Refined Cauchy/Littlewood identities and their applications to KPZ models

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Reference:

TI-Mucciconi-Sasamoto, Forum of Mathematics, Pi 11(e27) 1-101, 2023,

TI-Mucciconi-Sasamoto, arXiv:2204.08420

Cauchy identity

$$\sum_{\lambda\in\mathbb{Y}} s_\lambda(a_1,\cdots,a_n)s_\lambda(b_1,\cdots,b_n)=\prod_{i,j=1}^{n}\frac{1}{1-a_ib_j}.$$

- $\lambda\in\mathbb{Y}:=\{(\lambda_1,\lambda_2,\cdots)\in\mathbb{Z}_{\geq 0}|\lambda_1\geq\lambda_2\geq\cdots\geq 0\}$: partition (Young diagram)
- $s_\lambda(a_1,\cdots,a_n)$: Schur function, a symmetric polynomial of $a_1,\cdots,a_n$.
- Equivalent relation $\sum_{\lambda\in\mathbb{Y}}\mathbb{S}_{a,b}(\lambda)=1$.

  Schur measure $\mathbb{S}_{a,b}$: $0\leq a_i\leq 1,\ 0\leq b_i\leq 1,\ i=1,\cdots,n$

$$\mathbb{S}_{a,b}:=\frac{1}{Z_S}s_\lambda(a_1,\cdots,a_n)s_\lambda(b_1,\cdots,b_n),\ \text{Probability measure on } \mathbb{Y},$$

$$Z_S=\prod_{i,j=1}^{n}\frac{1}{1-a_ib_j}.$$

c.f. $s_\lambda(a_1,\cdots,a_n)\geq 0$ for $\forall\lambda\in\mathbb{Y}_n$.

Schur measure

$$\mathbb{S}_{a,b} := \frac{1}{Z_S} s_\lambda(a_1, \cdots, a_n) s_\lambda(b_1, \cdots, b_n), \quad Z_S = \prod_{i,j=1}^{n} \frac{1}{1 - a_i b_j}.$$

- KPZ models: Johansson 2000, $\cdots$

$$\mathbb{P}_{\mathrm{KPZ}}(X \le k) = \mathbb{P}_{\mathbb{S}_{a,b}}(\lambda_1 \le k), \text{for } k = 0, 1, \cdots$$

X : the current in the TASEP with the step initial condition,
particle position in the pushTASEP with the step initial condition,
energy of directed polymer model, height in the PNG model, etc.

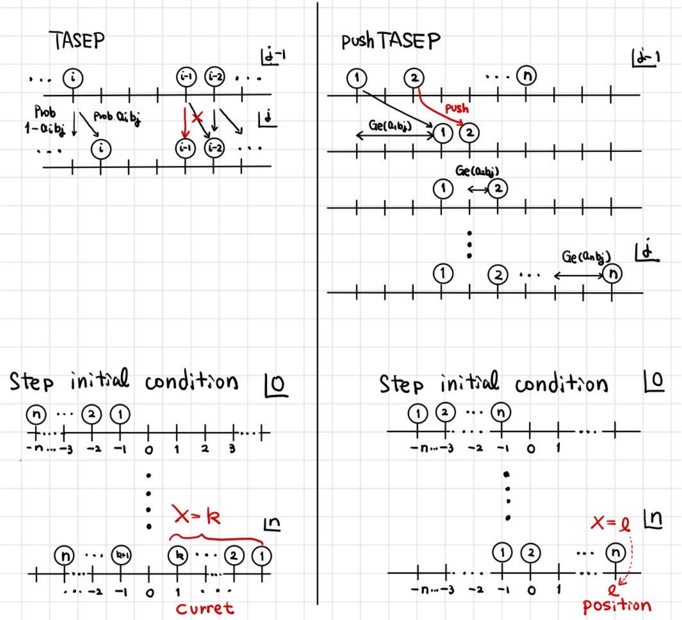
- Determinantal point process: Okounkov 1999

$m_k(x_1, \cdots, x_k) := \mathbb{P}_{\mathbb{S}_{a,b}}(\lambda = (\lambda_1, \cdots, \lambda_n) \supset (x_1, \cdots, x_k))$
: the k-point correlation function

$$m_k(x_1, \cdots, x_k) = \det (K(x_i, x_j))_{i,j=1,\cdots,k}, \quad \forall k \le n,$$

$$K(x, y) = \frac{1}{(2\pi \mathrm{i})^2} \oint_{|z|=r} \frac{\mathrm{d}z}{z^x} \oint_{|w|=r'} \frac{\mathrm{d}w}{w^{-y}} \frac{w}{z - w} \prod_{i=1}^{n} \frac{1 - a_i w}{1 - a_i z} \frac{1 - b_i/z}{1 - b_i/w}$$

TASEP and pushTASEP



Determinantal formula for the KPZ models

- Combining them, we have

$$\mathbb{P}_{\mathrm{KPZ}}(X \le c) = \det(1-K)_{\ell^2(k,k+1,\cdots)} : \text{Fredholm determinant formula}$$

$$:= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{x_1=c}^{\infty} \cdots \sum_{x_k=c}^{\infty} \det(K(x_i,x_j))_{i,j=1,\cdots,k}$$

- KPZ universality class:Johansson 2000

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{X-An}{Cn^{\frac{1}{3}}} \le s\right) = F_2(s) \text{ GUE Tracy-Widom distribution}$$

$$F_2(s) = \det(1-\mathcal{K}_{\mathrm{Ai}})_{L^2(s,\infty)} := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_s^{\infty} dx_1 \cdots \int_s^{\infty} dx_k \det(\mathcal{K}_{\mathrm{Ai}}(x_i,x_j))_{i,j=1}^{k},$$

$$\mathcal{K}_{\mathrm{Ai}}(x,y) := \int_0^{\infty} d\lambda Ai(x+\lambda)Ai(y+\lambda).$$

