

# Sharp Interface Limit for Glauber-Kawasaki Process

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# 1. Introduction

# Glauber-Kawasaki process

- Glauber-Kawasaki process on  $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d$ ,  $N, d \in \mathbb{N}$ .

Exclusion process + Glauber dynamics.

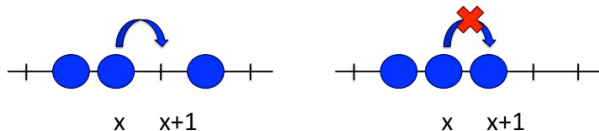
- State space:  $\{0, 1\}^{\mathbb{T}_N^d} \ni \eta = \{\eta(x)\}_{x \in \mathbb{T}_N^d}$ ; for  $x \in \mathbb{T}_N^d$ ,

$\eta(x) = 1 \cdots$  there is a particle at site  $x$ .

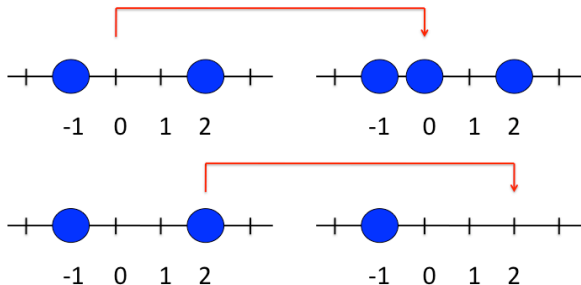
$\eta(x) = 0 \cdots$  there is no particle at site  $x$ .

# Model

- Symmetric simple exclusion process, rate  $N^2$ :



- Glauber dynamics, rate  $c(x, \eta)$ :



# Generator description

- Let  $\eta_t^N$  be a Markov process on  $\{0, 1\}^{\mathbb{T}_N^d}$  generated by

$$L_N f = N^2 L_E f + L_G f,$$

$$L_E f(\eta) = \sum_{j=1}^d \sum_{x \in \mathbb{T}_N^d} \{f(\eta^{x, x+e_j}) - f(\eta)\},$$

$$L_G f(\eta) = \sum_{x \in \mathbb{T}_N^d} c(x, \eta) \{f(\eta^x) - f(\eta)\}.$$

- In the above formula,

$$\eta^{x,y}(z) = \begin{cases} \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y, \\ \eta(z) & \text{otherwise,} \end{cases} \quad \eta^x(z) = \begin{cases} 1 - \eta(x) & \text{if } z = x, \\ \eta(z) & \text{if } z \neq x. \end{cases}$$

# Glauber jump rate $c(x, \eta)$

- For  $0 \leq \gamma < 1$ ,

$$c(x, \eta) = \frac{1}{d} \sum_{j=1}^d \left[ 1 + \gamma(1 - 2\eta(x))(\eta(x + e_j) + \eta(x - e_j) - 1) + \gamma^2(2\eta(x - e_j) - 1)(2\eta(x + e_j) - 1) \right],$$

where  $\gamma = \tanh \beta$ . ( $\beta$ : inverse temperature)

cf.  $d$ -dimensional nearest neighbor Ising model.

# Macroscopic behavior

- This model has been introduced by De-Masi, Ferrari and Lebowitz in 1986 to derive the reaction-diffusion equation:

$$\partial_t \rho = \Delta \rho + F(\rho), \quad \rho : [0, \infty) \times \mathbb{T}^d \rightarrow [0, 1].$$

- For  $\rho \in [0, 1]$ ,  $F(\rho)$  and  $W'(\rho) = -F(\rho)$  are given by

$$F(\rho) = B(\rho) - D(\rho)$$

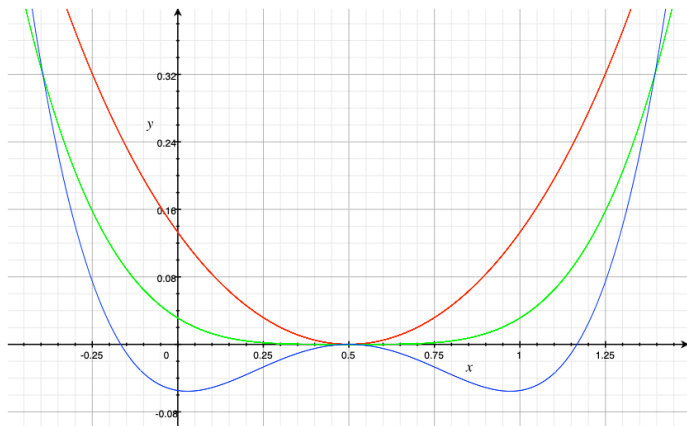
$$B(\rho) = E_{\nu_\rho}[c(0, \eta)(1 - \eta(0))]$$

