Exact Solution of the Macroscopic Fluctuation Theory for SEP

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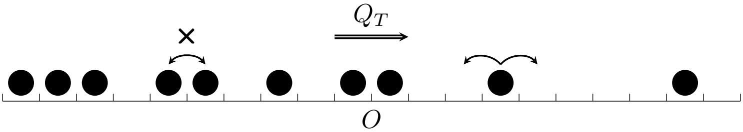
Outline

The total current in SEP

We shall consider the Symmetric Exclusion Process ($p = q = 1$) on $\mathbb{Z}$.

The initial conditions at $t = 0$ are two-sided Bernoulli: ρ_- on the left, ρ_+ on the right.

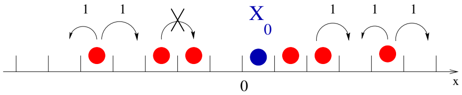
Time integrated current Q_T = total number of particles that have jumped from 0 to 1 *minus* the total number of particles that have jumped from 1 to 0 during the time interval $(0, T)$.

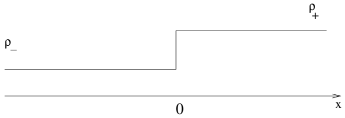


We are interested in the statistics of Q_T for a large time T (Large deviations).

A Tagged Particle in the SEP with step profile

Consider SEP with a step-like Bernoulli initial condition with density ρ_- (resp. ρ_+) to the left (resp. right). The tagged particle (or tracer) is initially located at 0. Let the system evolve: X_t denotes the position of the tracer at time t.





What is the statistics of the position of the tracer X_t and its asymptotics in the long time limit?

Because of the non-crossing condition, the statistics of the current and that of a tagged particle are 'simply' related.

Some classical results

We consider the Symmetric Exclusion Process, $(p = q = 1)$, on an *infinite one-dimensional lattice* $\mathbb{Z}$ with a finite density ρ of particles.

Suppose that we tag and observe a particle that was initially located at site 0 and monitor its position X_t with time.

On the average $\langle X_t \rangle = 0$ but how large are its fluctuations?

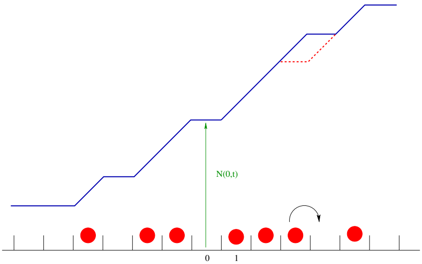
- If the particles were non-interacting (no exclusion constraint), each particle would diffuse normally $\langle X_t^2 \rangle = Dt$.
- Because of the exclusion condition, a particle displays, when $t \to \infty$, an anomalous diffusive behaviour:

$$\langle X_t^2 \rangle \simeq 2\frac{1-\rho}{\rho}\sqrt{\frac{Dt}{\pi}} \qquad \text{(Arratia, 1983)}$$

- The LDP is satisfied by X_t (Sethuraman and Varadhan, 2013).
- Finite time distribution of X_t? Rate Function of X_t?

Mapping to an interface model

We represent the exclusion process by an interface model



$N(0,t)$ represents the total current Q_t through $(0,1)$ in the duration t.

$$N(x,t) = N(0,t) + \begin{cases} \sum_{y=1}^{x} \eta_y(t)\,, & x > 0 \\ 0, & x = 0 \\ -\sum_{y=x+1}^{0} \eta_y(t)\,, & x < 0 \end{cases}$$

Note that $N(x,t)$ *is related to the KPZ height via* $h(x,t) = N(x,t) - \frac{x}{2}$

Tracer's position versus the height N(x,t)

Because the tracer is continuously moving, it is useful to relate its position X_t to the observable $N(x,t)$, which is fixed at position x.

Using particle number conservation, one can show

$$\mathrm{Prob}\left(X_t > x\right) = \mathrm{Prob}\left(N(x,t) > 0\right)$$

Or, equivalently,

$$\mathrm{Prob}\left(X_t \leq x\right) = \mathrm{Prob}\left(N(x,t) \leq 0\right)$$

This relates the statistical properties of X_t and those of the height $N(x,t)$. In particular, one can deduce the large deviation function and the cumulants of X_t from the corresponding quantities for $N(x,t)$.

