Outliers of perturbations of banded Toeplitz matrices

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Notations

$B \in \mathcal{M}_n(\mathbb{C})$
Eigenvalues of B: $\lambda_1(B), \lambda_2(B), \cdots, \lambda_n(B)$,

The empirical distribution of these eigenvalues:

$$\mu_B := \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(B)}$$

The singular values of B: $0 \le s_n(B) \le \cdots \le s_1(B)$.

Model investigated by Bordenave-C. 16

$$M_n = \sigma \frac{X_n}{\sqrt{n}} + A_n \in \mathcal{M}_n(\mathbb{C})$$

- $\sigma > 0$, $X_n = (X_{ij})_{i,j\geq 1}$ i.i.d complex variables such that $\mathbb{E}(X_{ij}) = 0, \quad \mathbb{E}(|X_{ij}|^2) = 1, \quad \mathbb{E}(|X_{ij}|^4) < \infty.$
(Note that, then, $\left\| \frac{X_n}{\sqrt{n}} \right\| \to_{n\to+\infty} 2$ (Bai-Yin-Krishnaiah 88))
- A_n is deterministic. $\sup_n \|A_n\| < +\infty$.

$$\forall z \in \mathbb{C}, \quad \mu_{(A_n - zI_n)(A_n - zI_n)^*} \xrightarrow{w} \nu_z \quad \text{as } n \to +\infty,$$

for some probability measure ν_z on $\mathbb{R}^+$.

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Dozier-Silverstein (2007)+Tao-Vu (2010) $\Longrightarrow$ a.s $\mu_{M_n} \xrightarrow[n\to+\infty]{w} \beta$

where β is a probability measure on $\mathbb{C}$ which is characterized by $(\nu_z)_{z\in\mathbb{C}}$.

Some examples of perturbations

$$C_n = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots \\ 1 & 0 & \cdots & \cdots \end{pmatrix} \quad \mathcal{N}_n = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & \cdots & \cdots \end{pmatrix}.$$

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$\Longrightarrow$ the limiting empirical spectral distribution β of $M_n = \sigma \frac{X_n}{\sqrt{n}} + A_n$ (uniquely determined by the ν_z) is the same.

$$M_n = \sigma \frac{X_n}{\sqrt{n}} + A_n \in \mathcal{M}_n(\mathbb{C}) \text{ where } A_n \in \{C_n, \mathcal{N}_n\}.$$

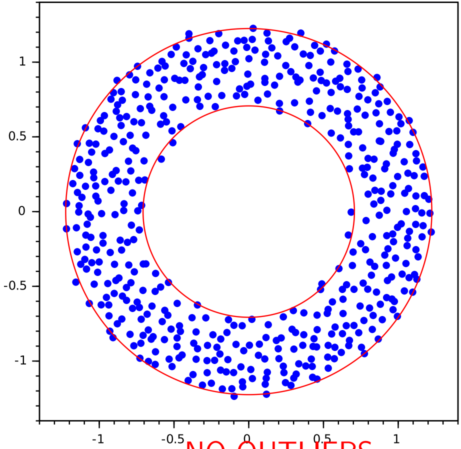
Sniady 2002$\Longrightarrow$ the limiting empirical spectral distribution β of M_n, has a natural interpretation in free probability theory.

$$\Longrightarrow \operatorname{supp}(\beta) = \left\{ z \in \mathbb{C} : \sqrt{(1-\sigma^2)_+} \leq |z| \leq \sqrt{1+\sigma^2} \right\}.$$

(Bordenave-Caputo-Chafai (2014), Zhong (2021), Bercovici-Zhong (2022))

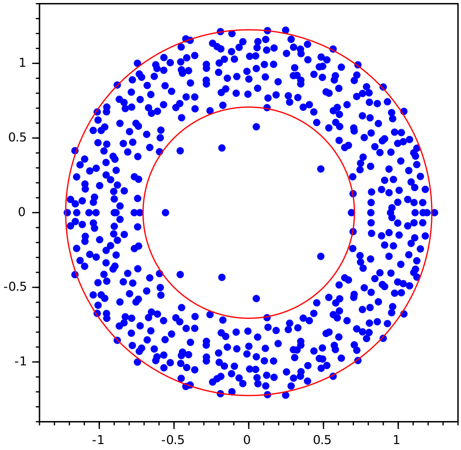
The eigenvalues of $M_n = \sigma \frac{X_n}{\sqrt{n}} + A_n$

$A_n = C_n, \quad n = 500, \quad \sigma^2 = 1/2$, X_{ij} real gaussian



NO OUTLIERS

Eigenvalues of $M_n = \sigma \frac{X_n}{\sqrt{n}} + \mathcal{N}_n$, $n = 500$, $\sigma^2 = 1/2$



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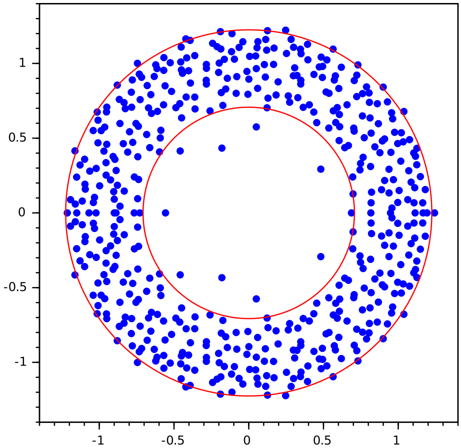

NO OUTLIERS in ${}^cB(0, \sqrt{1+\sigma^2})$

But OUTLIERS in $\mathring{B}(0, \sqrt{1-\sigma^2})$

Notion of well-conditioned perturbation

Definition

Let $\Gamma \subset \mathbb{C} \setminus \mathrm{supp}(\beta)$ be a compact set. A_n is well-conditioned in Γ if for any $z \in \Gamma$, there exists $\eta > 0$ such that for all n large enough, $s_n(A_n - zI_n) > \eta$.

