A statistical physics approach to the Sine_β process and other random point processes

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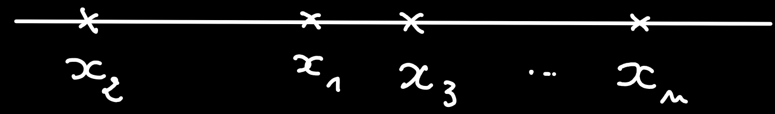
Outline of the talk

Joint work with D. Dereudre (Lille), A. Hardy (Lille, on leave) and T. Leblé (Paris Cité)

1. Introduction to the Sine_β process
2. Main result: DLR equation for Sine_β
3. Applications
   - → CLT for linear statistics
   - → Number rigidity
   - → Curvature bound for Dyson Brownian motion *
4. Perspectives

1. What is Sine_β ?

We start with a log-gas on $\mathbb{R}$ (or on $\mathbb{U}$)



$\gamma = \{x_1, \dots, x_n\}$

Energy:

$$\boxed{H_n(\gamma) = \frac{1}{2} \sum_{i \neq j} \overbrace{-\log}^{g} |x_i - x_j| + n \sum_{i=1}^{n} V(x_i)}$$

more natural on $\mathbb{R}^2$ — confining potential

Gibbs measure:

inverse temperature

$$\boxed{d\mathbb{P}^n_{V,\beta}(x_1, \dots, x_n) = \frac{1}{Z^n_{V,\beta}} e^{-\frac{\beta}{2} H_n(x_1, \dots, x_n)} dx_1 \dots dx_n}$$

Related to well-know models in random matrix theory (RMT):

- $\beta = 2$ $\quad V(x) = \frac{x^2}{2}$ on $\mathbb{R}$ $\longrightarrow$ Gaussian Unitary Ensemble (GUE)

  $V(x) = 0$ on $\mathbb{U}$ $\longrightarrow$ Circular Unitary Ensemble (CUE)
  (Haar on $\mathcal{U}_n$)

· <u>$\beta > 0$</u> $\quad V(x) = \frac{x^2}{2}$ on $\mathbb{R}$ $\longrightarrow$ GβE (tridiagonal model by Dumitriu - Edelman)

$V(x) = 0$ on $\mathbb{C}$ $\longrightarrow$ CβE (pentadiagonal model by Killip - Nenciu)

general V : not necessarily a suitable RM model but still a natural family of measures.

Microscopic behavior of the log-gas

→ $\underline{\beta = 2}$ is well known.

$\mathbb{P}^n_{V,2}$ seen as a point process is determinantal : its correlation functions $(\rho_k)_{1 \leq k \leq n}$ can be expressed as

$$\rho_k(x_1, \dots, x_k) = \det\left(K^n_V(x_i, x_j)\right)_{i,j=1}^k, \quad \forall\, 1 \leq k \leq n$$

In the case of the CUE, it is easy to check that:

$$K^n_V(\theta, \theta') = \frac{\sin\left(\frac{n}{2}(\theta - \theta')\right)}{\sin\left(\frac{\theta - \theta'}{2}\right)} \quad \text{if } \theta \neq \theta'$$

If we rescale : $\tilde{\gamma} := \left\{ \frac{n}{2\pi}\theta_1, \dots, \frac{n}{2\pi}\theta_n \right\}$, the new kernel is

$$\tilde{K}^n_V(\theta, \theta') = \frac{\sin(\pi(\theta - \theta'))}{n \sin\left(\frac{\pi}{n}(\theta - \theta')\right)} \xrightarrow[n \to \infty]{} \frac{\sin(\pi(\theta - \theta'))}{\pi(\theta - \theta')} \quad \text{Sine process}$$

⟶ bulk universality properties, projection kernel ...

→ $\underline{\beta > 0}$: much less is known

. Valko - Virag : existence of a limiting process for GβE zoomed
2009 in the bulk
. Killip - Nenciu : existence of a limiting process for zoomed CβE
. Nakano : the two limiting processes are the same

Sine$_\beta$ process.

. Brownian carousel : the number of points of Sine$_\beta$ in $[0, \lambda]$ is $\frac{1}{2\pi} \alpha_\lambda(\infty)$, with

$$\begin{cases} d\alpha_\lambda(t) = \lambda \frac{\beta}{4} e^{-\frac{\beta t}{4}} dt + \mathrm{Re}\left(\left(e^{i\alpha_\lambda(t)} - 1\right) dZ_t\right) \\ \alpha_\lambda(0) = 0 \end{cases}$$

Same 2d - BM
for all $\lambda > 0$.

. Valko - Virag (2017 - 2022) : description of some random operators whose spectrum is Sine$_\beta$.