It is thus enough to focus on $N(x,t)$.

Microscopic Approach

T. Imamura, K.M, T. Sasamoto, Phys. Rev. Lett. **118**, *160601 (2017)*

T. Imamura, K.M, T. Sasamoto, CMP **384***:1409, (2021).*

Exact expression of the generating function

We shall derive a formula for the characteristic function of the height $N(x,t)$, exact at any finite-time, in terms of a Fredholm determinant:

$$\boxed{\langle e^{\lambda N(x,t)} \rangle = \det(1 + \omega K_{t,x}) W_0(\lambda)}$$

where

$$\omega(\lambda) = \rho_+(e^{\lambda} - 1) + \rho_-(e^{-\lambda} - 1) + \rho_+\rho_-(e^{\lambda} - 1)(e^{-\lambda} - 1)$$

$$K_{t,x}(\xi_1, \xi_2) = \frac{\xi_1^{|x|} e^{\epsilon(\xi_1)t}}{\xi_1\xi_2 + 1 - 2\xi_2} \quad \text{with} \quad \epsilon(\xi) = \xi + \xi^{-1} - 2$$

$$W_0(\lambda) = \left(1 + \rho_\pm(e^{\pm\lambda} - 1)\right)^{|x|} \text{ with } \pm = \mathrm{sgn}(x)$$

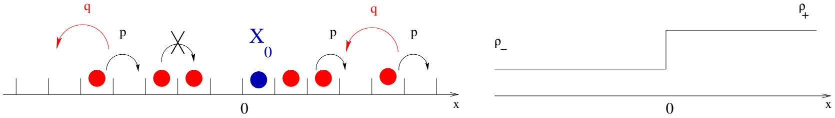
From this result, information about the tracer will be deduced.

We now outline the strategy to solve the problem.

A more general set-up: tracer and heights in ASEP

The moving tracer position X_t has been traded for the localized height variables $N(x,t)$, which form an infinite set of highly correlated observables. *We need to restore finiteness.*

It will prove useful to study the tracer problem in the more general setting of the *asymmetric exclusion process* with jump rates p and q with $p \leq q$:



This provides us with the extra-parameter τ:

$$\tau = \frac{p}{q} \leq 1$$

a. DUALITY for ASEP

For the Asymmetric Exclusion Process, with asymmetry parameter $\tau = p/q < 1$, the observable $N(x, t)$ satisfies a remarkable self-duality property.

For $x_1 < x_2 < \ldots < x_n$, τ-correlations of the type,

$$\phi(x_1, \ldots, x_n; t) = \langle \tau^{N(x_1,t)} \ldots \tau^{N(x_n,t)} \rangle$$

follow the same dynamical equations as the ASEP with a finite number n of particles located at $x_1, \ldots, x_n$.

Duality results from a quantum group invariance of the process (G. Schütz, T Imamura and T. Sasamoto, C. Giardina et al.)

However, it can be understood here in an elementary manner using stochastic (Poisson) calculus.

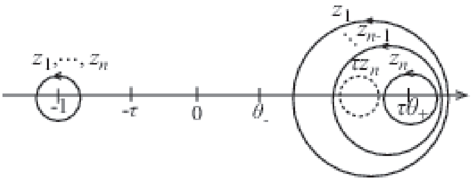
Thanks to duality, we have to deal with (a sequence of) problems with finite number of particles.

b. Integral formulas for the deformed correlations

Inspired by the fact that ASEP is integrable by "Bethe Ansatz", the τ-correlation functions can be expressed as multiple contour integrals in the complex plane:

$$\langle \tau^{\sum_i N(x_i,t)} \rangle = \tau^{\sum_i i - \frac{x_i}{2}} \prod_{i=1}^{n} \left(1 - \frac{r_-}{\tau^i r_+}\right) \int \cdots \int \prod_{i<j} \frac{z_i - z_j}{z_i - \tau z_j} \prod_{i=1}^{n} \frac{e^{\Lambda_{x_i,t}(z_i)}}{(1 - \frac{z_i}{\tau\theta_+})(z_i - \theta_-)} \mathrm{d}z_i$$

with $r_\pm = \rho_\pm(1-\rho_\mp)$, $\theta_\pm = \rho_\pm/(1-\rho_\pm)$ and $e^{\Lambda_{x,t}(z)} = \left(\frac{1+z}{1+z/\tau}\right)^x e^{-\frac{q(1-\tau)^2 z}{(1+z)(\tau+z)} t}$



c. Combinatorics of the nested complex integrals

• The contour integral formulas were initially inspired by the Bethe Ansatz (Schütz, Tracy-Widom). Yet, they are *NOT* just an Ansatz but *exact* representations of the correlators. The z_i's are dummy integration variables, not Bethe roots (there are no Bethe equations here).