$$D(\rho) = E_{\nu_\rho}[c(0, \eta)\eta(0)]$$

$$W(\rho) = (1 - 2\gamma)(\rho - 1/2)^2 + 2\gamma^2(\rho - 1/2)^4.$$



# Potential



red:  $\gamma = 1/4$ , green:  $\gamma = 1/2$ , blue:  $\gamma = 3/4$ .

# Empirical measure

- For  $\eta \in \{0, 1\}^{\mathbb{T}_N^d}$ ,

$$\pi^N(\eta; d\theta) = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta(x) \delta_{x/N}(d\theta).$$

$\delta_\theta$ : Dirac measure at  $\theta \in \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d = [0, 1)^d$ .

- Recall:  $L_N = N^2 L_E + L_G$ ; the diffusive scaling.
- Note:  $\pi_t^N(d\theta) = \pi^N(\eta_t^N; d\theta)$  is a random measure on  $\mathbb{T}^d$ .

# Hydrodynamic limit

## Theorem 1 (De Masi-Ferrari-Lebowitz, JSP, 1986)

Assume that for some measurable  $\rho_0 : \mathbb{T}^d \rightarrow [0, 1]$

$$\pi_0^N(d\theta) \rightarrow \rho_0(\theta)d\theta \text{ in prob. as } N \rightarrow \infty.$$

Then for any  $t \geq 0$ ,

$$\pi_t^N(d\theta) \rightarrow \rho(t, \theta)d\theta \text{ in prob. as } N \rightarrow \infty,$$

where  $\rho(t, \theta)$  is the unique weak sol. to

$$\begin{cases} \partial_t \rho = \Delta \rho + F(\rho), \\ \rho(0, \cdot) = \rho_0(\cdot). \end{cases}$$

## 2. Dynamical large deviations

- Fix a trajectory  $\pi(t, dx) = \phi(t, x)dx$ ,  $\phi : [0, T] \times \mathbb{T}^d \rightarrow [0, 1]$ .  
For any function  $H \in C^{1,2}([0, T] \times \mathbb{T}^d)$ , define

$$\begin{aligned}
 J^H(\pi) &= \int_{\mathbb{T}^d} \phi(T, x)H(T, x) dx - \int_{\mathbb{T}^d} \phi(0, x)H(0, x) dx \\
 &\quad - \int_0^T \int_{\mathbb{T}^d} \{ \phi \partial_t H + \phi \Delta H + \chi(\phi) |\nabla H|^2 \} dx dt \\
 &\quad - \int_0^T \int_{\mathbb{T}^d} \{ B(\phi) (e^H - 1) + D(\phi_t) (e^{-H} - 1) \} dx dt.
 \end{aligned}$$

where  $\chi(\alpha) = \alpha(1 - \alpha)$ .

- Let  $S : D([0, T], \mathcal{M}_+) \rightarrow [0, \infty]$  be the function defined by

$$S(\pi) = \begin{cases} \sup_{H \in C^{1,2}} J^H(\pi) & \text{if } \pi(t, dx) = \phi(t, x)dx, \\ \infty & \text{otherwise.} \end{cases}$$

# Previous works

- LDP:  $\mathbb{P}(\pi^n \sim \pi) \sim \exp\{-N^d S(\pi)\}$
- LDP for exclusion process  
[Kipnis-Olla-Varadhan, CPAM, 1989]
- LDP for Glauber-Kawasaki process  
[Jona Lasinio-Landim-Vares, PTRF, 1993]  
[Landim-T, AIHP, 2017]  
cf. [Farfan-Landim-T, PTRF, 2019]
- LDP for Zero-range process  
[Benois-Kipnis-Landim, SPA, 1995]  
[Fehrman-Gess, Inventiones, 2023]
- LDP for non-gradient exclusion process  
[Bertini-Faggionato-Gabrielli, AAP, 2013]