. universality with respect to V. (Bourgade - Erdös - Yau - Lin / Bekerman - Figalli - Guionnet)

2. Description of Sine_β as a Gibbs measure

We start back from

$$\boxed{d\mathbb{P}^n_{V,\beta}(x_1,\dots,x_n) = \frac{1}{Z^n_{V,\beta}}\, e^{-\frac{\beta}{2} H_n(x_1,\dots,x_n)}\, dx_1 \dots dx_n}$$

Can we make sense of the limit as $n \to \infty$ as an "infinite volume Gibbs measure"?

<u>Thm</u> (Dereudre, Hardy, Leblé, M. 2021) $\Lambda \subset \mathbb{R}$ compact set

configuration outside Λ — self interaction

$$\boxed{d\,\mathrm{Sine}_\beta(\eta \mid \gamma_{\Lambda^c}, \underbrace{|\gamma_\Lambda|}_{\text{nb of points in }\Lambda}) \propto \exp\Big(-\beta\big(H_\Lambda(\eta) + \underbrace{\mathcal{M}(\eta,\gamma_{\Lambda^c})}_{\text{interaction with the exterior}}\big)\Big)\, d\underbrace{\mathcal{B}_{|\gamma_\Lambda|}(\eta)}_{\text{Bernoulli process.}}}$$

"Canonical Dobrushin-Lanford-Ruelle equations"

DLR

More precisely, for any bounded measurable f,

$$\boxed{\begin{aligned} &\mathbb{E}_{\mathrm{Sine}_\beta}(f) = \int \Big[\int f(\{x_1,\dots,x_{|\gamma_\Lambda|}\} \cup \gamma_{\Lambda^c})\, \rho_{\Lambda^c}(x_1,\dots,x_{|\gamma_\Lambda|}) \prod_{i=1}^{|\gamma_\Lambda|} dx_i \Big] \mathrm{Sine}_\beta(d\gamma), \\ &\text{where } \rho_{\Lambda^c}(x_1,\dots,x_{|\gamma_\Lambda|}) = \frac{1}{Z_{\Lambda,\gamma_{\Lambda^c}}} \prod_{j<k}^{|\gamma_\Lambda|} |x_j - x_k|^\beta \prod_{i=1}^{|\gamma_\Lambda|} e^{-\sum_{i=1}^{|\gamma_\Lambda|} v_\beta(x_i,\gamma_{\Lambda^c})} \end{aligned}}$$

. $\beta = 2$, previous results by Bufetov (see also Kuijlaars - Miña Diaz)

Proof:

- use CβE as a reference model
- easy to show DLR for CβE (finite nb of particles)
- use convergence to Sine_β
- very delicate to make sense of the Move functions for large range interactions (subtle cancellations thanks to discrepancy estimates due to Leblé - Serfaty)

3. Applications and related results

3.1. Fluctuations of linear statistics for the Sine_β process

Thm (Leblé, 2022) If $\gamma \sim \mathrm{Sine}_\beta$, φ $\mathcal{C}^4$ and compactly supported,

$$\mathrm{Fluct}_\ell(\varphi)(x) = \int \varphi\left(\frac{x}{\ell}\right) (\gamma(dx) - dx) \xrightarrow[\ell \to \infty]{} \mathcal{N}(0, \sigma_\beta^2),$$

with $$\sigma_\beta^2 = \frac{1}{2\beta\pi^2} \iint \left(\frac{\varphi(x) - \varphi(y)}{x-y}\right)^2 dx\, dy$$

Strategy: analyze $$\mathcal{L}_{\varphi,\ell}(t) = \mathbb{E}\Big(\exp(t\, \mathrm{Fluct}_\ell(\varphi)(\gamma))\, \mathbb{1}_{\underbrace{\text{good event}}_{\text{on which } \exp(\cdots) \text{ is bounded}}}(\gamma)\Big)$$

starting point: DLR on $\Lambda \supset \mathrm{supp}\, \varphi$, compact

$$\mathcal{L}_{\varphi,\ell}(t) = \int \mathrm{Sine}_\beta(d\gamma) \frac{1}{Z_{\Lambda,\beta}(\gamma)} \exp\big(\underbrace{t\, \mathrm{Fluct}_\ell(\varphi)(\eta \cup \cancel{\gamma_{\Lambda^c}})}_{\text{Can be seen as a modification of } H_\Lambda}\big) \exp\big(-\beta(H_\Lambda(\eta) + M_\Lambda(\eta,\gamma))\big)\, dB_{|\gamma_\Lambda|}(\eta)$$

$\to$ analyze the change of equilibrium measure (transportation techniques)

3.2. Number rigidity

def : a point process $\mathbb{P}$ is number-rigid iff $\forall \Lambda$ compact set, $\exists$ measurable function f_Λ such that $\mathbb{P}$ a.s. $|\gamma_\Lambda| = f_\Lambda(\gamma_{\Lambda^c})$.

ex : → Poisson point processes are not number-rigid.
→ some DPP are know to be rigid, in particular Sine_2 (Bufetov).