Theorem (Bordenave-C. (2016))

Assume that A_n is well-conditioned in Γ, Then, a.s. for all n large enough, $M_n = \sigma \frac{X_n}{\sqrt{n}} + A_n$ has no eigenvalue in Γ.

When $\operatorname{supp}(\beta) = \left\{ \mathrm{z} \in \mathbb{C} : \sqrt{(1-\sigma^2)_+} \leq |\mathrm{z}| \leq \sqrt{1+\sigma^2} \right\}.$

- The circulant matrix is well-conditioned in $\mathbb{C} \setminus \operatorname{supp}(\beta)$:

$C_n = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots \\ 1 & 0 & \cdots & \cdots \end{pmatrix}$ since the singular values of $C_n - zI_n$ are equal to

$|e^{\frac{2i\pi l}{n}} - z| \geq |1 - |z|| > 0$, for $l = 1, \ldots, n$, for any z outside $\operatorname{supp}(\beta)$.

$\Longrightarrow$ No outliers of $\sigma \frac{X_n}{\sqrt{n}} + C_n$

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• The nilpotent matrix $\mathcal{N}_n$ is well conditioned out of the maximum circle since for $|z| > 1$, $s_n(\mathcal{N}_n - zI_n) \geq |z| - 1 \Longrightarrow$ No outliers of $\sigma \frac{X_n}{\sqrt{n}} + \mathcal{N}_n$ out of the maximum circle.

BUT the nilpotent matrix $\mathcal{N}_n$ is badly conditioned in the inner disk since for $|z| < 1$, $s_n(\mathcal{N}_n - zI_n) = o(1)$.

Proposition (Bordenave, C. 2016)

$0 < \sigma < 1$. *Assume that the* X_{ij}*'s are Gaussian. The point process of eigenvalues of* $M_N = \sigma \frac{X_n}{\sqrt{n}} + \mathcal{N}_n$ *in* $\mathring{B}(0, \sqrt{1-\sigma^2})$ *converges weakly to the point process of the zeroes of the Gaussian analytic function* $g(z)$ *on* $\mathring{B}(0, \sqrt{1-\sigma^2})$ *with kernel given by, for* $z, w \in \Gamma$*,*

$$K(z,w) = \frac{\varphi(z,w)^2}{1-\sigma^2\varphi(z,w)} \text{ where } \varphi(z,w) = \frac{1}{1-z\bar{w}}.$$

This means that for any compactly supported continuous function ψ *in* $\mathring{B}(0, \sqrt{1-\sigma^2})$*,*

$$\sum_{i=1}^{n} \psi(\lambda_i(M_n)) \xrightarrow{w}_{n\to+\infty} \sum_{x \text{zeroes of } g} \psi(x).$$

Gaussian analytic functions (Hough, Krishnapur, Peres, Virag (2009)).

A Gaussian analytic function on $\Gamma \subset \mathbb{C}$ is a random analytic function g such that for every $z_1, \ldots, z_p$ in Γ, $(g(z_1), \ldots, g(z_p))$ is a centered complex Gaussian vector with $\mathbb{E}(g(z_i)g(z_j)) = 0$. The distribution of g is characterized by its kernel

$$K(z; w) = \mathbb{E}g(z)\overline{g(w)}.$$

The nilpotent matrix $\mathcal{N}_n$ is a particular case of banded Toeplitz matrices. $T_n(\mathbf{a})$: a $n \times n$ banded Toeplitz matrix, with symbol $\mathbf{a}\colon \mathbb{S}^1 \to \mathbb{C}$ given by the Laurent polynomial

$$\mathbf{a}(\lambda) = \sum_{k=-r}^{s} a_k \lambda^k, \quad \lambda \in \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\},$$

$r \geq 0$, $s > 0$ and the a_k's are complex numbers with $a_s \neq 0$

$$T_n(\mathbf{a}) = \begin{pmatrix} a_0 & a_1 & \cdots & a_s & 0 & \cdots & 0 \\ a_{-1} & a_0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ a_{-r} & \ddots & \ddots & \ddots & \ddots & \ddots & a_s \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & a_0 & a_1 \\ 0 & \cdots & 0 & a_{-r} & \cdots & a_{-1} & a_0 \end{pmatrix}.$$

$X_N = (X_{ij})_{i,j\geq 1}$ i.i.d complex variables such that

$$\mathbb{E}(X_{ij}) = 0, \quad \mathbb{E}(|X_{ij}|^2) = 1, \quad \mathbb{E}(|X_{ij}|^4) < \infty.$$

Spectrum of $M_n = T_n(\mathbf{a}) + \sigma \frac{X_n}{\sqrt{n}}$ when n is large?

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Spectrum of $M_n = T_n(\mathbf{a}) + \sigma \frac{X_n}{\sqrt{n}}$ when n is large?