Littlewood identity

$$\sum_{\lambda \in \mathbb{Y}_p} s_\lambda(a_1, \cdots, a_n) = \prod_{1 \le i < j \le n} \frac{1}{1 - a_i a_j}$$

- $\mathbb{Y}_p := \{(\lambda_1, \lambda_2, \cdots) \in \mathbb{Y} | \lambda_1 = \lambda_2, \ \lambda_3 = \lambda_4 \cdots\}$
- cf: the Cauchy identity

$$\sum_{\lambda \in \mathbb{Y}} s_\lambda(a_1, \cdots, a_n) s_\lambda(b_1, \cdots, b_n) = \prod_{i,j=1}^{n} \frac{1}{1 - a_i b_j}.$$

- Pfaffian Schur measure $\mathbb{PS}_a$: $0 \le a_i \le 1, \ i = 1, \cdots, n$

$$\mathbb{PS}_a := \frac{1}{Z_{PS}} s_\lambda(a_1, \cdots, a_{n+1}), \ \text{Probability measure on } \mathbb{Y}_p,$$

$$Z_{PS} = \prod_{1 \le i < j \le n+1} \frac{1}{1 - a_i a_j}$$

Pfaffian Schur measure

$$\mathbb{PS}_a := \frac{1}{Z_{PS}} s_\lambda(a_1, \cdots, a_{n+1}), \;\; Z_{PS} = \prod_{1 \le i < j \le n+1} \frac{1}{1 - a_i a_j}$$

- KPZ models in half space: Borodin-Rains 2005, $\cdots$

$$\mathbb{P}_{\mathrm{KPZ}}(X = k) = \mathbb{P}_{\mathbb{PS}_a}(\lambda_1 = k), \text{for } k = 0, 1, \cdots$$

X : position of the nth particle in pushTASEP with particle creation

current in the TASEP in half space, etc.

- Pfaffian point process: Borodin-Rains 2005

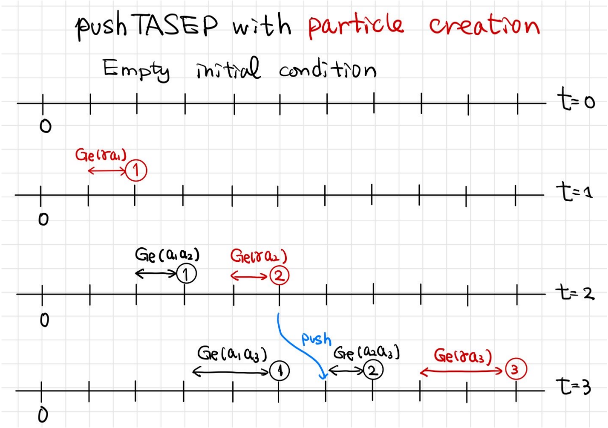
$m_k(x_1, \cdots, x_k) := \mathbb{P}_{\mathbb{PS}_a}(\lambda = (\lambda_1, \cdots, \lambda_n) \supset (x_1, \cdots, x_k))$

: the k-point correlation function

$$m_k(x_1, \cdots, x_k) = \mathrm{pf}\,(K(x_i, x_j))_{i,j=1,\cdots,k}, \quad \forall k \le n,$$

- Combining them, we have

$$\mathbb{P}_{\mathrm{KPZ}}(X \le k) = \mathrm{Pf}(1 - K)_{\ell^2(k,k+1,\cdots)} \text{ : Fredholm Pfaffian formula}$$



X: position of the nth particle at $t = n$. Set $a_{n+1} = \gamma$ in $\mathbb{PS}_a$, then

$$\mathbb{P}_{\mathrm{KPZ}}(X = c) = \mathbb{P}_{\mathbb{PS}_a}(\lambda_1 = c), \text{for } c = 0, 1, \cdots.$$

Kardar-Parisi-Zhang (KPZ) equation in 1dim full space

$$\partial_t \mathcal{H} = \frac{1}{2}\partial_x^2 \mathcal{H} + \frac{1}{2}(\partial_x \mathcal{H})^2 + \dot{W}, \quad x \in \mathbb{R}, t \in \mathbb{R}_+, \quad \mathbb{E}\left(\dot{W}(x,t)\dot{W}(y,s)\right) = \delta_{x-y}\delta_{t-s}$$

- Bertini-Giacomin 1995, Hairer 2014, Gubinelli-Imkeller-Perkowski 2015 Well posedness
- Sasamoto-Spohn 2010, Amir-Corwin-Quastel 2010: $\gamma_t := (t/2)^{1/3}$, $\mathcal{Z}(x,0) = \delta_x$

$$\mathbb{E}\left[e^{-e^{\mathcal{H}(0,t)+\frac{t}{24}-\gamma_t s}}\right] = \det\left(1 - \mathcal{K}_{\mathrm{Ai}}^{(\gamma_t)}\right)_{L^2((s,\infty))}$$

$$\mathcal{K}_{\mathrm{Ai}}^{(\alpha)}(x_1,x_2) = \int_{-\infty}^{\infty} dy \frac{1}{1+e^{-\alpha y}} \mathrm{Ai}(x_1+y)\mathrm{Ai}(x_2+y),$$

$\rightsquigarrow$ Integrable probability: Borodin-Corwin 2011 Macdonald processes, Borodin-Corwin-Sasamoto 2014 Markov duality/Bethe ansatz $\cdots$

- Question:

Can we construct the discrete models having the similar determinantal sturucture ?

Why does the determinant appears?

