Dunkl processes, random matrices and Hurwitz numbers

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Plan

1. Cépa and Lépingle work : β-Dyson particles.
2. Extension to radial Dunkl processes.
3. Reflected Brownian motion in Weyl chambers.
4. Intertwining operator and simple Hurwitz numbers.

Cépa and Lépingle work : β-Dyson particles

β-Dyson particles

1. Solution when it exists of

$$d\lambda_i(t) = dB_i(t) + \beta \sum_{j \neq i} \frac{dt}{\lambda_i(t) - \lambda_j(t)}, \quad 1 \leq i \leq N.$$

2. It does for $\beta \in \{1/2, 1, 2\}$: eigenvalues of matrix-valued Brownian motions (symmetric real, Hermitian complex, self-dual quaternionic).
3. **No collision** between particles for these values of β.

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Cépa and Lépingle's Theorems

Theorem

For any initial data $(\lambda_i(0))_{i=1}^n$, the β-Dyson differential system has a unique strong global (in time) solution.

Theorem

If $\beta \geq 1/2$ then the first collision time is a.s. infinite. Otherwise, it is a.s. finite.

Remark

Similar results for a particle system on the circle and its hyperbolic version : **general framework**.

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Multivalued SDE

- D : convex closed domain in $\mathbb{R}^N$.
- $\Phi : \mathbb{R}^N \to \mathbb{R}$: lsc convex such that
  1. C^1 in $\mathrm{Int}(D)$.
  2. blows-up on $\mathrm{Int}(D)^c$.
- $n(x), x \in D \setminus \mathrm{Int}(D),$: outward normal vector at x.

Theorem

For any initial data $X(0) \in D$, the 'multivalued' SDE

$$dX(t) = dB(t) - \nabla\Phi(X(t))dt - n(X(t)) \underbrace{dL(t)}_{Continuous\ boundary\ process}$$

has a unique strong global solution valued in D. Moreover,

$$\mathbb{E}\left[\int_0^t 1_{\{X_s \in \partial D\}} ds\right] = 0,$$

and

$$\mathbb{E}\left[\int_0^t |\nabla\Phi(X_s)| ds\right] < \infty.$$

Application to β-Dyson particles

- $$\Phi(x) = -\beta \sum_{1\le i<j\le N} \ln(x_i - x_j), \quad \underbrace{x_1 > \cdots > x_N}_{\mathrm{Int}(D)},$$

  and $\Phi = +\infty$ otherwise.
- The boundary process vanishes.

Other choices :

1. Particles on the circle :

$$\Phi(x) = -\beta \sum_{1\le i<j\le N} \ln\sin(x_i - x_j), \quad x_N + 2\pi > x_1 > \cdots > x_N.$$

2. Hyperbolic version :

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Extension to radial Dunkl processes

Root systems

- $(\mathbb{R}^N, \langle , \rangle)$.
- Reflection orthogonal to $\alpha \neq 0$:

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{\langle \alpha, \alpha\rangle}\alpha.$$

Definition

- *A* **root system** *R is a collection of vectors in $\mathbb{R}^d \setminus \{0\}$ such that $\sigma_\alpha(R) = R$ for any $\alpha \in R$.*
- *It is reduced if*

$$\mathbb{R}\alpha \cap R = \{\pm\alpha\}, \quad \forall \alpha \in R.$$

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Positive system

1. Pick $v \notin R, v \neq 0$:

$$R_+ := \{\alpha \in R, \langle \alpha, v \rangle > 0\}.$$

2. $R = R_+ \cup R_-$.

Example (Type A)

$R = \{\pm(e_i - e_j), 1 \leq i < j \leq N\}$.
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Theorem

Unique subset $S \subset R_+$ s.t. every root system is a **positive LC** *of vectors in S.*

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S is the simple system associated to R_+. $\alpha \in S$ is a simple root and $|S|$ is the rank of R.

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Weyl chamber and reflection groups

(R, R_+, S) :

Definition

The Weyl chamber is the cone :

$$C := \{x \in \mathbb{R}^d, \langle \alpha, x \rangle > 0, \alpha \in R_+\}$$
$$= \{x \in \mathbb{R}^d, \langle \alpha, x \rangle > 0, \alpha \in S\}.$$

Definition

The reflection group is generated by $\{\sigma_\alpha, \alpha \in R\}$.