• The nesting conditions on the contours are crucial : they are the one that encode the combinatorial sum over permutations of the Bethe Ansatz.

• The key step is to disentangle recursively the contours by evaluating the residues at non-essential singularities.

• In the symmetric limit, $\tau \to 1$, reordering the residue expansions with the help of some combinatorial formulas (à la Tracy and Widom) allows one to rewrite the characteristic function of $N(x, t)$ as a Fredholm determinant.

d. Long time asymptotics

The characteristic function of the height $N(x,t)$ is given, at any finite-time, by the Fredholm determinant:

$$\langle e^{\lambda N(x,t)} \rangle = \det(1 + \omega K_{t,x}) W_0(\lambda)$$

In the long time limit, the asymptotics analysis of this determinant, shows that this characteristic function behaves as

$$\langle e^{\lambda N(x,t)} \rangle \sim e^{-\sqrt{t}\mu(\xi,\lambda)}$$

where $\xi = -\frac{x}{\sqrt{4t}}$. The function $\mu(\xi,\lambda)$ is the cumulant generating function of $N(x,t)$:

$$\mu(\xi,\lambda) = \sum_{n=1}^{\infty} \frac{(-\omega)^n}{n^{3/2}} A(\sqrt{n}\,\xi) + \xi \log \frac{1 + \rho_+(e^{\lambda} - 1)}{1 + \rho_-(e^{-\lambda} - 1)}$$

with $A(u) = \xi + \int_{\xi}^{\infty} \mathrm{erfc}\,(u)\mathrm{d}u$ and

$$\omega(\lambda) = \rho_+(e^{\lambda} - 1) + \rho_-(e^{-\lambda} - 1) + \rho_+\rho_-(e^{\lambda} - 1)(e^{-\lambda} - 1)$$

From micro to Macro

• Exact solutions at the microscopic level require high-brow technology. However, at the level of large deviations, the cumulant generating function, $\mu(0,\lambda)$, is given by a rather simple expression.

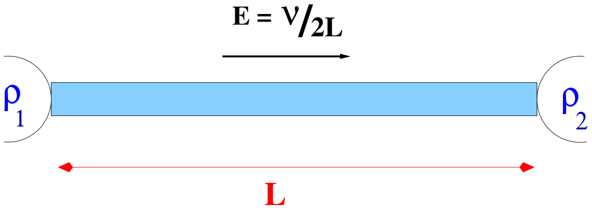
• We obtain the distribution of the height, current, tagged particle position in the long time limit. However, we have gained no knowledge on how large deviations (i.e. rare fluctuations) are dynamically generated.

• Time-dependent aspects seem to be out of reach of Bethe Ansatz (i.e. Integrable Probability) methods.

• A more physical picture, that would bypass combinatorics and asymptotics, and based on a more intuitive and direct approach, would be welcome.

Macroscopic Fluctuation Theory

Nonequilibrium systems: a Hydrodynamic Approach



Starting from the microscopic level, define local density $\rho(x,t)$ and current $j(x,t)$ with macroscopic space-time variables $x = i/L, t = s/L^2$ (diffusive scaling).

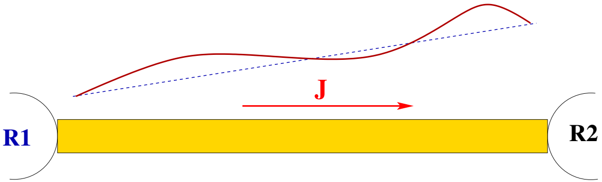
The typical evolution of the system is given by a hydrodynamic Burgers-type equation (De Masi, Ferrari, Kipnis Lebowitz, Olla, Presutti, Spohn, Varadhan...):

$$\partial_t \rho = \nabla\left(D(\rho)\nabla\rho\right) - \nu\nabla\sigma(\rho)$$

$D = 1$ and $\sigma = 2\rho(1-\rho)$ being transport coefficients.