By the Avram-Parter Theorem,

$\forall z \in \mathbb{C}, \quad \mu_{(T_n(\mathbf{a})-zI_n)(T_n(\mathbf{a})-zI_n)^*} \xrightarrow[n\to+\infty]{w} \nu_z$ (the push-foward by $w \mapsto |\,\mathbf{a}(w) - z|$ of the uniform measure on $\mathbb{S}^1$)

$\Longrightarrow$ (By Dozier-Silverstein+ Tao-Vu+Sniady)

μ_{M_N} converges weakly almost surely to β,

$$\mathrm{supp}(\beta) = \mathbf{a}(\mathbb{S}^1) \cup \left\{ z \in \mathbb{C} \setminus \mathbf{a}(\mathbb{S}^1) : \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{|\,\mathbf{a}(e^{i\theta}) - z|^2} \geq \sigma^{-2} \right\}.$$

(Bordenave-Caputo-Chafai, Zhong, Bercovici-Zhong (2022).)

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What about outliers?

$\mathbf{a}\colon \mathbb{S}^1 \to \mathbb{C}$, $\mathbf{a}(w) = \sum_{k=-r}^{s} a_k w^k$, $\quad w \in \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Definition

If $\mathbf{a}(w) \neq 0$ for all $w \in \mathbb{S}^1$, then $\mathbf{a}(w)$ traces out a continuous and closed curve in $\mathbb{C} \setminus \{0\}$ as w moves once around the couterclockwise oriented unit circle; the winding number of $\mathbf{a}$ = the number of times this curve surrounds the origin counterclockwise. Note that clockwise motion counts as negative and the winding number may be negative.

Infinite dimensional Toeplitz operator $T(\mathbf{a})$ on $\ell^2(\mathbb{N})$,

$$(T(\mathbf{a})x)_i = \sum_{l=-r}^{s} a_l x_{l+i}, \text{ for } i \in \mathbb{N}, \quad \text{where } x = (x_1, x_2, \ldots)$$

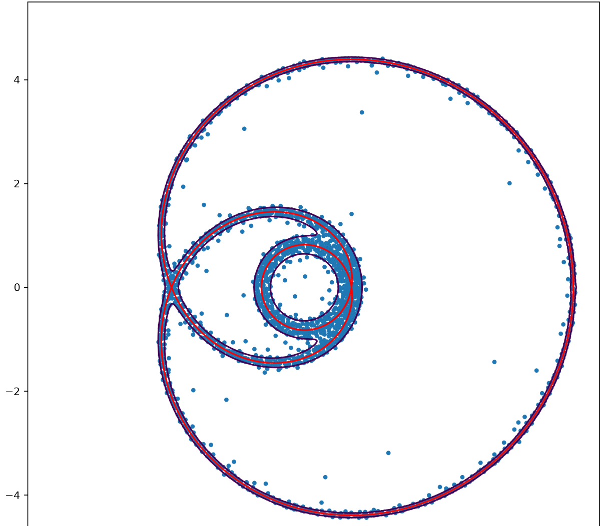
(we set $x_i = 0$ for non-positive integer values of i).

$$\mathrm{sp}\, T(\mathbf{a}) = \mathbf{a}(\mathbb{S}^1) \cup \left\{ z \in \mathbb{C} \setminus \mathbf{a}(\mathbb{S}^1) : \mathrm{wind}(\mathbf{a} - z) \neq 0 \right\},$$

Bottcher-Grudsky 05

The eigenvalues of $M_n = \sigma \frac{X_n}{\sqrt{n}} + T_n(\mathbf{a})$

$$\mathbf{a}(w) = w + 2w^2 + 2w^3,\ w \in \mathbb{S}^1,\ n = 1000, \sigma = 0.9$$



2 different phenomena occur

whether we consider outliers

$\nearrow$ outside $\mathrm{sp}(T(\mathbf{a}))$

$\searrow$ inside $\mathrm{sp}(T(\mathbf{a}))$

$T(\mathbf{a})$: the infinite dimensional Toeplitz operator.

Outside $\text{sp}(T(\mathbf{a}))$

Theorem (Bordenave-C.-Chapon 2024)

Let Γ be a compact set in a connected component of $\mathbb{C}\backslash(\mathrm{supp}(\beta) \cup \mathrm{sp}\, T(\mathbf{a}))$. Then, almost surely, for all n large enough, there are no eigenvalues of $M_n = \sigma \frac{X_n}{\sqrt{n}} + T_n(\mathbf{a})$ in Γ.

$$\Downarrow$$

Almost surely, for large n, no outlier of $M_n = \sigma \frac{X_n}{\sqrt{n}} + T_n(\mathbf{a})$ outside any ϵ-fattening of sp $T(\mathbf{a})$ ($T(\mathbf{a})$: the infinite Toeplitz operator)

Remark.

Basak-Zeitouni 2020 studied the outliers of a vanishing perturbation of a Toeplitz matrix:

$$\tilde{M}_n = T_n(a) + \frac{X_n}{n^\gamma} \quad \text{with } \gamma > 1/2.$$

$\left(\left\|\frac{X_n}{n^\gamma}\right\| \to_{n\to+\infty} 0 \text{ since } \left\|\frac{X_n}{\sqrt{n}}\right\| \to_{n\to+\infty} 2.\right)$.

In that case, the support of the limiting empirical spectral distribution is $\mathbf{a}(\mathbb{S}^{\mathbf{1}})$.