Proposition

- $|W|$ *is finite.*
- $\overline{C}$ *is a fundamental domain for the* W*-action on* $\mathbb{R}^N$ *:*

$$\mathbb{R}^d = \bigcup_{w \in W} w\overline{C}.$$

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Examples

Type A :

1. $$C_A = \{x_1 > \cdots > x_N\}.$$
2. $W = S_N$.

Type B :

1. $R = \{\pm(e_i - e_j), 1 \leq i < j \leq d, \pm e_i, 1 \leq i \leq N\}$.
2. $R_+ = \{(e_i - e_j), 1 \leq i < j \leq N, e_N\}$.
3. $$C_B = \{x_1 > \cdots > x_N > 0\}.$$
4. $W = S_N \rtimes (\mathbb{Z}_2)^N$.

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Multiplicity function

Definition

A multiplicity function is a map $k : R \to \mathbb{C}$ *s.t.*

$$k(\alpha) = k(w\alpha), \quad w \in W, \alpha \in R.$$

$\Rightarrow$ Takes as many values as the orbit space $|R/W|$.

Examples (Types A et B)

- $|R_A/W_A| = 1 \rightsquigarrow \beta$,
- $|R_B/W_B| = 2 \rightsquigarrow (\beta, \delta)$.

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$(R, R_+, S, W, \overline{C}, k \geq 0)$, R reduced.

Definition

X is the $\overline{C}$-valued diffusion with generator :

$$\mathscr{L}_X(f)(x) = \frac{1}{2}\Delta(f)(x) + \sum_{\alpha \in R_+} \frac{k(\alpha)}{\langle x, \alpha \rangle} \partial_\alpha(f)(x), \quad x \in C,$$

$$\partial_\alpha(f)(x) = 0, \quad x \in \alpha^\perp.$$

The radial Dunkl process

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Type B : Wishart and Laguerre processes

- $1 \leq i \leq N$, and any $k_0, \beta \geq 0$,

$$d\lambda_i(t) = dB_i(t) + \frac{k_0}{\lambda_i(t)}dt + \frac{\beta}{2}\sum_{j \neq i}\left[\frac{1}{\lambda_i(t) - \lambda_j(t)} + \frac{1}{\lambda_i(t) + \lambda_j(t)}\right],$$

in $C_B = \{\lambda_1 > \lambda_2 \cdots > \lambda_N > 0\}$.

- **Wishart** and **Laguerre** processes (singular values of real/complex rectangular Brownian matrices) :

$$\beta \in \{1, 2\}, \quad k_0 = \beta(n - N + 1)/2, \quad n \geq N.$$

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$B = (B_i)_{i=1}^d$: Brownian motion.

Theorem (Demni)

1. *Assume* $k(\alpha) > 0 \forall \alpha \in R$. *Then for any* $X_0 \in \overline{C}$,

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admits a unique strong (global in time) solution.

2. *If* $X_0 = x \in C$ *and* $0 \leq k(\alpha) < 1/2$ *for some* α *then*

$$\mathbb{P}_x(\langle \alpha, X_t \rangle = 0) = 1.$$

Extension of Cépa-Lépingle results

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Consequences

1. Improve existence and uniqueness results for eigenvalues of matrix-valued processes.
2. Extend to the affine setting :
   - Weyl group (finite) $\rightsquigarrow$ Affine Weyl group (infinite).
   - Weyl chamber $\rightsquigarrow$ Weyl alcove.
   - Eigenvalues of Brownian motions in compact groups and matrix-valued Jacobi processes.
   - Non compact case : Heckman-Opdam processes
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The opposite direction : Reflected Brownian motion in Weyl chambers

Motivation

- What happens when $k(\alpha) = 0$ for some (any) $\alpha \in R$?
- Brownian motions in convex polyhedra : widely studied by Williams et al..
- A lack of a concrete construction : defined as a solution of a martingale problem.
- Extension of Tanaka's formula ($N = 1$) :

$$d|W|_t = dB_t + \frac{1}{2}L_t^0(|W|) = dB_t + L_t^0(W),$$

  to higher dimensions $N \geq 2$.
- Concrete construction using folding operators.

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Folding operators

For any $\alpha \in R_+$, we set :

$$f_\alpha(x) = x + 2\frac{(\langle \alpha, x\rangle)^-}{\langle \alpha, \alpha\rangle}\alpha.$$

$$\begin{aligned} f_\alpha(x) &= x, & \langle \alpha, x\rangle \geq 0, \\ &= \sigma_\alpha(x), & \langle \alpha, x\rangle \leq 0. \end{aligned}$$

f_α projects onto the positive half-space $\{\langle \alpha, x\rangle \geq 0\}$.