How can Fluctuations be taken into account?

The General Large Deviations Problem



The Probability to observe an atypical local current $j(x,t)$ and density profile $\rho(x,t)$ during $0 \leq s \leq L^2 T$ (i.e. diffusive scaling, L is the size of the system) assumes a Large Deviation behaviour

$$\Pr\{j(x,t), \rho(x,t)\} \sim \mathrm{e}^{-L\,\mathcal{I}(j,\rho)}$$

Knowing $\mathcal{I}(j,\rho)$, one could deduce the large deviations of the current and of the density profile. For instance, $\Phi(j) = \min_\rho\{\mathcal{I}(j,\rho)\}$.

Is there a Principle which gives this large deviation functional for driven diffusive systems out of equilibrium?

The MFT Action

For a weakly-driven diffusive system, the large deviation form of the probability to observe a current $j(x, t)$ and a density profile $\rho(x, t)$ during a time T, is given by

$$\Pr\{j(x,t), \rho(x,t)\} \sim e^{-S_{MFT}(j,\rho)},$$

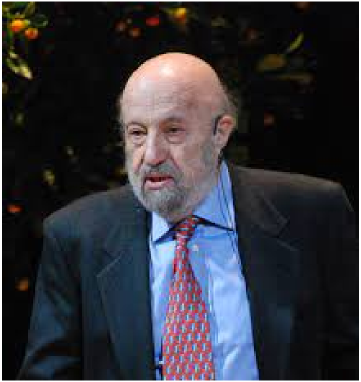
with

$$S_{MFT}(j, \rho) = \int_0^T dt \int_{-\infty}^{+\infty} \frac{(j + D(\rho)\nabla\rho)^2 \, dx}{2\sigma(\rho)}$$

under the constraint $\partial_t \rho = -\nabla . j$

(L. Bertini, D. Gabrielli, A. De Sole, G. Jona-Lasinio and C. Landim).

For a given problem, only the dominant paths will dominate the probability measure. They can be obtained by optimizing this action under constraints.



The macroscopic fluctuation theory generalizes the linear response fluctuation theory of Onsager and Machlup (1953)

Unfortunately, solving these equations was not a straightforward task. Exact results were first obtained at the microscopic level and, then, coarse-grained.

MFT from Fluctuating Hydrodynamics

Heuristically, this action S_{MFT} results from the Langevin PDE

$$\frac{\partial \rho}{\partial t} = -\frac{\partial j}{\partial x} \quad \text{with} \quad j = -D(\rho)\frac{\partial \rho}{\partial x} + \sqrt{\sigma(\rho)}\xi(x,t)$$

The transport coefficients, $D(\rho)$ (bulk diffusivity) and $\sigma(\rho)$ (conductivity) must be determined at the level of microscopic physics.

For SEP, the LDP was established by Kipnis, Olla and Varadhan (1989). The transport coefficients of SEP are:

$$D(\rho) = 1 \quad \text{and} \quad \sigma(\rho) = 2\rho(1-\rho)$$

Using this framework, one can in principle calculate the large deviations directly at the macroscopic level by solving the full, time-dependent, MFT equations.

The Equations of Macroscopic Fluctuation Theory

The optimization of the action is a variational problem that leads to Euler-Lagrange equations. By a Legendre transform, a Hamiltonian structure is obtained by using a pair variables (ρ, H), conjugate to (ρ, j). Here, $\rho(x,t)$ is the density-field and $H(x,t)$ is a conjugate (momentum) field. The dynamics is given by

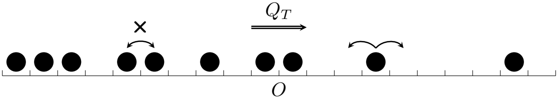
$$\partial_t \rho = \partial_x [D(\rho)\partial_x \rho] - \partial_x[\sigma(\rho)\partial_x H]$$

$$\partial_t H = -D(\rho)\partial_{xx} H - \frac{1}{2}\sigma'(\rho)(\partial_x H)^2$$

with Hamiltonian $\mathcal{H} = \sigma(\rho)(\partial_x H)^2/2 - D(\rho)\partial_x \rho \partial_x H.$

The information of the microscopic dynamics relevant at macroscopic scale is embodied in the transport coefficients D and σ other details are 'blurred' in this continuous hydrodynamic limit.