The authors proved that there are for large n, no outliers of $\tilde{M}_n$ outside any ϵ-fattening of sp $T(\mathbf{a})$ either.
See also Sjöstrand-Vogel 2021 (in the Gaussian case).

Main ideas to prove "no outlier outside sp $T(\mathbf{a})$"

If $n > r + s$, we write

$T_n(\mathbf{a}) = C_n(\mathbf{a}) + B_n$, B_n has a rank $\leq r+s$, $C_n(\mathbf{a})$ is the circulant matrix

$$C_n(\mathbf{a}) = \begin{pmatrix}
a_0 & a_1 & \cdots & a_s & 0 & \cdots & 0 & a_{-r} & \cdots & a_{-1} \\
a_{-1} & a_0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & a_{-r} \\
a_{-r} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & a_s \\
a_s & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & a_1 \\
a_1 & \cdots & a_s & 0 & \cdots & 0 & a_{-r} & \cdots & a_{-1} & a_0
\end{pmatrix}.$$

$$T_n(\mathbf{a}) = C_n(\mathbf{a}) + B_n$$

where

$$B_n = -\begin{pmatrix} 0_{r\times s} & 0_{r\times(n-r-s)} & D_r \\ 0_{(n-r-s)\times s} & 0_{(n-r-s)\times(n-r-s)} & 0_{(n-r-s)\times r} \\ E_s & 0_{s\times(n-r-s)} & 0_{s\times r} \end{pmatrix},$$

$$D_r = \begin{pmatrix} a_{-r} & a_{-r+1} & \cdots & a_{-1} \\ 0 & a_{-r} & \cdots & a_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{-r} \end{pmatrix}, \quad E_s = \begin{pmatrix} a_s & 0 & \cdots & 0 \\ a_{s-1} & a_s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_s \end{pmatrix}.$$

We write $B_n = -P_n Q_n$, with $P_n, Q_n^\top \in M_{n,r+s}(\mathbb{C})$

$$P_n = \begin{pmatrix} 0_{r\times s} & I_r \\ 0_{(n-s-r)\times s} & 0_{(n-s-r)\times r} \\ E_s & 0_{s\times r} \end{pmatrix}, \quad Q_n = \begin{pmatrix} I_s & 0_{s\times(n-s-r)} & 0_{s\times r} \\ 0_{r\times s} & 0_{r\times(n-s-r)} & D_r \end{pmatrix}.$$

$$\mathrm{supp}(\beta) = \mathbf{a}(\mathbb{S}^1) \cup \left\{ z \in \mathbb{C} \setminus \mathbf{a}(\mathbb{S}^1) : \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|\,\mathbf{a}(e^{i\theta}) - z|^2} \geq \sigma^{-2} \right\}.$$

$$C_n(\mathbf{a}) = F \mathrm{diag}(\mathbf{a}(\omega_n^{l-1}), l = 1, \ldots, n) F^*, \ \omega_n = e^{\frac{2i\pi}{n}},$$

$$F_n = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \cdots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \ldots & \omega_n^{(n-1)(n-1)} \end{pmatrix}.$$

$\Longrightarrow$ $C_n(\mathbf{a})$ is well-conditioned in any compact set in $\mathbb{C} \setminus \mathrm{supp}(\beta)$, (Bordenave-C. 2016)

$\Longrightarrow$ almost surely, for large n, $\sigma \frac{X_n}{\sqrt{n}} + C_n(\mathbf{a})$ has no outlier.

First key ingredient: Sylvester's identity

We fix a compact set $\Gamma \subset \mathbb{C} \setminus \mathrm{supp}(\beta)$.

$$T_n(\mathbf{a}) = C_n(\mathbf{a}) + P_n Q_n, \; P_n \in M_{n,r+s}(\mathbb{C}), \; Q_n \in M_{r+s,n}(\mathbb{C}).$$

a.s. for n large enough, $\sigma \frac{X_n}{\sqrt{n}} + C_n(\mathbf{a})$ has no eigenvalue in Γ.

$$Y_n := \sigma \frac{X_n}{\sqrt{n}}, \; R_n^{Y+C}(z) = (zI_n - Y_n - C_n(\mathbf{a}))^{-1}, \; \forall z \in \Gamma.$$

$$\begin{aligned}
\det(z - M_n) &= \det(z - Y_n - T_n(\mathbf{a})) \\
&= \det(z - Y_n - C_n(\mathbf{a}) - P_n Q_n) \\
&= \det(z - Y_n - C_n(\mathbf{a}))\det(I_n - P_n Q_n R_n^{Y+C}(z)) \\
&= \det(z - Y_n - C_n(\mathbf{a}))\det(I_{r+s} - Q_n R_n^{Y+C}(z) P_n)
\end{aligned}$$

Sylvester's determinant identity: for $A \in M_{p,q}(\mathbb{C}), B \in M_{q,p}(\mathbb{C})$, $\det(I_p + AB) = \det(I_q + BA)$.

$$R_n^{Y+C}(z) = (zI_n - Y_n - C_n(\mathbf{a}))^{-1}$$

$$\det(zI_n - M_n) = \det(zI_n - Y_n - C_n(\mathbf{a}))\det(I_{r+s} - Q_n R_n^{Y+C}(z) P_n)$$

Since $\det(zI_n - Y_n - C_n(\mathbf{a})) \neq 0$ on Γ,

$z \in \Gamma$ is an eigenvalue of M_n if and only if z is a zero of the determinant of a $(r+s) \times (r+s)$ matrix:

$$f_n(z) = \det(I_{r+s} + Q_n R_n^{Y+C}(z) P_n).$$

$\Longrightarrow$ try to find the asymptotic of this function!