# 3. Motion by mean curvature

Glauber-Kawasaki process

$$L_N = N^2 L_{Kaw} + L_{Gla}$$

$$\xrightarrow{N \rightarrow \infty}$$

Hydrodynamic equation

$$\begin{cases} \partial_t \rho = \Delta \rho + F(\rho) \\ \rho(0, \cdot) = \chi_{\Gamma_0}(\cdot) \end{cases}$$



Glauber-Kawasaki process

$$L_N = N^2 L_{Kaw} + K L_{Gla} \xrightarrow{N \rightarrow \infty}$$

Hydrodynamic equation

$$\begin{cases} \partial_t \rho = \Delta \rho + K F(\rho) \\ \rho(0, \cdot) = \chi_{\Gamma_0}(\cdot) \end{cases}$$

Hydrodynamic equation

$$\begin{cases} \partial_t \rho = \Delta \rho + KF(\rho) \\ \rho(0, \cdot) = \chi_{\Gamma_0}(\cdot) \end{cases}$$

$K \rightarrow \infty$

Motion by mean curvature

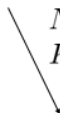
$\{\Gamma_t : t \in [0, T]\}$ : closed & smooth hypersurfaces

$$\begin{cases} V = \kappa \\ \Gamma_t|_{t=0} = \Gamma_0 \end{cases}$$

$V$ : inward normal velocity,  $\kappa$ :  $(d-1)$  × mean curvature

Glauber-Kawasaki process

$$L_N = N^2 L_{Kaw} + K_N L_{Gla}$$

$$\begin{array}{l} N \rightarrow \infty \\ K_N \rightarrow \infty \end{array}$$


Motion by mean curvature

$\{\Gamma_t : t \in [0, T]\}$ : closed & smooth hypersurfaces

$$\begin{cases} V = \kappa \\ \Gamma_t|_{t=0} = \Gamma_0 \end{cases}$$

$V$ : inward normal velocity,  $\kappa$ :  $(d-1) \times$  mean curvature

## Previous works

- [Katsoulakis-Souganidis, ARMA, 1994] ... level-set method
- [Bonaventura, Nonl Anal, 1995] ... correl. function method
- [Funaki-T, JSP, 2019] ... entropy method
- [EFHPS, CMP, 2022] ... entropy method for Glauber+Zero-range process

$$\partial_t \rho = \Delta P(\rho) + F(\rho)$$

- [F-van Meurs-S-T, JSP, 2023] ... entropy method for Glauber+gradient Kawasaki process

Question Can we study the sharp interface limit for the large deviation functional in the quasilinear setting?

## 4. Sharp interface limit for LD functional

- Fix a density evolution  $\phi : [0, T] \times \mathbb{T}^d \rightarrow [0, 1]$ .  
For any function  $H \in C^{1,2}([0, T] \times \mathbb{T}^d)$ , define

$$\begin{aligned}
 J^H(\phi) &= \int_{\mathbb{T}^d} \phi(T, x) H(T, x) dx - \int_{\mathbb{T}^d} \phi(0, x) H(0, x) dx \\
 &\quad - \int_0^T \int_{\mathbb{T}^d} \{ \phi \partial_t H + P(\phi) \Delta H + \sigma(\phi) |\nabla H|^2 \} dx dt \\
 &\quad - \int_0^T \int_{\mathbb{T}^d} \{ B(\phi) (e^H - 1) + D(\phi_t) (e^{-H} - 1) \} dx dt.
 \end{aligned}$$

where  $\sigma(\alpha) = P(\alpha)\alpha(1 - \alpha)$  and  $S(\phi) = \sup_{H \in C^{1,2}} J^H(\phi)$ .

- $S$  is the (expected) LD rate functional from the HDL of the Glauber+ gradient Kawasaki process.

We consider the smooth functions

$$P, B, D, W : [0, 1] \rightarrow \mathbb{R}$$

satisfying the following conditions:

(A1)  $P(0) = 0$  and  $P'(\rho) > 0$  for any  $\rho \in [0, 1]$ .

(A2)  $B(\rho) + D(\rho) > 0$  for any  $\rho \in [0, 1]$  and  $B - D = -W'$ .

(A3)  $W$  is a double-well potential: that is, there exist exactly three critical points  $0 < \rho_- < \rho_* < \rho_+ < 1$  such that  $W(\rho_{\pm}) < W(\rho)$  for any  $\rho \neq \rho_{\pm}$  and  $W''(\rho_{\pm}) > 0$ .