Height fluctuations at macroscopic scale



In the continuous limit:
$N(X,T) = \int_0^\infty [\rho(x,T) - \rho(x,0)]dx - \int_0^X [\rho(x,T)]dx$. And for large T, we have (LDP):

$$\left\langle e^{\lambda N(X,T)} \right\rangle \simeq e^{\sqrt{T}\mu(\xi,\lambda)}$$

What are the **profile** ρ and the conditioning **momentum field** H required to generate a given large fluctuation of the local height $N(X,T)$?

We want to extract this function μ macroscopically.

We must average $e^{\lambda N(X,T)}$ under the MFT measure.

MFT equations for current fluctuations

Thus, we must solve the PDE's (for SEP, $D = 1$, $\sigma = 2\rho(1-\rho)$) :

$$\begin{aligned} \partial_t \rho &= \partial_x \left[\partial_x \rho - 2\rho(1-\rho)\partial_x H\right] \\ \partial_t H &= -\partial_{xx} H - (1-2\rho)(\partial_x H)^2 \end{aligned}$$

With non-local boundary conditions:

$$H(x,T) = \lambda\theta(x - X)$$

$$H(x,0) = \lambda\theta(x) + \log \frac{\rho(x,0)(1-\bar{\rho}(x))}{\bar{\rho}(x)(1-\rho(x,0))}$$

where $\bar{\rho}(x) = \rho_-\theta(-x) + \rho_+\theta(x)$ is the mean-initial step profile. The condition at $t = 0$ expresses the fact that the initial profile fluctuates with two-sided Bernoulli measure.

Knowing the optimal profile ρ^* solving this system, the CGF will be obtained from

$$\sqrt{T}\frac{d\mu}{d\lambda} = N(X,T) = \int_X^\infty \rho^*(x,T) - \int_0^\infty \rho^*(x,0)dx$$

• The MFT equations describe the non-equilibrium behaviour of many diffusive interacting particle systems (dynamical transitions, shocks...).

• Mathematical/Numerical difficulties : well-posedness; non-local boundary conditions.

• Time-dependent equations were solved only in noninteracting case and for years no analytic time-dependent solutions of these coupled PDE's were known.

• Recently, several exact results for closely related problems of optimal fluctuation paths have appeared: Krajenbrink and Le Doussal (weak-noise KPZ); Bettelheim, Smith and Meerson (KMP); Grabsch, Poncet, Rizkallah, Illien and Bénichou (Single Files) and Moriya-M-Sasamoto (SEP).

SOLVING THE MFT BY

INVERSE SCATTERING

H. Moriya, KM and T. Sasamoto, Exact solution of the macroscopic fluctuation theory for SEP, Phys. Rev. Lett. **129***, 040601 (2022)*

H. Moriya, KM, T. Sasamoto: Exact solutions to macroscopic fluctuation theory through classical integrable systems, submitted, (2024).

O. Bénichou, A. Grabsch, KM, H. Moriya and T. Sasamoto, in preparation (2024).

Reflecting Brownian Motions: Cole-Hopf mapping

In the limiting case of very low density, the simple exclusion process reduces to a system of Brownian Motions with specular reflection (RBM) and the MFT equations read:

$$\partial_t \rho = \partial_x [\partial_x \rho - 2\rho\partial_x H]$$
$$\partial_t H = -\partial_{xx} H - (\partial_x H)^2$$

These equations are solved by mapping them to two decoupled heat equations thanks to the Cole-Hopf transformation:

$$u(x,t) = \rho \mathrm{e}^{-H}$$
$$v(x,t) = \mathrm{e}^{H}$$

In these new variables, the above equations become

$$\partial_t u = \partial_{xx} u$$
$$\partial_t v = -\partial_{xx} v$$

The particles are in fact non-interacting: this is a "free" model.

