

Three proofs of quadratic reciprocity and their impact on twentieth century mathematics

Clemens Berger

Université Côte d'Azur

Colloque Laboratoire J. A. Dieudonné
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- 1 Quadratic reciprocity
- 2 Combinatorial proof
- 3 Algebraic proof
- 4 Cyclotomic proof

Definition (Legendre symbol for odd prime p and coprime x)

$$\left(\frac{x}{p}\right) = \begin{cases} +1 & \text{if } x \text{ is a square in } \mathbb{F}_p^\times \\ -1 & \text{if } x \text{ is not a square in } \mathbb{F}_p^\times \end{cases}$$

Lemma (Euler's criterion)

$$\left(\frac{x}{p}\right) = x^{\frac{p-1}{2}} \text{ in } \mathbb{F}_p \text{ so that } \left(\frac{x}{p}\right) \left(\frac{y}{p}\right) = \left(\frac{xy}{p}\right) \text{ in } \mathbb{F}_p.$$

Proof.

$$X^{p-1} - 1 = (X^{\frac{p-1}{2}} + 1)(X^{\frac{p-1}{2}} - 1) \text{ in } \mathbb{F}_p[X]. \quad \square$$

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Theorem (Quadratic reciprocity law – Euler, Legendre, Gauss)

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Example (Is 14 a square in \mathbb{F}_{41} ?)

- $\left(\frac{2}{41}\right) = +1$
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has referenced more than 300 proofs of the quadratic reciprocity law among which 8 by Gauss.
We outline three of them, a combinatorial, an algebraic, and a cyclotomic proof.

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Remark (exécutaire testamentaire)

En 1866, la tuberculose emporte Riemann, et sa mort prématurée survient en pleine phase de grande créativité. Peu avant son décès, imitant Dirichlet, il avait fait de Dedekind le dépositaire de ses manuscrits, lequel s'attelle presque tout de suite à l'entreprise colossale que représente l'organisation des écrits de Riemann et la sélection des textes publiables ... Il est difficile d'évaluer le travail d'exécutaire testamentaire qu'accomplit Dedekind, d'autant qu'il avait vingt-huit ans quand Dirichlet est mort et quarante-cinq ans quand les Œuvres complètes (1876) de Riemann sont publiées, et que ces dix-sept ans correspondent à ses années les plus productives sur le plan mathématique. Dedekind a su faire passer les contributions de ses deux amis avant les siennes. (wikipedia)

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Proof.

m_x has even/odd number of orbits iff x is/is not square in \mathbb{F}_p . \square

Corollary (complementary laws of quadratic reciprocity)

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- $\left(\frac{\Delta_f}{K}\right) = \begin{cases} +1 & \text{if } \Delta_f \text{ is a square in } K \text{ (i.e. } \sqrt{\Delta_f} \in K) \\ -1 & \text{if } \Delta_f \text{ is not a square in } K \text{ (i.e. } \sqrt{\Delta_f} \notin K) \end{cases}$

Proposition (assume $\text{Gal}(L/K)$ is cyclic with generator $\phi_{L/K}$)

$\left(\frac{\Delta_f}{K}\right) = \text{sgn}(\phi_{L/K})$ where $\phi_{L/K}$ permutes the roots of f .

Proof.

$$\phi_{L/K}(\sqrt{\Delta_f}) = \text{sgn}(\phi_{L/K})\sqrt{\Delta_f}. \quad \square$$

Lemma ($p^* = (-1)^{\frac{p-1}{2}} p$)

$$\left(\frac{p}{q}\right) \left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2} \frac{q-1}{2}} \text{ if and only if } \left(\frac{p^*}{q}\right) = \left(\frac{q}{p}\right)$$

Theorem (Dedekind 1877)

$$\left(\frac{p^*}{q}\right) = \left(\frac{\Delta_{X^p-1}}{\mathbb{F}_q}\right) = \left(\frac{q}{p}\right)$$

Proof.