Two generalizations of the Cauchy identity

- Cauchy identity

$$\sum_{\lambda} s_\lambda(a_1, \cdots, a_n) s_\lambda(b_1, \cdots, b_n) = \prod_{i,j=1}^{n} \frac{1}{1 - a_i b_j},$$

- $s_\mu \to P_\mu$: q-Whittaker polynomial

$$\sum_{\mu \in \mathbb{Y}_n} b_\mu(q) P_\mu(a; q) P_\mu(b; q) = \prod_{i,j=1}^{n} \frac{1}{(a_i b_j; q)_\infty},$$

where $b(\mu) = \prod_{1 \le j} \frac{1}{(q;q)_{\mu_j - \mu_{j+1}}}$,
the q-Pochhammer symbols: $(x; q)_n := \prod_{i=1}^{n} (1 - x q^{i-1})$,
$(x; q)_\infty := \prod_{i=1}^{\infty} (1 - x q^{i-1})$

- $s_\lambda \to s_{\lambda/\rho}$: the skew Schur polynomial

$$\sum_{\lambda, \rho} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b) = \frac{1}{(q; q)_\infty} \prod_{i,j=1}^{n} \frac{1}{(a_i b_j; q)_\infty}$$

- In $q = 0$, both of them goes to the original Cauchy identity.

q-Whittaker measure

$0 < q < 1,\ 0 < a_i,\ \ b_i < 1,\ \ i = 1, \cdots, n,\ \mu \in \mathbb{Y}_n$

$$\mathbb{W}^{(q)}_{a,b}(\mu) := \frac{1}{Z_{qW}} b_\mu(q) P_\mu(a_1, \cdots, a_n; q) P_\mu(b_1, \cdots, b_n; q),\ \ Z_{qW} = \prod_{i,j=1}^{n} \frac{1}{(a_i b_j; q)_\infty}$$

- introduced by Borodin-Corwin 2014
- In $q = 0$, $\mathbb{W}^{(q)}_{a,b}(\mu) = \mathbb{S}_{(a,b)}$: Schur measure
- KPZ models:

  TASEP, pushTASEP$\to$ q-TASEP Borodin-Corwin 2014, O'Connell-Pei 2013,

  q-PushTASEPMatveev-Petrov2015

  Directed polymer models $\to$ Log-Gamma polymer model

  Corwin-O'Connell-Seppalainen-Zygouras 2014

  O'Connell-Yor model O'Connell 2012

  PNG model $\to$ t-PNG model Aggarwal, Borodin, and Wheeler 2021
- Determinantal formulas Borodin-Cowin 2014$\cdots$

Periodic Schur measure

$0 < q < 1,\ 0 < a_i,\ \ b_i < 1,\ \ i = 1, \cdots, n,\ \lambda \in \mathbb{Y}_n,\ a := (a_1, \cdots, a_n),\ \ b = (b_1, \cdots, b_n)$

$$\mathbb{S}_{a,b}^{(q)}(\lambda) = \frac{1}{Z_{S(q)}} \sum_{\rho(\subset\lambda)} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b),\ \ Z_{S(q)} := \frac{1}{(q;q)_\infty} \prod_{i,j=1}^{n} \frac{1}{(a_i b_j; q)_\infty}$$

- Introduced by Borodin (2007)
- The case $q = 0$: $\mathbb{S}_{a,b}^{(0)}(\mu) = \mathbb{S}_{(a,b)}$: Schur measure
- Determinantal point process Borodin 2007

  Let $\lambda \sim \mathbb{S}_{a,b}^{(q)}$, $S \sim \mathrm{Theta}(\zeta, q)$ and λ and S are independent. Then $\tilde{\lambda} := (\lambda_1 + S, \lambda_2 + S, \cdots)$ is the determinantal point process $\sim$ free fermions at positive temperature Betea-Bouttier (2019)

Two generalizations of the Cauchy identity

- Cauchy identity

$$\sum_{\lambda} s_\lambda(a_1,\cdots,a_n)s_\lambda(b_1,\cdots,b_n)=\prod_{i,j=1}^{n}\frac{1}{1-a_ib_j},$$

- $s_\mu \to P_\mu$: q-Whittaker polynomial

$$\sum_{\mu\in\mathbb{Y}_n} b_\mu(q)P_\mu(a;q)P_\mu(b;q)=\prod_{i,j=1}^{n}\frac{1}{(a_ib_j;q)_\infty}.$$

- $s_\lambda \to s_{\lambda/\rho}$: the skew Schur polynomial

$$\sum_{\lambda,\rho} q^{|\rho|}s_{\lambda/\rho}(a)s_{\lambda/\rho}(b)=\frac{1}{(q;q)_\infty}\prod_{i,j=1}^{n}\frac{1}{(a_ib_j;q)_\infty}$$

- Using $1/(q;q)_\infty=\sum_\nu q^{|\nu|}$, we have the following identity

$$\sum_{\mu,\nu} q^{|\nu|}b_\mu(q)P_\mu(a;q)P_\mu(b;q)=\sum_{\lambda,\rho} q^{|\rho|}s_{\lambda/\rho}(a)s_{\lambda/\rho}(b)$$

Refined Cauchy identity

Theorem [TI-Mucciconi-Sasamoto, Forum Math. Pi, 11, e-27, 2023]

For $\ell = 0, 1, 2, \cdots$, we have

$$\sum_{\substack{\mu,\nu \\ \mu_1+\nu_1=\ell}} q^{|\nu|} b_\mu(q) P_\mu(a;q) P_\mu(b;q) = \sum_{\substack{\lambda,\rho \\ \lambda_1=\ell}} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b)$$