This idea to study outliers of random matrices thanks to Sylvester's formula and resolvent computations was initiated by Benaych-Georges and Rao 2011.

Similarly,

$$T_n(\mathbf{a}) = C_n(\mathbf{a}) + P_n Q_n,\ P_n \in M_{n,r+s}(\mathbb{C}),\ Q_n \in M_{r+s,n}(\mathbb{C}).$$

$C_n(\mathbf{a})$ has no eigenvalue in $\Gamma \subset \mathbb{C} \setminus \mathrm{supp}(\beta)$ since $\mathbf{a}(\mathbb{S}^1) \subset \mathrm{supp}(\beta)$.

$$\forall z \in \Gamma, \quad R_n^C(z) = (zI_n - C_n(\mathbf{a}))^{-1}.$$

$$\begin{aligned} \det(zI_n - T_n(\mathbf{a})) &= \det(zI_n - C_n(\mathbf{a}) - P_n Q_n) \\ &= \det(zI_n - C_n(\mathbf{a}))\det(I_n - P_n Q_n R_n^C(z)) \\ &= \det(zI_n - C_n(\mathbf{a}))\det(I_{r+s} - Q_n R_n^C(z) P_n) \end{aligned}$$

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- eigenvalues of $M_n = Y_n + T_n(\mathbf{a})$ in $\Gamma \leftrightarrow$ zeroes of f_n

$$f_n(z) = \det(I_{r+s} + Q_n R_n^{Y+C}(z) P_n)$$

- eigenvalues of $T_n(\mathbf{a})$ in $\Gamma \leftrightarrow$ zeroes of g_n

$$g_n(z) = \det(I_{r+s} + Q_n R_n^{C}(z) P_n) = \frac{\det(zI_N - T_n(\mathbf{a}))}{\det(zI_N - C_n(\mathbf{a}))}$$

- eigenvalues of $M_n = Y_n + T_n(\mathbf{a})$ in $\Gamma \leftrightarrow$ zeroes of f_n

$$f_n(z) = \det(I_{r+s} + Q_n R_n^{Y+C}(z) P_n)$$

- eigenvalues of $T_n(\mathbf{a})$ in $\Gamma \leftrightarrow$ zeroes of g_n

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- $$R_n^{Y+C} = R_n^C + R_n^C Y_n R_n^{Y+C}$$

$$\downarrow$$

$$\begin{aligned}\Sigma_n(z) &:= Q_n R_n^{Y+C}(z) P_n - Q_n R_n^C(z) P_n \\ &= \sum_{k\geq 1} Q_n \left(R_n^C(z) Y_n\right)^k R_n^C(z) P_n \underset{n\to+\infty}{\longrightarrow} 0 \text{ uniformly in z in } \Gamma.\end{aligned}$$

$$a.s. \ \sup_{z\in\Gamma} |f_n(z) - g_n(z)| \to_{n\to+\infty} 0.$$

Second key ingredient: Szegö's strong limit Theorem

Note that $zI_n - T_n(\mathbf{a}) = T_n(z - \mathbf{a}) = T_n(\mathbf{b})$.

$$\mathbf{b}(t) = z - \mathbf{a}(t).$$

Since $z \notin \mathbf{a}(\mathbb{S}^1)$ and $z \notin \mathrm{sp}\, T(\mathbf{a})$ we have

$$\forall t \in \mathbb{S}^1,\ \mathbf{b}(t) \neq 0, \text{ and } \mathrm{wind}(\mathbf{b}) = 0.$$

$\Longrightarrow$ we can apply Szegö's strong limit theorem that describes the limiting behaviour of $\det T_n(\mathbf{b})$.

• eigenvalues of $M_n = Y_n + T_n(\mathbf{a})$ in $\Gamma \leftrightarrow$ zeroes of f_n

$$f_n(z) = \det(I_{r+s} + Q_n R_n^{Y+C}(z) P_n)$$

• eigenvalues of $T_n(\mathbf{a})$ in $\Gamma \leftrightarrow$ zeroes of g_n

$$g_n(z) = \det(I_{r+s} + Q_n R_n^{C}(z) P_n) = \frac{\det(zI_N - T_n(\mathbf{a}))}{\det(zI_N - C_n(\mathbf{a}))}$$

• a.s. $\sup_{z\in\Gamma} |f_n(z) - g_n(z)| \to_{n\to+\infty} 0$
• Since $\Gamma \subset \mathbb{C} \setminus \operatorname{sp} T(\mathbf{a})$, by Szegö's strong limit theorem, for n large enough,

$$\inf_{z\in\Gamma} |g_n(z)| = \inf_{z\in\Gamma} \left| \frac{\det(z - T_n(\mathbf{a}))}{\det(z - C_n(\mathbf{a}))} \right| > \frac{1}{2^{r+s+rs+1}}$$

Last key ingredient: Rouché's Theorem

- eigenvalues of $M_n = Y_n + T_n(\mathbf{a})$ in $\Gamma \leftrightarrow$ zeroes of f_n
- a.s. $\sup_{z\in\Gamma} |f_n(z) - g_n(z)| \to_{n\to+\infty} 0$
- $$\inf_{z\in\Gamma} |g_n(z)| = \inf_{z\in\Gamma} \left| \frac{\det(z - T_n(\mathbf{a}))}{\det(z - C_n(\mathbf{a}))} \right| > \frac{1}{2^{r+s+rs+1}}$$

$\Longrightarrow$ a.s. for all large n, for all $z \in \Gamma$, $|f_n(z) - g_n(z)| < |g_n(z)|$

so that by Rouché's theorem, for large n, f_n and g_n have the same number of zeroes in Γ. Since g_n has no zeroes in Γ, we deduce that, a.s., there are no eigenvalues of M_n in Γ for n large enough.