(A4)  $P$ -balance condition:

$$\int_{\rho_-}^{\rho_+} W'(\rho) P'(\rho) d\rho = 0.$$

- Let us introduce the scaled LD functional: for each  $\varepsilon > 0$ ,

$$\begin{aligned}
 J_\varepsilon^H(\phi) &= \frac{1}{\varepsilon} \int_{\mathbb{T}^d} \phi(T, x) H(T, x) dx - \frac{1}{\varepsilon} \int_{\mathbb{T}^d} \phi(0, x) H(0, x) dx \\
 &\quad - \frac{1}{\varepsilon} \int_0^T \int_{\mathbb{T}^d} \{ \phi \partial_t H + P(\phi) \Delta H + \sigma(\phi) |\nabla H|^2 \} dx dt \\
 &\quad - \frac{1}{\varepsilon^3} \int_0^T \int_{\mathbb{T}^d} \{ B(\phi) (e^H - 1) + D(\phi_t) (e^{-H} - 1) \} dx dt,
 \end{aligned}$$

and set  $S_\varepsilon(\phi) = \sup_{H \in C^{1,2}} J_\varepsilon^H(\phi)$ .

Question What is the limit of  $S_\varepsilon(\phi)$  when  $\phi$  creates a separating interface  $\{\Gamma_t\}_{t \in [0, T]}$ ?



- Let  $\Gamma = \{\Gamma_t\}_{t \in [0, T]}$  be a family of smooth hypersurfaces.
- $V_t$ : the inward normal velocity of  $\Gamma_t$
- $\kappa_t$ : the mean curvature of  $\Gamma_t$
- $d(t, \cdot)$ : the signed distance from  $\Gamma_t$
- Define the action functional  $S_{\text{ac}}(\Gamma)$  by

$$S_{\text{ac}}(\Gamma) = \int_0^T \int_{\Gamma_t} \frac{(V_t - \theta \kappa_t)^2}{4\mu} d\mathcal{H}^{d-1} dt,$$

$\mathcal{H}^{d-1}$ : the  $(d - 1)$ -dimensional Hausdorff measure of  $\mathbb{T}^d$ .

- We choose the constants  $\theta$  and  $\mu$  later.  
 $\theta$ : mobility,  $\mu$ : transport coefficient

- Let  $\bar{u} : \mathbb{R} \rightarrow (0, 1)$  solve the following one-dimensional ODE:

$$\begin{cases} (P(\bar{u}))'' + B(\bar{u}) - D(\bar{u}) = 0 & \text{in } \mathbb{R}, \\ \bar{u}(\pm\infty) = \rho_{\pm}, \quad \bar{u}(0) = \frac{\rho_+ + \rho_-}{2}. \end{cases}$$

- For each bounded  $Q : [0, T] \times \mathbb{T}^d \times \mathbb{R} \rightarrow \mathbb{R}$ , set

$$\phi_{\varepsilon}(t, x) = \bar{u} \left( \frac{d(t, x)}{\varepsilon} + \varepsilon Q \left( t, x, \frac{d(t, x)}{\varepsilon} \right) \right).$$

- Note that for each  $t \in [0, T]$ ,

$$\lim_{\varepsilon \rightarrow 0} \phi_{\varepsilon}(t, \cdot) = \begin{cases} \rho_+ & \text{on the exterior of } \Gamma_t, \\ \rho_- & \text{on the interior of } \Gamma_t. \end{cases}$$

- Define the linear operator  $L_{\bar{u}}$  by

$$L_{\bar{u}}\psi(\xi) = [2\sigma(\bar{u}(\xi))\psi'(\xi)]' - [B(\bar{u}(\xi)) + D(\bar{u}(\xi))]\psi(\xi).$$

- Let  $\bar{v} = P(\bar{u})$  and  $\nu$  be the constant defined by

$$\nu = \langle \bar{v}', (-L_{\bar{u}})\bar{v}' \rangle_{L^2}/2.$$

- Define the constants  $\theta_1, \theta_2$  by

$$\theta_1 = \int_{\rho_-}^{\rho_+} \sqrt{2\widetilde{W}(\rho)} d\rho, \quad \theta_2 = \int_{\rho_-}^{\rho_+} P'(\rho)\sqrt{2\widetilde{W}(\rho)} d\rho, \quad (1)$$

where the function  $\widetilde{W}$  is defined as

$$\widetilde{W}(\rho) = \int_{\rho_-}^{\rho} W'(\tilde{\rho})P'(\tilde{\rho}) d\tilde{\rho}. \quad (2)$$