- Equivalent relation: $\mathbb{P}(\mu_1 + \chi = \ell) = \mathbb{P}(\lambda_1 = \ell),\ \ell = 0, 1, 2, \cdots$ where $\mu \sim \mathbb{W}^{(q)}_{a,b},\ \lambda \sim \mathbb{S}^{(q)}_{a,b},\ \chi \sim q\mathrm{Geo}(q)$, i.e. $\mathbb{P}(\chi = k) = q^k (q;q)_k/(q;q)_\infty$, $k = 0, 1, \cdots$
- LHS: KPZ $\mathbb{P}(X \le \ell) = \mathbb{P}(\mu_1 \le \ell)$
- RHS:DPP $\mathbb{P}(\lambda_1 + S \le k) = \det(1 - K)$
- Combining them we have $\mathbb{P}(X + \chi + S \le \ell) = \det(1 - K)$

Bijective approach

TI-Mucciconi-Sasamoto, Forum Math. Pi, 11, e27, 2023

$$(V, W; \kappa; \nu) \Leftrightarrow (P, Q)$$

with the properties $\mu_1 + \nu_1 = \lambda_1, \quad H(V) + H(W) + |\kappa| + |\nu| = |\rho|$.

- $V, W \in \mathrm{VST}(\mu, n)$: Vertically strict tablueax, e.g.

1	2	2	3
2	5	3	
3			

  $H(V)$: energy of V
- $\kappa = (\kappa_1, \cdots, \kappa_{\mu_1}) \in \mathcal{K}(\mu) := \{\kappa \in \mathbb{N}_0^{\mu_1} : \kappa_i \geq \kappa_{i+1} \text{ if } \mu'_i = \mu'_{i+1}\}$
- $\nu \in \mathbb{Y}$
- $P, Q \in \mathrm{SYT}(\lambda/\rho, n)$: Semistandard Young tableaux, e.g.

			2
	1	3	3
2	2	5	
3			

As a corollary we get

$$\sum_{\substack{\mu,\nu \in \mathcal{P} \\ \mu_1+\nu_1 \leq n}} q^{|\nu|} b_\mu(q) P_\mu(a; q) P_\mu(b; q) = \sum_{\substack{\lambda,\rho \in \mathcal{P} \\ \rho \subset \lambda, \lambda_1 \leq n}} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b).$$

Bijective approach

- $V, W \in \mathrm{VST}(\mu, n)$: Vertically strict tablueax, $H(V)$: energy of V

  Sanderson 2000, Schlling-Tingley 2012: Demazure character formula

$$P_\mu(a_1, \cdots, a_n; q) = \sum_{V \in \mathrm{VST}(\mu)} q^{H(V)} a_1^{\sharp 1(V)} \cdots a_n^{\sharp n(V)}$$

- $\kappa = (\kappa_1, \cdots, \kappa_{\mu_1}) \in \mathcal{K}(\mu) := \{\kappa \in \mathbb{N}_0^{\mu_1} : \kappa_i \geq \kappa_{i+1} \text{ if } \mu'_i = \mu'_{i+1}\}$

$$b_\mu(q) = \sum_{\kappa \in \mathcal{K}(\mu)} q^{|\kappa|}$$

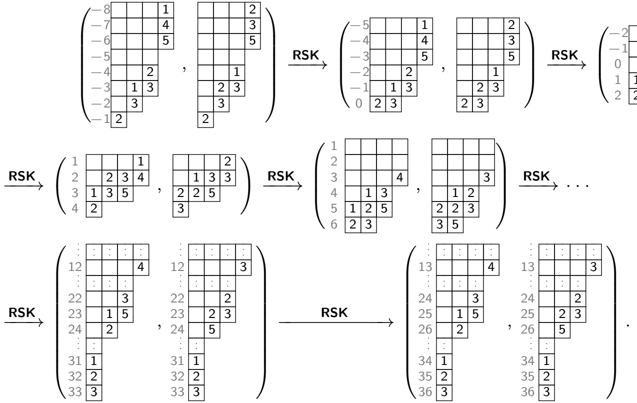
- $P, Q \in \mathrm{SYT}(\lambda/\rho, n)$: Semistandard Young tableaux,

$$s_{\lambda/\rho}(a_1, \cdots, a_n) = \sum_{P \in \mathrm{SYT}(\lambda/\rho, n)} a_1^{\sharp 1(P)} \cdots a_n^{\sharp n(P)}$$

As a corollary we get

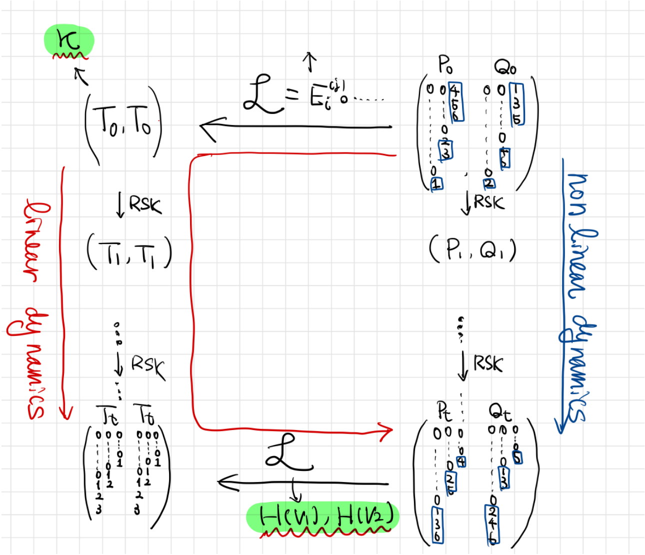
$$\sum_{\substack{\mu, \nu \in \mathcal{P} \\ \mu_1 + \nu_1 \leq n}} q^{|\nu|} b_\mu(q) P_\mu(a; q) P_\mu(b; q) = \sum_{\substack{\lambda, \rho \in \mathcal{P} \\ \rho \subset \lambda, \lambda_1 \leq n}} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b).$$