$\Longrightarrow$ No outlier of M_n outside any ϵ-fattening of sp $T(\mathbf{a})$.

What about Outliers in sp $T(\mathbf{a})$?

For $\Gamma \subset \operatorname{sp} T(\mathbf{a}) \cap (\mathbb{C} \setminus \operatorname{supp}(\beta))$, still true:

- eigenvalues of $M_n = Y_n + T_n(\mathbf{a})$ in $\Gamma \leftrightarrow$ zeroes of f_n

$$f_n(z) = \det(I_{r+s} + Q_n R_n^{Y+C}(z) P_n)$$

- $$\sup_{z \in \Gamma} |f_n(z) - g_n(z)| \to_{n \to +\infty} 0,$$

where $g_n(z) = \det(I_{r+s} + Q_n R_n^{C}(z) P_n) = \dfrac{\det(zI_N - T_n(\mathbf{a}))}{\det(zI_N - C_n(\mathbf{a}))}$.

BUT for any compact set $\Gamma \subset \operatorname{sp} T(\mathbf{a}) \setminus \mathbf{a}(\mathbb{S}^1)$, there exists $\delta > 0$,

$$\sup_{z \in \Gamma} |g_n(z)| = \sup_{z \in \Gamma} \left| \frac{\det(T_n(\mathbf{a}) - z)}{\det(C_n(\mathbf{a}) - z)} \right| = O(e^{-n\delta}) \text{ as } n \to \infty,.$$

$\Longrightarrow$ Rouché's theorem no longer applies!

Random analytic functions

$D \subset \mathbb{C}$: a connected open set.
$\mathcal{H}(D)$: the space of complex analytic functions in D, endowed with the distance

$$d(f,g) = \sum_{j\geq 1} 2^{-j} \frac{\|f-g\|_{K_j}}{1+\|f-g\|_{K_j}}, \ \|f-g\|_{K_j} = \sup_{z\in K_j} |f(z)-g(z)|$$

where $(K_j)_{j\geq 1}$ is an exhaustion by compact sets of D.
$(\mathcal{H}(D), d)$ is a complete separable metric space.
The space $\mathcal{H}(D)$ is equipped with the (topological) Borel σ-field.

A random analytic function on D: an $\mathcal{H}(D)$-valued random variable on a probability space.

Let $D \subset \mathbb{C}$ be an open connected set. For a nonzero analytic function $f \in \mathcal{H}(D)$, we denote by Z_f the set of zeroes of f and define the zero process of f as

$$\xi_f = \sum_{z \in Z_f} m_z \delta_z$$

where m_z is the multiplicity of a zero z.

Theorem (Shirai (2012))

Suppose that a sequence of random analytic functions $\{X_n\}$ converges in law to X in $\mathcal{H}(D)$. Then the zero process ξ_{X_n} converges in law to the zero process ξ_X provided that $X \neq 0$ almost surely, i.e. for all continuous function φ with compact support in D,

$$\int \varphi d\xi_{X_n} \xrightarrow[n\to\infty]{weakly} \int \varphi d\xi_X .$$

For $\Gamma \subset \text{sp}\, T(\mathbf{a}) \cap (\mathbb{C} \setminus \text{supp}(\beta))$:
eigenvalues of $M_n = Y_n + T_n(\mathbf{a})$ in $\Gamma \leftrightarrow$ zeroes of f_n

$$f_n(z) = \det(I_{r+s} + Q_n R_n^{Y+C}(z) P_n),$$

a.s $\sup_{z\in\Gamma} |f_n(z)| \to 0$.

Our goal: find some scaling parameter $\theta_n > 0$ and some nonzero random analytic function f such that, in $\mathcal{H}(\overset{\circ}{\Gamma})$,

$$\theta_n f_n \underset{n\to+\infty}{\overset{w}{\to}} f.$$

Then, from Shirai's result, we could deduce the convergence in law of the zero process $\xi_{\theta_n f_n}$ (that is the point process of eigenvalues of M_n in $\overset{\circ}{\Gamma}$) towards ξ_f.

Ideas: Central limit Theorem and further expansion of the determinant

Eigenvalues of $M_n = Y_n + T_n(\mathbf{a})$ in $\Gamma \leftrightarrow$ zeroes of f_n

$$\begin{aligned} f_n(z) &= \det(I_{r+s} + Q_n R_n^{Y+C}(z) P_n) \\ &= \det(\underbrace{I_{r+s} + Q_n R_n^{C}(z) P_n}_{H_n(z)} + \Sigma_n(z)) \\ &= \det(\underbrace{H_n(z)}_{\text{deterministic}} + \underbrace{\Sigma_n(z)}_{\text{random}}), \end{aligned}$$

$$a.s. \ \sup_{z \in \Gamma} \|\Sigma_n(z)\| \underset{n \to +\infty}{\longrightarrow} 0.$$

$$\forall q \in \mathbb{N}, \ n^q \sup_{z \in \Gamma} \|H_n(z) - H(z)\| \to_{n \to 0} 0,$$

$H(z)$: a deterministic $(r+s) \times (r+s)$ matrix only involving z and the symbol $\mathbf{a}$.