- Then define the mobility  $\mu$  and the transport coefficient  $\theta$  by

$$\mu := \nu/\theta_1^2, \quad \theta := \theta_2/\theta_1.$$

## Theorem 2 (Kagaya-T, 24+)

Let  $\Gamma = \{\Gamma_t\}_{t \in [0, T]}$  be a family of smooth hypersurfaces. For smooth functions  $Q : [0, T] \times \mathbb{T}^d \times \mathbb{R} \rightarrow \mathbb{R}$ , define

$$\phi_\varepsilon(t, x) = \bar{u} \left( \frac{d(t, x)}{\varepsilon} + \varepsilon Q \left( t, x, \frac{d(t, x)}{\varepsilon} \right) \right).$$

Then there exists  $\hat{Q} = \hat{Q}(\Gamma)$  such that, choosing  $Q = \hat{Q}$ , it holds that

$$\lim_{\varepsilon \rightarrow 0} S_\varepsilon(\phi_\varepsilon) = S_{\text{ac}}(\Gamma).$$

## Related results

- Stochastic reaction-diffusion equation

$$I_\varepsilon(\rho) = \frac{1}{4} \int_0^T \int_{\mathbb{T}^d} (\partial_t \rho - \Delta \rho + \varepsilon^{-2} W'(\rho))^2 dx dt.$$

[Kohn-Otto-Reznikoff-Vanden Eijnden, 2007, CPAM]

- Glauber dynamics for Ising systems with Kac potentials
- Glauber+Kawasaki process ( $P(u) = u$ )  
[Bertini-Buttà-Pisante, 2019, AHP]
- In any cases, the limit functional is given by

$$S_{\text{ac}}(\Gamma) = \int_0^T \int_{\Gamma_t} \frac{(V_t - \theta \kappa_t)^2}{4\mu} d\mathcal{H}^{d-1} dt.$$

## Some remarks

- For the Glauber dynamics for Ising systems with Kac potentials, the physical constants  $\theta, \mu$  can be explicitly calculated in terms of the functions appearing in equilibrium statistical mechanics.
- Recall  $\bar{v} = P(\bar{u})$ .  $\bar{v}$  satisfies the ODE

$$\begin{cases} \bar{v}'' + B \circ P^{-1}(\bar{v}) - D \circ P^{-1}(\bar{v}) = 0 & \text{in } \mathbb{R}, \\ \bar{v}(\pm\infty) = P(\rho_{\pm}), \quad \bar{v}(0) = P\left(\frac{\rho_+ + \rho_-}{2}\right). \end{cases}$$

- Note  $\bar{u} = \bar{v}$  in the case of  $P(u) = u$  ([BBP, 19, AHP]).  
→ We must care about the difference between  $\bar{u}$  and  $\bar{v}$ .

# Sketch of the proof

- For each smooth  $\phi_\varepsilon$ , one can solve the equation in  $H$

$$\partial_t \phi_\varepsilon = \Delta P(\phi_\varepsilon) - \nabla(\sigma(\phi_\varepsilon)\nabla H) + \frac{1}{\varepsilon^2} (B(\phi_\varepsilon)e^H - D(\phi_\varepsilon)e^{-H}).$$

- Let  $H_\varepsilon$  be the solution. Using  $H_\varepsilon$ ,  $S_\varepsilon(\phi_\varepsilon)$  is expressed as

$$\begin{aligned} S_\varepsilon(\phi_\varepsilon) &= \frac{1}{\varepsilon} \int_0^T \int_{\mathbb{T}^d} \sigma(\phi_\varepsilon) |\nabla H_\varepsilon|^2 \, dxdt \\ &\quad + \frac{1}{\varepsilon^3} \int_0^T \int_{\mathbb{T}^d} B(\phi_\varepsilon) (1 - e^{H_\varepsilon} + H_\varepsilon e^{H_\varepsilon}) \, dxdt \\ &\quad + \frac{1}{\varepsilon^3} \int_0^T \int_{\mathbb{T}^d} D(\phi_\varepsilon) (1 - e^{-H_\varepsilon} - H_\varepsilon e^{-H_\varepsilon}) \, dxdt. \end{aligned}$$

- So we need to expand  $S_\varepsilon(\phi_\varepsilon)$  in a proper manner.
- For this purpose, we propose the expansion of the form

$$\frac{1}{\varepsilon} \int_0^T \int_{\mathbb{T}^d} A(t, x, d(x, t)/\varepsilon) dx dt + \text{error},$$

where  $A : [0, T] \times \mathbb{T}^d \times \mathbb{R} \rightarrow \mathbb{R}$  is some function.