A generalization of the Cole-Hopf mapping for SEP

The following novel non-local transformation

$$u(x,t) = \left(\frac{\partial \rho}{\partial x} - \rho(1-\rho)\frac{\partial H}{\partial x} \right) \exp\left[-\int_{-\infty}^{x} dy (1-2\rho)\partial_y H \right],$$

$$v(x,t) = -\frac{\partial H}{\partial x} \exp\left[\int_{-\infty}^{x} dy (1-2\rho)\partial_y H \right]$$

maps the MFT equations to the *Ablowitz-Kaup-Newell-Segur (AKNS) system:*

$$\partial_t u(x,t) = \partial_{xx} u(x,t) - 2u(x,t)^2 v(x,t)$$

$$\partial_t v(x,t) = -\partial_{xx} v(x,t) + 2u(x,t)v(x,t)^2$$

The AKNS equations correspond to the NLS equation in imaginary time.

Transformation of the boundary conditions

The boundary conditions transform also well under the generalized Cole-Hopf (but still remain non-local in time):

$$u(x,0) = \omega\delta(x)$$
$$v(x,T) = \delta(x - X)$$

with the scaling variable:

$$\omega = (e^{\lambda} - 1)\rho_-(1-\rho_+) + (e^{-\lambda} - 1)\rho_+(1-\rho_-)$$

The AKNS equations have an infinite number of conserved quantities in involution. They are classically integrable in the sense of Liouville.

The AKNS equations can be solved by using the Inverse Scattering Theory.

Classical Integrability I: Lax Pair

Consider the following auxiliary linear problem:

$$\begin{cases} \frac{\partial}{\partial x}\Psi(x,t) & = U(x,t;k)\Psi(x,t) \\ \frac{\partial}{\partial t}\Psi(x,t) & = V(x,t;k)\Psi(x,t) \end{cases}$$

with $\Psi^T(x,t) = (\psi_1(x,t), \psi_2(x,t))$; $U(x,t)$ and $V(x,t)$ are the matrices:

$$U = \begin{pmatrix} -ik & v(x,t) \\ u(x,t) & ik \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 2k^2 + uv & 2ik\, v - \partial_x v \\ 2ik\, u + \partial_x u & -2k^2 - uv \end{pmatrix}$$

The compatibility of these equations, $\partial_t\partial_x\Psi = \partial_x\partial_t\Psi$, is ensured by the zero curvature condition:

$$\boxed{\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U,V] = 0}$$

This condition is ensured if the functions u and v satisfy the AKNS system.

Classical Integrability II: Scattering

The first equation of the pair reads, in components:

$$\begin{cases} \frac{\partial}{\partial x}\psi_1(x,t) & = -ik\psi_1 + v(x,t)\psi_2 \\ \frac{\partial}{\partial x}\psi_2(x,t) & = u(x,t)\psi_1 + ik\psi_2 \end{cases}$$

This is a linear scattering problem on $\mathbb{R}$, for any given value of the time t, in which $u(x,t)$ and $v(x,t)$ that solve AKNS appear as potentials.

Because these potentials vanish at infinity, asymptotic states are well-defined: ψ_1 and ψ_2 behave as plane waves at $x = \pm\infty$.

Therefore, incoming/outgoing plane waves from $x \to -\infty$

$$\phi(x;k) \sim \begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix} \qquad \text{and} \qquad \bar{\phi}(x;k) \sim -\begin{pmatrix} 0 \\ e^{ikx} \end{pmatrix}$$

will scatter at $x \to +\infty$ as follows

$$\phi(x;k) \sim \begin{pmatrix} a(k,t)e^{-ikx} \\ b(k,t)e^{ikx} \end{pmatrix} \qquad \text{and} \qquad \bar{\phi}(x;k) \sim \begin{pmatrix} \bar{b}(k,t)e^{-ikx} \\ -\bar{a}(k,t)e^{ikx} \end{pmatrix}$$

The functions $a, \bar{a}, b, \bar{b}$ are the scattering amplitudes associated to this (Dirac) scattering process.

Classical Integrability III: Diagonalization

Using the second equation of the Lax pair, which describes the time dynamics of Ψ and the asymptotic plane-wave expressions, the time evolution of the scattering amplitudes is obtained explicitly:

$$a(k,t) = a(k,0), \quad b(k,t) = b(k,0)e^{-4k^2 t}$$
$$\bar{a}(k,t) = \bar{a}(k,0), \quad \bar{b}(k,t) = \bar{b}(k,0)e^{4k^2 t}$$

Key feature: **The dynamics drastically simplifies in terms of the scattering amplitudes.** (The scattering amplitudes are the action-angle variables of the dynamics.)