Notation: $U_n(z) \underset{n\to+\infty}{\simeq} V_n(z)$ means that the Lévy-Prokhorov distance between the finite dimensional distributions of U_n and those of V_n goes to zero as n goes to infinity.

Fundamental result

$$\Sigma_n(z) = Q_n R_n^{Y+C}(z) P_n - Q_n R_n^C(z) P_n = \sum_{k\geq 1} Q_n \left(R_n^C(z) \frac{X_n}{\sqrt{n}} \right)^k R_n^C(z) P_n.$$

Central limit theorem. Under some technical assumptions on the common distribution of the X_{ij}'s and if $\mathbb{E}(X_{ij}^2) = 0$ (this last assumption to ease the exposition).

$$\sqrt{n}\Sigma_n(z) \underset{n\to+\infty}{\simeq} Z_n(z) := \sigma Q_n R_n^C(z) X_n R_n^C(z) P_n + W_{r+s}(z),$$

$W_{r+s}(z)$ a Gaussian analytic matrix independent with X_n.

Proof based on a multivariate central limit theorem for $\left(n^{-(k_l-1)/2} \mathrm{Tr}(B_{l,0} X_n B_{l,1} X_n \cdots B_{l,k_l-1} X_n) \right)_{l\in L}$, L some finite set of integer numbers, $\forall l \in L$, $B_l = (B_{l,0}, B_{l,1}, \ldots,) \in M_n(\mathbb{C})^{\mathbb{N}}$ sequence of deterministic matrices such that $B_{l,0}$ is a rank one projector.

eigenvalues of $M_n = Y_n + T_n(\mathbf{a})$ in $\Gamma \leftrightarrow$ zeroes of f_n

$$f_n(z) = \det(H_n(z) + \Sigma_n(z)) = \det H_n(z) + \sum_{k=1}^{r+s} \frac{1}{k!} D^k \det(H_n(z))(\Sigma_n(z), \ldots, \Sigma_n(z))$$

$$\forall q \in \mathbb{N},\ n^q \sup_{z \in \Gamma} \|H_n(z) - H(z)\| \to_{n \to 0} 0,$$

Assume now that $r = 0$. If $\mathrm{wind}(\mathbf{a} - z) = \delta > 0$, then for any $k < \delta$, $D^k \det(H(z)) \equiv 0$ and $D^\delta \det(H(z)) \neq 0$.

Central limit theorem.

$$\sqrt{n}\Sigma_n(z) \underset{n\to+\infty}{\simeq} Z_n(z) := \sigma Q_n R_n^C(z) X_n R_n^C(z) P_n + W_{r+s}(z),$$

$W_{r+s}(z)$ a Gaussian analytic matrix independent with X_n.

Assume now that $r = 0$. If $\text{wind}(\mathbf{a} - z) = \delta > 0$, then for any $k < \delta$, $D^k \det(H(z)) \equiv 0$ and $D^\delta \det(H(z)) \neq 0$.

$$\det(H_n + \Sigma_n) = \sum_{k=0}^{\delta-1} \frac{1}{k!} D^k \det(H_n)(\Sigma_n, \dots, \Sigma_n)$$
$$+ \frac{1}{\delta!} D^\delta \det(H_n)(\Sigma_n, \dots, \Sigma_n)$$
$$+ \sum_{k>\delta} \frac{1}{k!} D^k \det(H_n)(\Sigma_n, \dots, \Sigma_n, \Sigma_n, \dots, \Sigma_n).$$

Central limit theorem.

$$\sqrt{n}\Sigma_n(z) \underset{n\to+\infty}{\simeq} Z_n(z) := \sigma Q_n R_n^C(z) X_n R_n^C(z) P_n + W_{r+s}(z),$$

$W_{r+s}(z)$ a Gaussian analytic matrix independent with X_n.

Assume now that $r = 0$. If $\text{wind}(\mathbf{a} - z) = \delta > 0$, then for any $k < \delta$, $D^k \det(H(z)) \equiv 0$ and $D^\delta \det(H(z)) \neq 0$.

$$\begin{aligned}(\sqrt{n})^\delta \det(H_n + \Sigma_n) = (\sqrt{n})^\delta \textstyle\sum_{k=0}^{\delta-1} \frac{1}{k!} D^k \det(H_n)(\Sigma_n, \dots, \Sigma_n) \\ + \frac{1}{\delta!} D^\delta \det(H_n)(\sqrt{n}\Sigma_n, \dots, \sqrt{n}\Sigma_n) \\ + \textstyle\sum_{k>\delta} \frac{1}{k!} D^k \det(H_n)(\sqrt{n}\Sigma_n, \dots, \sqrt{n}\Sigma_n, \Sigma_n, \dots, \Sigma_n).\end{aligned}$$

$$(\sqrt{n})^\delta f_n(z) \underset{n\to+\infty}{\simeq} \frac{1}{\delta!} D^\delta \det(H(z))(Z_n(z), \dots, Z_n(z))$$