Skew RSK dynamics



- Simirlar to the Box-ball system (BBS)
- $(\sqcup_{\lambda,\rho}\mathrm{SYT}(\lambda/\rho, n) \times \mathrm{SYT}(\lambda/\rho, n), \{E_j\ (i), F_j^{(i)}\}_{i=0,1,j=1,\cdots,n})$: Affine bicrystal structure
- $E_j^{(i)} \circ \mathbf{RSK} = \mathbf{RSK} \circ E_j^{(i)}$, $F_j^{(i)} \circ \mathbf{RSK} = \mathbf{RSK} \circ F_j^{(i)}$

$(V, W, \kappa, \nu) \leftrightarrow (P, Q)$



Two generalizations of the Littlewood identity

- Littlewood identity

$$\sum_{\lambda' \in \mathbb{Y}_{\text{even}}} s_\lambda(a_1, \cdots, a_n) = \prod_{1 \le i < j \le n} \frac{1}{1 - a_i a_j}$$

- $s_\mu \to P_\mu$: q-Whittaker polynomial, $b^{\text{ev}}_\mu(q) = \prod_{j=2,4,6,\cdots} \frac{1}{(q;q)_{\mu_j - \mu_{j+1}}}$

$$\sum_{\mu' : \text{even}} b^{\text{ev}}_\mu(q) P_\mu(a; q) = \prod_{1 \le i < j \le n} \frac{1}{(a_i a_j; q)_\infty}$$

Half q-Whittaker measure: $\mathbb{HW}^{(q)}_a(\mu) := \frac{1}{Z_{hqW}} b^{\text{ev}}_\mu(q) P_\mu(a; q)$

- $s_\lambda \to s_{\lambda/\rho}$: the skew Schur polynomial

$$\sum_{\lambda', \rho' : \text{even}} q^{|\rho|/2} s_{\lambda/\rho}(a) = \frac{1}{(q;q)_\infty} \prod_{1 \le i < j \le n} \frac{1}{(a_i a_j; q)_\infty}$$

Free boundary Schur measure: $\mathbb{FBS}^{(q)}_a(\lambda) := \frac{1}{Z_{fbS}} \sum_{\rho' : \text{even}} q^{|\rho|/2} s_{\lambda/\rho}(a)$

Half q-Whikkater measure (special case)

$$\mathbb{HW}_a^{(q)}(\mu) := \frac{1}{Z_{hqW}} b_\mu^{\mathrm{ev}}(q) P_\mu(a;q), \ Z_{hqW} = \prod_{1\le i<j\le n} \frac{1}{(a_i a_j;q)_\infty}.$$

- Introduced by Barraquand-Borodin-Corwin 2020
- KPZ models on half space $\mathbb{P}_{KPZ}(X \le k) = \mathbb{P}_{\mathbb{HW}^{(q)}(a)}(\mu_1 \le k)$

  X=the position of the rightmost particle in q-pushTASEP with particle creation

  the current at the origin in ASEP on half line

  the free energy of the log-Gamma polymer model in half space

  the KPZ equation in half space
- The Fredholm Pffafian formulas are less known.

Free boundary Schur measure (special case)

$$\mathbb{FBS}_a^{(q)}(\lambda) = \frac{1}{Z_{fbS}} \sum_{\rho:\rho' \text{ is even}} q^{|\rho|/2} s_{\lambda/\rho}(a).$$

- Introduced by Betea-Bouttier-Nejjar-Vuletic 2018
- Pfaffian point process Betea-Bouttier-Nejjar-Vuletic 2018 Let $\lambda \sim \mathbb{FBS}_a^{(q)}$, $\tilde{S} \sim \text{Theta}(\zeta^2, q^2)$ and λ and $\tilde{S}$ are independent.
  Then $(\lambda_1 + 2\tilde{S}, \lambda_2 + 2\tilde{S}, \cdots)$ is a Pfaffian point process
- We have

$$\mathbb{P}[\lambda_1 + 2\tilde{S} \le s] = \text{Pf}(J - L)_{\ell^2(\mathbb{Z}'_{>s})} : \text{Fredholm Pfaffian}$$

Two generalizations of the Littlewood identity

- Littlewood identity

$$\sum_{\lambda' \in \mathbb{Y}_{\text{even}}} s_\lambda(a_1, \cdots, a_n) = \prod_{1 \le i < j \le n} \frac{1}{1 - a_i a_j}$$

- $s_\mu \to P_\mu$: q-Whittaker polynomial, $b^{\text{ev}}_\mu(q) = \prod_{j=2,4,6,\cdots} \frac{1}{(q;q)_{\mu_j - \mu_{j+1}}}$

$$\sum_{\mu' : \text{even}} b^{\text{ev}}_\mu(q) P_\mu(a;q) = \prod_{1 \le i < j \le n} \frac{1}{(a_i a_j; q)_\infty}$$

- $s_\lambda \to s_{\lambda/\rho}$: the skew Schur polynomial

$$\sum_{\lambda', \rho' : \text{even}} q^{|\rho|/2} s_{\lambda/\rho}(a) = \frac{1}{(q;q)_\infty} \prod_{1 \le i < j \le n} \frac{1}{(a_i a_j; q)_\infty}$$

- From the above two identities, we have

$$\sum_{\mu', \nu' : \text{even}} q^{|\nu|/2} b^{\text{ev}}_\mu(q) P_\mu(a;q) = \sum_{\lambda', \rho' : \text{even}} q^{|\rho|/2} s_{\lambda/\rho}(a)$$