If we know the scattering amplitudes at initial time, they are determined at all times. Then, the potentials $u(x,t)$ and $v(x,t)$ can be reconstructed at any time by the inverse-scattering procedure (Gelfand-Levitan-Marchenko).

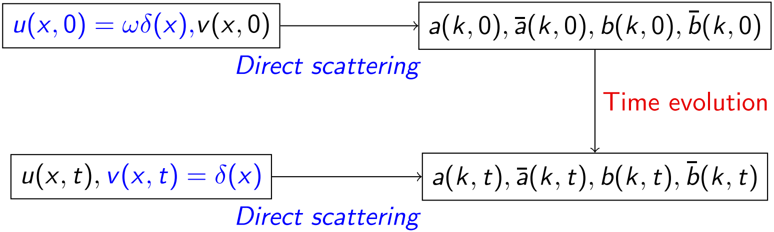
Solving MFT by Inverse Scattering

We wish to apply ISM to Simple Exclusion (MFT). However, we have non-local boundary conditions (*not* Cauchy initial conditions):

$u(x,0) = \omega\delta(x)$	but	$v(x,0)$ is unknown
$v(x,T) = \delta(x)$	but	$u(x,T)$ is unknown

1. Calculate implicitly the scattering amplitudes both at $t = 0$ and $t = T$ in terms of the unknown potentials $v(x,0)$ and $u(x,T)$, knowing that the other potential is a Dirac function.

2. Match the scattering data at initial and final times using the 'trivial' action-angle dynamics (integrability).



The scattering amplitudes

The Dirac scattering problem at initial time with $u(x,0) = \omega\delta(x)$ and $v(x,0)$ unspecified is elementary to solve. One finds (in terms of the half Fourier-transforms of v):

$$a(k,0) = 1 + \omega\hat{v}_+(k), \quad b(k,0) = \omega$$
$$\bar{a}(k,0) = 1 + \omega\hat{v}_-(k), \quad \bar{b}(k,0) = -\left[\hat{v}(k) + \omega\hat{v}_+(k)\hat{v}_-(k)\right]$$

Similarly, the Dirac scattering problem at final time with $u(x,T)$ unknown and with $v(x,T) = \delta(x)$ gives

$$a(k,T) = 1 + \hat{u}_+(k), \quad b(k,T) = (\hat{u}(k) + \hat{u}_+(k)\hat{u}_-(k))e^{-2ikX}$$
$$\bar{a}(k,T) = 1 + \hat{u}_-(k), \quad \bar{b}(k,T) = -e^{-2ikX}$$

From the simple evolution of the scattering data, we deduce that $\hat{u}_\pm = \omega\hat{v}_\pm$ and

$$\hat{u}(k) + \hat{u}_+(k)\hat{u}_-(k) = \omega e^{-4k^2T+2ikX}$$

Equation for the density profile

Hence, we have shown that the half Fourier transform of the final profile

$$\hat{u}_{\pm}(k) = \int_{\mathbb{R}_{\mp}} u(x+X,T)e^{-2ikx}dx$$

satisfies a scalar Riemann–Hilbert factorization problem:

$$\boxed{(\hat{u}_{+}(k)+1)(\hat{u}_{-}(k)+1) = 1 + \omega e^{-4k^2T+2ikX}}$$

where $1+\hat{u}_{\pm}$ is analytic on the upper (respectively lower) complex plane, with a given product along $\mathbb{R}$.

This Riemann–Hilbert problem is solved by using the Cauchy Formula (after taking logarithms) and we obtain:

$$\hat{u}_{\pm}(k) + 1 = \exp\left[-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-\omega e^{-4k^2T+2ikX})^n}{n}\mathrm{erfc}(\mp i\sqrt{4nT}(k-\frac{iX}{4T})\right]$$

Cumulant Generating Function of the current

Calculating the height $N(X,T)$ from the optimal profiles at $t = 0$ and $t = T$ yields its Cumulant Generating Function (CGF).
In the long time limit, $\langle e^{\lambda N(X,T)} \rangle \simeq e^{\sqrt{T}\mu(\xi,\lambda)}$, with

$$\mu(\xi,\lambda) = \sum_{n=1}^{\infty} \frac{(-\omega)^n}{n^{3/2}} A(\sqrt{n}\,\xi) + \xi \log \frac{1+\rho_+(e^{\lambda}-1)}{1+\rho_-(e^{-\lambda}-1)}$$