Is Sine_β rigid ?

"Standard method" (Ghosh-Peres) : control of the variance of linear statistics.

($\forall \Lambda, \varepsilon > 0$, $\exists$ compactly support $f_{\Lambda,\varepsilon}$ s.t $f_{\Lambda,\varepsilon}|_\Lambda = 1$ and $\mathrm{Var}\left(\sum f_{\Lambda,\varepsilon}(x_i)\right) \leq \varepsilon$

$\Rightarrow$ $\mathbb{P}$ is number-rigid)

Chhaibi-Najnudel (2021) ⟶ Sine_β is rigid.

Our approach : any process $\mathbb{P}$ satisfying canonical DLR (and tolerant !) is number-rigid DHLM.

<u>Related results</u>: • Dereudre - Vasseur 2023. Non rigidity for β-circular Riesz gases.

$\rightarrow$ replace $g(x) = -\log|x|$ by $g(x) = \|x\|^s$, $d-1 < s < d$

($s = d-2$ = Coulomb).

D-V have shown canonical DLR for the infinite-volume process and used it to show that it is not rigid.

• DLR for Airy_β (non-stationary process!)
work in progress Lambert - M. - Paquette

3.3. Curvature of Dyson Brownian motion

Hermitian Brownian motion: $H_n(t) = \begin{pmatrix} B_{11}(t) & & \frac{B_{kl}(t) + i\tilde{B}_{kl}(t)}{\sqrt{2}} \\ & \ddots & \\ \frac{B_{kl}(t) - i\tilde{B}_{kl}(t)}{\sqrt{2}} & & B_{nn}(t) \end{pmatrix}$

It is well known (Dyson) that the process of its eigenvalues satisfies

$$d\lambda_k = d\beta_k + \sum_{l \neq k} \frac{1}{\lambda_k - \lambda_l} dt$$

Dyson BM = independent standard BM conditioned not to intersect.

Can be generalized:

$$d\lambda_k = d\beta_k + \frac{\beta}{2} \sum_{l \neq k} \frac{1}{\lambda_k - \lambda_l} dt \left(+ V(\lambda_k)\, dt \right)$$

One can make sense of the "infinite Dyson Brownian motion"

$$dX_k = dB_k + \frac{\beta}{2} \lim_{r\to\infty} \sum_{\substack{j\neq k \\ |X_j - X_k| \leq r}} \frac{1}{X_k - X_j}\, dt$$

[Osada ($\beta = 1, 2, 4$), Tsai ($\beta > 0$)]

In particular, this infinite system can be seen as a diffusion on the space of locally finite configurations,

and $\underline{\text{Sine}_\beta \text{ is an invariant measure}}$ of this dynamics.

Recall the <u>Bakry - Emery</u> curvature bound:

If a Riemannian manifold (M, g) has Ricci curvature $\mathrm{Ric} \geq K$, for $K \in \mathbb{R}$

then $\forall\, u \in W^{1,2}(M)$ $\quad |\nabla T_t u|^2 \leq e^{-Kt}\, T_t |\nabla u|^2$, $\quad$ (**)

with $\{T_t\}$ the heat semi-group on (M, g).

"Thm" (Suzuki, 2023)

If we consider the differential structure induced by the (infinite) β Dyson Brownian motion, there is an analogue of (**) for the corresponding semi-gp with $K \geq 0$.

rk: this implies other inequalities (local Poincaré ineq, local log-Sobolev ineq).

DLR is a key input in this study. : first consider the dynamics restricted to the (finite number) of particles in a ball B_R.

The invariant measure is the law of η knowing the config outside: exactly as defined by DLR eq., with density

$$\prod_{i<j} |x_i - x_j|^\beta \prod_{i=1}^{k} \lim_{R \to \infty} \prod_{\substack{y \in \gamma_{\Lambda^c} \\ |y| \leq R}} \left| 1 - \frac{x_i}{y} \right|^\beta$$

Suzuki shows geodesical convexity of the log of this density.

4. A few perspectives

→ better understanding of the solution of DLR equations (unicity, properties ...)

→ show DLR for other processes of interest (other Riesz gas, Coulomb $d \geq 2$, Airy$_\beta$ ...)

→ other applications.

Thanks for you attention!