Refined Littlewood identity

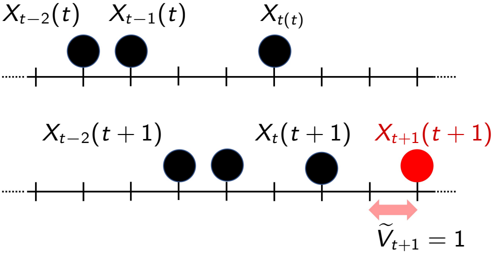
Theorem [TI-Mucciconi-Sasamoto, Forum Math. Pi, 11, e-27, 2023]

For $\ell = 0, 1, 2, \cdots$, we have

$$\sum_{\substack{\mu', \nu' : \text{even} \\ \mu_1 + \nu_1 = \ell}} q^{|\nu|/2} b_\mu^{\text{ev}}(q) P_\mu(a; q) = \sum_{\substack{\lambda', \rho' : \text{even} \\ \lambda_1 = \ell}} q^{|\rho|/2} s_{\lambda/\rho}(a)$$

- Equivalent relation: $\mathbb{P}(\mu_1 + \chi = \ell) = \mathbb{P}(\lambda_1 = \ell), \ \ell = 0, 1, 2, \cdots$ where $\mu \sim \mathbb{HW}_a^{(q)}, \ \lambda \sim \mathbb{FBS}_a^{(q)}, \ \chi \sim q\text{Geo}(q)$, i.e. $\mathbb{P}(\chi = k) = q^k (q; q)_k / (q; q)_\infty$, $k = 0, 1, \cdots$
- LHS: KPZ $\mathbb{P}(X \le \ell) = \mathbb{P}(\mu_1 \le \ell)$
- RHS:PPP $\mathbb{P}(\lambda_1 + 2\tilde{S} \le k) = \text{pf}(J - L)$
- Combining them we have $\mathbb{P}(X + \chi + 2\tilde{S} \le \ell) = \text{pf}(J - L)$

q-pushTASEP with particle creation



$X_j(t),\ j = 1, \cdots, N,\ t \in \mathbb{Z}_{\geq 0}$: position of the jth particle at time t

$(X_1(t) < \cdots < X_N(t))$

- Introduced by Barraquand-Borodin-Corwin 2020
- Assume there are t particles $X_1(t) < \cdots < X_t(t)$ at time t. Then at $t+1$, update all t particles folloiwng the rule of $q-$pushTASEP with $V_{k,t} \sim q\mathrm{Geo}(a_k a_{t+1})$.
- Once the position of the rightmost particle $X_t^{\mathrm{hs}}(t+1)$ has been determined, a new particle is added to its right and $X_{t+1}(t+1) = X_t(t+1) + 1 + \widetilde{V}_{t+1}$, where $\widetilde{V}_{t+1} \sim q\mathrm{Geo}(\gamma a_{t+1})$. (We set $X_0(1) = 0$)

Pfaffian formula for the q-push TASEP with particle creation

TI-Mucciconi-Sasamoto arxiv:2204.08420 Cor.5.8

$$\mathbb{P}\left(X_N^{\mathrm{hs}}(t) - N + \chi + 2\tilde{S} \le s\right) = \mathrm{Pf}\,(J - L)_{\ell^2(\mathbb{Z}'_{>s})},$$

where $X_N^{\mathrm{hs}}(t)$: Nth particle positon in q-pushTASEP with particle creation
$\chi \sim q\mathrm{Geo}(q)$, $\tilde{S} \sim \mathrm{Theta}(\zeta^2, q^2)$

- Idea of proof:

$$X_N(t) - N + \chi + 2\tilde{S} \overset{\mathcal{D}}{=} \mu_1 + \chi + 2\tilde{S}$$
$$\overset{\mathcal{D}}{=} \lambda_1 + 2\tilde{S} = \mathrm{Pf}\,(J - L)_{\ell^2(\mathbb{Z}'_{>s})}$$

- $X_N^{\mathrm{hs}}(t)$ is related to other important KPZ models in half space: the log-gamma polymer model and the stochastic heat equation (the KPZ equation)

Correlation kernel

$$L(x,y) = \begin{pmatrix} k(x,y) & -\nabla_y k(x,y) \\ -\nabla_x k(x,y) & \nabla_x \nabla_y k(x,y) \end{pmatrix},$$

where the difference operator ∇_x is defined by $\nabla_x f(x) = \frac{1}{2}[f(x+1) - f(x-1)]$ and

$$k(x,y) = \frac{1}{(2\pi \mathrm{i})^2} \oint_{|z|=r} \frac{\mathrm{d}z}{z^{x+3/2}} \oint_{|w|=r} \frac{\mathrm{d}w}{w^{y+5/2}} F(z)F(w)\kappa^{\mathrm{hs}}(z,w),$$

$$F(z) = \frac{(\gamma/z;q)_\infty}{(\gamma z;q)_\infty} \prod_{i=1}^{N} \frac{(a_i/z;q)_\infty}{(a_i z;q)_\infty},$$

$$\kappa^{\mathrm{hs}}(z,w) = \frac{(q,q,w/z,qz/w;q)_\infty}{(1/z^2,1/w^2,1/zw,qwz;q)_\infty} \frac{\vartheta_3(\zeta^2 z^2 w^2;q^2)}{\vartheta_3(\zeta^2;q^2)},$$

$$\vartheta_3(\zeta;q) = (q,-\sqrt{q}\zeta,-\sqrt{q}/\zeta;q)_\infty.$$

Goal: KPZ equation in half space

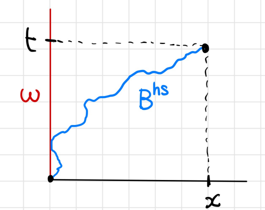
$$\partial_t \mathcal{H}^{\mathrm{hs}} = \frac{1}{2}\partial_x^2 \mathcal{H}^{\mathrm{hs}} + \frac{1}{2}\left(\partial_x \mathcal{H}^{\mathrm{hs}}\right)^2 + \dot{W}, \quad x \in \mathbb{R}_{\geq 0}, t \in \mathbb{R}_+,$$

$$\partial_x \mathcal{H}^{\mathrm{hs}}(x,t)\big|_{x=0} = \omega, \quad \omega \in \mathbb{R} \text{ boundary parameter}$$

