_\lambda^n \ell^{\lambda,\epsilon}(\tau) = \sum_{\mathbf{p}\in\mathbb{N}^{E_\tau},\, |\mathbf{p}|=n} \ell^{\lambda,\epsilon}(\tau^{\mathbf{p}}).$$

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Then one has

$$\sum_{\tau^{\mathbf{p}}} \frac{\ell^{1,\epsilon}(\tau^{\mathbf{p}})}{S(\tau^{\mathbf{p}})} \chi_a(\tau)(\mathsf{c}_0) \left(a(\mathsf{c}_0) - 1\right)^{|\mathbf{p}|} \mathcal{F}(\tau)(\mathsf{c}_0)$$

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$$\overset{\text{Lemma}}{=} \sum_{\tau} \frac{\chi_a(\tau)(c_0) \mathcal{F}(\tau)(c_0)}{S(\tau)} \ell^{a(c_0),\epsilon}(\tau).$$

ご清聴ありがとうございました

Thank you for your attention!