- Then the co-area formula shows

$$\lim_{\varepsilon \rightarrow 0} S_\varepsilon(\phi_\varepsilon) = \int_0^T \int_{\Gamma_t} \int_{\mathbb{R}} A(t, x, \xi) dt dx \mathcal{H}^{d-1}(d\xi).$$



- Letting  $d_\varepsilon(t, x) = d(t, x)/\varepsilon$ , we can expand  $\phi_\varepsilon, H_\varepsilon$  in  $\varepsilon$  as

$$\phi_\varepsilon = \bar{u}(d_\varepsilon) + \varepsilon \bar{u}'(d_\varepsilon) Q_{d_\varepsilon} + \text{error},$$

$$H_\varepsilon(t, x) = \varepsilon \widehat{H}_1(t, x, d_\varepsilon) + \text{error},$$

where  $\widehat{H}_1$  is the solution to

$$\begin{aligned} L_{\bar{u}} \widehat{H}_1(t, x, \xi) &= \bar{v}'(\xi) \Delta d(t, x) + 2\bar{v}''(\xi) \partial_\xi Q(t, x, \xi) \\ &\quad + \bar{v}'(\xi) \partial_\xi^2 Q(t, x, \xi) - \bar{u}'(\xi) \partial_t d(t, x) \\ &=: F_Q(t, x, \xi), \end{aligned}$$

where

$$L_{\bar{u}} \psi(\xi) = [2\sigma(\bar{u}(\xi))\psi'(\xi)]' - [B(\bar{u}(\xi)) + D(\bar{u}(\xi))] \psi(\xi).$$

- When  $\bar{u} = \bar{v}$ , the right-hand side becomes

$$\begin{aligned} & \bar{u}'(\xi) (\Delta d(t, x) - \partial_t d(t, x)) + 2\bar{u}''(\xi) \partial_\xi Q(t, x, \xi) \\ & + \bar{u}'(\xi) \partial_\xi^2 Q(t, x, \xi). \end{aligned}$$

- If we choose  $Q(t, x, \xi) = (\Delta d(t, x) - \partial_t d(t, x)) q(\xi)$  for some function  $q(\xi)$ , finding a solution to

$$L_{\bar{u}} \widehat{H}_1(t, x, \xi) = F_Q(t, x, \xi)$$

is reduced to the one to

$$L_{\bar{u}} \widehat{\psi}(\xi) = \bar{u}'(\xi) + 2\bar{u}''(\xi) q'(\xi) + \bar{u}'(\xi) q''(\xi),$$

which is solvable since  $L_{\bar{u}}$  is invertible.

- After simple (but quite tedious) computations,

$$\lim_{\varepsilon \rightarrow 0} S_\varepsilon(\phi_\varepsilon) = \frac{1}{2} \int_0^T \int_{\Gamma_t} \int_{\mathbb{R}} (-L_{\bar{u}} \hat{H}_1) \hat{H}_1 d\xi d\mathcal{H}^{d-1}(x) dt.$$

- Solving the minimizing problem

$$\min \int_{\mathbb{R}} F_Q(-L_{\bar{u}})^{-1} F_Q d\xi,$$

for each  $t, x$ , and choosing  $Q$  as the minimizer  $\hat{Q}$ , we get

$$\lim_{\varepsilon \rightarrow 0} S_\varepsilon(\phi_\varepsilon) = \int_0^T \int_{\Gamma_t} \frac{(V_t - \theta \kappa_t)^2}{4\mu} d\mathcal{H}^{d-1} dt = S_{\text{ac}}(\Gamma).$$

# 5. Summary of the talk

# Summary of the talk

## Previous result

- Dynamical large deviations [Landim-T, AIHP, 2018]
- Motion by mean curvature  
[Funaki-T, 2019, JSP]  
[Funaki-van Meurs-Sethuraman-T, 2023, JSP]

## Main result

- Sharp interface limit of the LD rate functional for the Glauber+gradient Kawasaki process  
generalization of [BBP, 2019, AHP] to the quasilinear setting

## Future work

- $\Gamma$ -convergence of rate functionals.
- LDP from the MMC.

**Thank you for your attention.**