- Wu 2020, Parekh 2019 proved well-posedness.
- Wu 2020: Cole-Hopf transformation $\mathcal{Z}^{\mathrm{hs}}(x,t) = e^{\mathcal{H}^{\mathrm{hs}}(x,t)}$

$$\partial_t \mathcal{Z}^{\mathrm{hs}} = \frac{1}{2}\partial_x^2 \mathcal{Z}^{\mathrm{hs}} + \mathcal{Z}^{\mathrm{hs}}\dot{W}, \qquad x \in \mathbb{R}_{\geq 0}, t \in \mathbb{R}_+,$$

$$\mathcal{Z}^{\mathrm{hs}}(x,0) = \delta_0(x), \qquad (\partial_x - \omega)\,\mathcal{Z}^{\mathrm{hs}}(x,t)\big|_{x=0} = 0$$

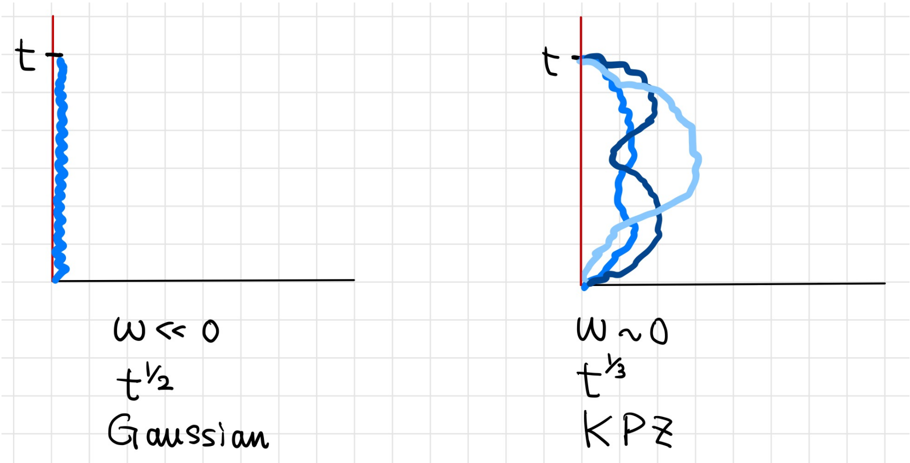


- Feynman-Kac formula

$$\mathcal{Z}^{\mathrm{hs}}(x,t) = \mathbb{E}\left[:\exp: \left\{ \int_0^t \left(\dot{W}(B^{\mathrm{hs}}(s),s) - \omega\delta_0(B^{\mathrm{hs}}(s)) \right) \mathrm{d}s \right\} \middle| B^{\mathrm{hs}}(0)=0,\ B^{\mathrm{hs}}(t)=x \right],$$

- Main result: Fredholm Pfaffian formula for $\mathcal{H}^{\mathrm{hs}}(0,t) = \log \mathcal{Z}^{\mathrm{hs}}(0,t)$

Transition between Gaussian and KPZ



Main result: Pfaffian formula for the half space SHE

Theorem TI-Mucciconi-Sasamoto arxiv:2204.08420 Th.1.3.

When $\omega > -1/2$, we have

$$\mathbb{E}\left[e^{-e^{-s+\log Z^{\mathrm{hs}}(0,t)+t/24}}\right] = \mathrm{Pf}\,[J - \mathcal{L}]_{\mathbb{L}^2(s,+\infty)}$$

$$\mathcal{L}(X,Y) = \begin{pmatrix} \mathcal{K}^{\mathrm{hs}}(X,Y) & -\partial_y \mathcal{K}^{\mathrm{hs}}(X,Y) \\ -\partial_x \mathcal{K}^{\mathrm{hs}}(X,Y) & \partial_x \partial_y \mathcal{K}^{\mathrm{hs}}(X,Y) \end{pmatrix},$$

$$\mathcal{K}^{\mathrm{hs}}(X,Y) = \int_{\mathrm{i}\mathbb{R}+d} \frac{\mathrm{d}Z}{2\pi\mathrm{i}} \int_{\mathrm{i}\mathbb{R}+d} \frac{\mathrm{d}W}{2\pi\mathrm{i}} e^{\frac{t}{2}\left(\frac{Z^3}{3}+\frac{W^3}{3}\right)-ZX-WY} \times \frac{\Gamma(\frac{1}{2}+\omega-Z)}{\Gamma(\frac{1}{2}+\omega+Z)} \frac{\Gamma(\frac{1}{2}+\omega-W)}{\Gamma(\frac{1}{2}+\omega+W)} \Gamma(2Z)\Gamma(2W) \frac{\sin[\pi(Z-W)]}{\sin[\pi(Z+W)]}$$

and we assume $0 < d < \min(1/2, 1/2+\omega)$.