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Group-geometric facts

1. The Weyl group is finite $\Leftrightarrow$ It admits a longest element w_0 (with respect to $\sigma_\alpha, \alpha \in R_+$, length $= |R_+|$).
2. w_0 admits different equivalent (braid relations) reduced expressions.
3. To any reduced expression $w_0 = \sigma_{\alpha_1} \ldots \sigma_{\alpha_{|R_+|}}$, we associate

$$f_{w_0} = f_{\alpha_1} \ldots f_{\alpha_{|R_+|}}.$$

Proposition (Demni-Lépingle)

f_{w_0} is independent of the reduced expression and takes values in $\overline{C}$.

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Tanaka-type formula

B : $\mathbb{R}^N$-valued Brownian motion.

Theorem (Demni-Lépingle)

1. *There exists a $\mathbb{R}^N$ Brownian motion G such that*

$$f_{w_0}(B_t) = f_{w_0}(B_0) + G_t + \frac{1}{2}\sum_{\alpha\in S} L_t^0\left(\langle\alpha, f_{w_0}(B)\rangle\right)\alpha.$$

2. *If $\alpha \in S$ is the unique simple root in its orbit $W\alpha$ then*

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Example : B_2

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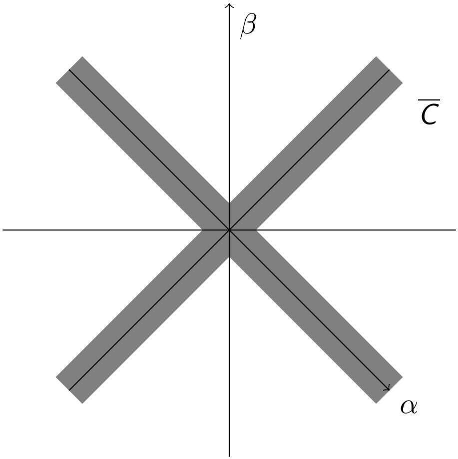

Dunkl Intertwining operator and simple Hurwitz numbers

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$$D_\xi(k)f(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha)\langle \alpha, \xi \rangle \frac{f(\sigma_\alpha x) - f(x)}{\langle \alpha, x \rangle}.$$

$$\boxed{k = 0}$$

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For any **reduced** *root system, the algebra generated by* $\{D_\xi(k, R), \xi \in \mathbb{R}_N \setminus \{0\}\}$ *is commutative.*

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Seek a (linear) isomorphism V_k such that

1. $V_k 1 = 1$ (normalization).
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Probabilistic interpretation

- Intertwining of generators :

$$\boxed{\Delta_k V_k = V_k \Delta}.$$

- Rank-one case :
  1. $k = 1$: Pitman Theorem (from Brownian motion to 3-Bessel process).
  2. $k \geq 0$: explicit integral representation.
- **Higher ranks** and $k = 1$: Extended Pitman Theorem through Duistermaat-Heckman measure (Biane-Bougerol-O'connell).
- Dihedral and A-type root systems.

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Final answer

Theorem (Deléaval-D-Youssfi)

If $k \in M_{reg}$ *then for any* $p \in \mathscr{P}_n, n \geq 1,$,

$$V_k(p)(x) = \sum_{w_1,\dots,w_n \in W} C_n(w_n) C_{n-1}(w_n^{-1} w_{n-1}) \dots C_1(w_2^{-1} w_1) \partial_{w_n x} \dots \partial_{w_1 x}(p)(x),$$

where for any $w \in W$,

$$C_n(w) := \sum_{m=0}^{\infty} \frac{c_m(w)}{(n+\gamma)^{m+1}}, \quad \gamma := \sum_{\alpha \in R_+} k(\alpha),$$

and

$$c_m(w) = \sum_{\sigma_{\alpha_1} \dots \sigma_{\alpha_m} = w} k(\alpha_1) \dots k(\alpha_m).$$

$k \equiv 1$: Simple Hurwitz numbers

- $k \equiv 1$:

$$c_m(w) = |\text{number of factorisations of } w \text{ into } m \text{ reflections}|.$$

- $W = S_N$: simple Hurwitz numbers.
- W-invariant Dunkl theory is connected to Harish-Chandra integrals over compact groups.

Thanks ! ! ! ! !