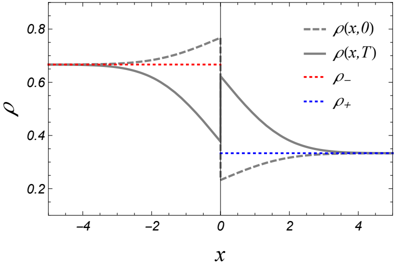
with $\mathrm{A(u)} = \xi + \int_{\xi}^{\infty} \mathrm{erfc}\,(\mathrm{u})\mathrm{du}$ and $\omega = (\mathrm{e}^{\lambda}-1)\rho_-(1-\rho_+) + (\mathrm{e}^{-\lambda}-1)\rho_+(1-\rho_-)$

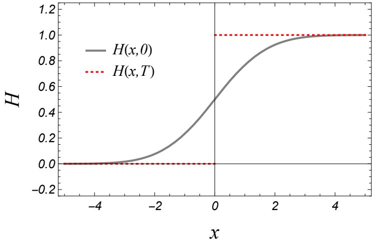
This is the same formula that was obtained by previously using Integrable Probabilities, *it is now derived directly at the macroscopic level*.

Optimal profiles describing the dynamical evolution that generates a given fluctuation (rare event) that were out of reach by the microscopic techniques are now found by solving the MFT at hydrodynamic scale.

Optimal Profiles and Control Fields

Gathering all the pieces and going back to the variables of the original MFT equations, explicit formulas for the optimal fields (ρ^*, H^*) are obtained.





Optimal profiles for the total current at $X = 0$ of ρ (left) and H (right) at $t = 0$ and at $t = T$, with $\rho_+ = 1/3$, $\rho_- = 2/3$, $\lambda = 1$ and $T = 1$.

Conclusions

A major challenge in non-equilibrium physics is to determine the large deviations, considered to be the relevant generalizations of the thermodynamic potentials (Free Energy) far from equilibrium.

Interacting particle processes (such as the exclusion process) are ideal toy-models to investigate these questions with a large variety of methods:

• Microscopic scale: Combinatorics, Matrix representation, Bethe Ansatz, Integrable Probabilities...

• Coarse-grained level: hydrodynamic limits, fluctuating hydrodynamics (SPDE), Macroscopic Fluctuation Theory for optimal paths (PDE)...

Finding explicit time-dependent solutions of the MFT has been a challenge since this theory was proposed (2001). Very recently, several exact results appeared: Krajenbrink-Le Doussal (weak-noise KPZ); Bettelheim-Smith-Meerson (KMP); Grabsch et al. and MMS (SEP).

These exact results are based on the Inverse Scattering Method, originally developed to study non-linear dispersive hydrodynamics.

Applications of the ISM to non-equilibrium statistical mechanics seems very promising. The relation between microscopic and macroscopic integrability is very intriguing.

Finite time distribution of the tracer

The distribution function of the tracer X_t is given, at all times, in terms of a Fredholm determinant:

$$\mathbb{P}[X_t \leq x] = \int_{C_0} \frac{dz}{1-z} \det(1 + \omega K_{x,t})_{L_2(C_0)} W_0(z)$$

where

$$\omega(z) = \rho_+(z^{-1} - 1) + \rho_-(z - 1) + \rho_+\rho_-(z^{-1} - 1)(z - 1)$$

$$K_{t,x}(\xi_1, \xi_2) = \frac{\xi_1^{|x|} e^{\epsilon(\xi_1)t}}{\xi_1\xi_2 + 1 - 2\xi_2} \quad \text{with} \quad \epsilon(\xi) = \xi + \xi^{-1} - 2$$

$$W_0(\lambda) = \left(1 + \rho_\epsilon(z^{-\epsilon} - 1)\right)^{|x|} \text{ with } \epsilon = \text{sgn}(x)$$

The ω variable expresses fundamental symmetries of the model : parity and time-reversal.

The Kernel $K_{t,x}$ originates from the Bethe Ansatz.

The function W_0 carries 'Poisson-like' boundary conditions.

C_0 is a small enough complex contour around 0 (poles from the denominator of the kernel are excluded).