- First rigorous derivation of the Pfaffian formula for the half space SHE.
- Krajenbrink-Le Doussal 2020: Equivalent formula obtained by the replica method

Main result: Baik-Rains transition in the half-space SHE

Theorem [TI-Mucciconi-Sasamoto arxiv:2204.08420 Th. 1.7]

- if $\omega \geq -1/2$, we have

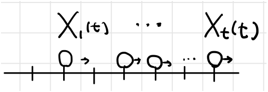
$$\lim_{t\to+\infty} \mathbb{P}\left[\frac{\log \mathcal{Z}^{\mathrm{hs}}(0,t) + t/24}{2^{-1/3}t^{1/3}} \leq r\right] = \begin{cases} F_4(s), \text{GSE Tracy-Widom} & \omega > -\frac{1}{2} \\ F_1(s), \text{GOE Tracy-Widom} & \omega = -\frac{1}{2} \end{cases}$$

- if $\omega < -1/2$, we have

$$\lim_{t\to+\infty} \mathbb{P}\left[\frac{\log \mathcal{Z}^{\mathrm{hs}}(0,t) + f_\omega t}{\sigma_\omega t^{1/2}} \leq r\right] = \int_{-\infty}^{r} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\,\mathrm{d}u,$$

- First rigorous derivation of the Baik-Rains transition for full range of parameter ω.
- Current distribution of the half-line open ASEP.
  - Barraquand-Borodin-Corwin-Wheeler 2018: $\omega = -1/2$
  - Jimmy He 2023 (arXiv:2303.16335): full range of ω

Outline



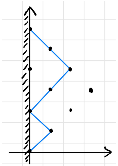
Refined Littlewood identity

$$\mathbb{P}(\mu_1+\chi=\ell)=\mathbb{P}(\lambda_1=\ell)$$

q-pushTASEP with particle creation

$$\mathbb{P}\left(X_N^{\mathrm{hs}}(t)-N+\chi+2\tilde{S}\le s\right)=\mathrm{Pf}\,(J-L)_{\ell^2(\mathbb{Z}'_{>s})}$$

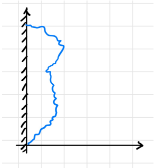
$\downarrow$ Barraquand-Borodin-Corwin 2020
$q\to 1$ scaling limit



Half space Log-gamma polymer

$$\mathbb{E}\left[e^{-e^{-\varsigma+\log Z^{\mathrm{hs}}(N,N)}}\right]=\mathrm{Pf}(J-\mathbf{L})_{\mathbb{L}^2(\varsigma,+\infty)},$$

$\downarrow$ Wu 2020
continuum limit



Half space stochastic heat equation

$$\mathbb{E}\left[e^{-e^{-s+\log Z^{\mathrm{hs}}(0,t)+t/24}}\right]=\mathrm{Pf}\,[J-\mathcal{L}]_{\mathbb{L}^2(s,+\infty)}$$

$\downarrow$ $t\to\infty$ scaling limit

Baik-Rains transition

Summary

We have proved two relations in TI-Mucciconi-Sasamoto, Forum Math. Pi, 11, e27, 2023

- Refined Cauchy identity

$$(1) \sum_{\substack{\mu,\nu \\ \mu_1+\nu_1=\ell}} q^{|\nu|} b_\mu(q) P_\mu(a;q) P_\mu(b;q) = \sum_{\substack{\lambda,\rho \\ \lambda_1=\ell}} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b)$$

$$\Leftrightarrow (1)' \mathbb{P}(\mu_1 + \chi_q = \ell) = \mathbb{P}(\lambda_1 = \ell)$$

$$\mu \sim \mathbb{W}^{(q)}_{a,b}(\mu) \qquad \lambda \sim \mathbb{PS}^{(q)}_{a,b}(\lambda)$$

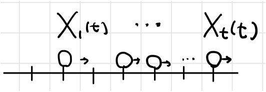
- Refined Littlewood identity

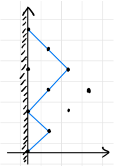
$$(2) \sum_{\substack{\mu',\nu' \text{:even} \\ \mu_1+\nu_1=\ell}} q^{|\nu|/2} b^{\text{ev}}_\mu(q) P_\mu(x;q) = \sum_{\substack{\lambda',\rho' \text{:even} \\ \lambda_1=\ell}} q^{|\rho|/2} s_{\lambda/\rho}(x)$$

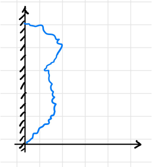
$$\Leftrightarrow (2)' \mathbb{P}(\mu_1 + \chi_{q/2} = \ell) = \mathbb{P}(\lambda_1 = \ell)$$

$$\mu \sim \mathbb{HW}^{(q)}_{a,b}(\mu) \qquad \lambda \sim \mathbb{FBS}^{(q)}_{a}(\lambda)$$

Summary







Refined Littlewood identity

$$\mathbb{P}(\mu_1 + \chi = \ell) = \mathbb{P}(\lambda_1 = \ell)$$

q-pushTASEP with particle creation

$$\mathbb{P}\left(X_N^{\mathrm{hs}}(t) - N + \chi + 2\tilde{S} \le s\right) = \mathrm{Pf}\,(J - L)_{\ell^2(\mathbb{Z}'_{>s})}$$

$\downarrow$ Barraquand-Borodin-Corwin 2020
$q \to 1$ scaling limit

Half space Log-gamma polymer

$$\mathbb{E}\left[e^{-e^{-\varsigma + \log Z^{\mathrm{hs}}(N,N)}}\right] = \mathrm{Pf}(J - \mathbf{L})_{\mathbb{L}^2(\varsigma, +\infty)},$$

$\downarrow$ Wu 2020
continuum limit

Half space stochastic heat equation

$$\mathbb{E}\left[e^{-e^{-s + \log Z^{\mathrm{hs}}(0,t) + t/24}}\right] = \mathrm{Pf}\,[J - \mathcal{L}]_{\mathbb{L}^2(s, +\infty)}$$

$\downarrow$ $t \to \infty$ scaling limit

Baik-Rains transition