$\uparrow$ whose zeroes are the eigenvalues of M_n

Convergence towards zero of the Lévy-Prokhorov distance between the finite dimensional distributions of $(\sqrt{n})^\delta f_n = \det(H_n + \Sigma_n)$ and $\frac{1}{\delta!} D^\delta \det(H)(Z_n, \ldots, Z_n))$

$$+$$

Tightness (by using Shirai's Criterion for random analytic functions)

$$+$$

Shirai's theorem on zero processes

lead the following

Theorem (Bordenave-C.-Chapon 2024)

Assume that $r = 0$ and that the common distribution of the X_{ij}'s is symmetric, has moments of any order and $\mathbb{E}(X_{ij}^2) = 0$. Let $\Gamma \subset (\mathbb{C} \setminus \mathrm{supp}(\beta)) \cap \{z \in \mathbb{C} \setminus \mathbf{a}(\mathbb{S}^1), \mathrm{wind}(\mathbf{a} - z) = \delta \neq 0\}$ be a connected compact set. Then the Lévy-Prokhorov distance between the point process of eigenvalues of M_n in $\mathring{\Gamma}$ and the point process of zeroes of the random analytic function

$$z \mapsto D^{\delta} \det(H(z))(Z_n(z), \ldots, Z_n(z))$$

in $\mathring{\Gamma}$ goes to zero as n goes to infinity, where

$$Z_n(z) := \sigma Q_n R_n^C(z) X_n R_n^C(z) P_n + W_s(z),$$

$\{(W_s(z))_{p,q}, 1 \leq p, q \leq s, z \in \Gamma\}$ is a Gaussian process, independent from X_n, whose covariance depends on σ and $\mathbf{a}$.

$$\mathbb{E}((W_s)_{pq}(z)(W_s)_{p'q'}(z')) = 0,$$

$$\mathbb{E}((W_s)_{pq}(z)\overline{(W_s)_{p'q'}(z')})$$

$$= \frac{\sigma^4 \frac{1}{2i\pi}\int_{\mathbb{S}^1} \frac{dw}{w(z-\mathbf{a}(w))\overline{(z'-\mathbf{a}(w))}}}{1-\sigma^2\frac{1}{2i\pi}\int_{\mathbb{S}^1} \frac{dw}{w(z-\mathbf{a}(w))\overline{(z'-\mathbf{a}(w))}}} \frac{1}{2i\pi}\int_{\mathbb{S}^1} \frac{w^{p-p'-1}dw}{(z-\mathbf{a}(w))\overline{(z'-\mathbf{a}(w))}}$$

$$\times \frac{1}{2i\pi}\int_{\mathbb{S}^1} \frac{(E_s M(w) E_s^*)_{q'q}dw}{w(z-\mathbf{a}(w))\overline{(z'-\mathbf{a}(w))}},$$

$$(M(w))_{p,q} := w^{p-q},$$

$$E_s = \begin{pmatrix} a_s & 0 & \cdots & 0 \\ a_{s-1} & a_s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_s \end{pmatrix}.$$

Define for any $w \in \mathbf{a}(\mathbb{S}^1)$, $M(w)$ as the $(r+s) \times (r+s)$ matrix whose entries are $(M(w))_{p,q} = w^{p-q}$. Set

$$\mathcal{D}_{r+s} := \begin{pmatrix} I_s & 0 \\ 0 & D_r \end{pmatrix}, \quad \mathcal{E}_{r+s} := \begin{pmatrix} E_s & 0 \\ 0 & I_r \end{pmatrix}.$$

$$H(z) = I_{r+s} + \frac{1}{2i\pi}\mathcal{D}_{r+s}\int_{\mathbb{S}^1} \frac{1}{w(z-\mathbf{a}(w))} \begin{bmatrix} w^s I_s & 0 \\ 0 & w^{-r} I_r \end{bmatrix} M(w)dw\, \mathcal{E}_{r+s}.$$

$$D_r = \begin{pmatrix} a_{-r} & a_{-r+1} & \cdots & a_{-1} \\ 0 & a_{-r} & \cdots & a_{-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{-r} \end{pmatrix}, \quad E_s = \begin{pmatrix} a_s & 0 & \cdots & 0 \\ a_{s-1} & a_s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_s \end{pmatrix}.$$

Remarks

- The limit process of outliers turns out to be non-universal and to possess different laws in different parts of space corresponding to $\{z \in \mathbb{C} \setminus \mathbf{a}(\mathbb{S}^1), \mathrm{wind}(\mathbf{a} - z) = \delta \neq 0\}$, $-r \leq \delta \leq s$.
- In $\{z \in \mathbb{C} \setminus \mathbf{a}(\mathbb{S}^1), \mathrm{wind}(\mathbf{a} - z) = 1\}$ the outliers correspond, in the case of additive Gaussian noise, to zeros of a Gaussian analytic function, but there are regions where this is not the case.
- Similar phenomena were previously brought out by Basak-Zeitouni 2020 when they studied the outliers of:

$$\tilde{M}_n = T_n(a) + \frac{X_n}{n^\gamma} \text{ with } \gamma > 1/2 \ (\left\| \frac{X_n}{n^\gamma} \right\| \to_{n \to +\infty} 0).$$

  See also Sjöstrand-Vogel 2016 for bi-diagonal matrices.
- When $r = 0$, our method using Sylvester's determinant identity can be used to study Basak-Zeitouni's framework.
- Work in progress for $r \neq 0$

Thank you very much for your attention!