$\Delta_f = (-1)^{\binom{n}{2}} \prod_i f'(\alpha_i)$ so $\Delta_{X^p-1} = (-1)^{\binom{p}{2}} \prod_{i=1}^p p\alpha_i^{p-1} \equiv p^*$.
 $X^p - 1$ splits in \mathbb{F}_{q^r} if $p|q^r - 1$ and $\text{sgn}(\phi_{\mathbb{F}_{q^r}/\mathbb{F}_q}) = \text{sgn}(m_q)$. \square

Theorem (Stickelberger 1898, Hensel 1905, Swan 1962)

For $f \in \mathbb{F}_q[X]$ sth. $\Delta_f \neq 0$ one has $\left(\frac{\Delta_f}{\mathbb{F}_q}\right) = (-1)^{\deg(f) - \#\text{irrfact}(f)}$

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Definition (Dirichlet character)

$\chi : \mathbb{F}_p^\times \rightarrow \mathbb{C}^*$ sth. $\chi(xy) = \chi(x)\chi(y)$ and $\chi(1) = 1$. For pointwise multiplication, Dirichlet characters mod p form a cyclic group.

Remark

The Legendre symbol is the only Dirichlet character of order 2.

Definition (Gauss sums for $\zeta = e^{2\pi i/p}$)

- $\tau_\chi = \sum_{k=1}^{p-1} \chi(k) \zeta^k$
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Proposition (Discrete Fourier Transform)

$$\hat{\chi} = \tau_{\chi} \bar{\chi} \text{ and } \hat{\hat{\chi}} = \tau_{\chi} \tau_{\bar{\chi}} \chi \text{ and } \tau_{\chi} \tau_{\bar{\chi}} = p\chi(-1).$$

Proof.

$$\chi(s)\hat{\chi}(s) = \chi(s) \sum_{k=1}^{p-1} \chi(k)\zeta^{sk} = \sum_{k=1}^{p-1} \chi(sk)\zeta^{sk} = \tau_{\chi}.$$

Therefore, $\hat{\hat{\chi}} = \tau_{\chi} \hat{\chi} = \tau_{\chi} \tau_{\bar{\chi}} \chi$

$\hat{\hat{\chi}} = V^2 \chi = p\tilde{\chi}$ where V is defined below and $\tilde{\chi}(i) = \chi(p-i)$. \square

Corollary (Eisenstein 1844)

$$(\tau_p)^2 = p \left(\frac{-1}{p} \right) = p^* \text{ and } (\tau_p)^q \equiv \left(\frac{q}{p} \right) \tau_p \pmod{q\mathbb{Z}[\zeta]}$$

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$$(\tau_p)^q = \left(\sum_{k=1}^{p-1} \left(\frac{k}{p} \right) \zeta^k \right)^q \equiv \sum_{k=1}^{p-1} \left(\frac{k}{p} \right) \zeta^{qk} = \left(\frac{q}{p} \right) \tau_p \quad \square$$

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Theorem (Gauss 1818, proof by Motose 2003)

$$\tau_p = \begin{cases} \sqrt{p} & \text{if } p \equiv 1 \pmod{4} \\ i\sqrt{p} & \text{if } p \equiv 3 \pmod{4} \end{cases}$$

Definition (cyclotomic Vandermonde matrix)

For $V = \begin{pmatrix} \zeta & \zeta^2 & \dots & \zeta^{p-1} \\ \zeta^2 & \zeta^4 & \dots & \zeta^{2(p-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta^{p-1} & \zeta^{2(p-1)} & \dots & \zeta^{(p-1)^2} \end{pmatrix}$ one has $V \begin{pmatrix} \chi(1) \\ \vdots \\ \chi(p-1) \end{pmatrix} = \begin{pmatrix} \hat{\chi}(1) \\ \vdots \\ \hat{\chi}(p-1) \end{pmatrix}$.

Remark (Variation on Wilson's Theorem)

$$\left(\frac{\frac{p-1}{2}!}{p}\right) = (-1)^{\#\{\text{positive non-squares in } \mathbb{F}_p\}} = \left(\frac{-2}{p}\right) = (-1)^{\frac{(p-1)(p-3)}{8}}$$

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Proof.

$$\begin{aligned} \det(V) &= \zeta^{\binom{p-1}{2}} \prod_{j>i} (\zeta^j - \zeta^i) = \prod_{j=p-i} (\zeta^j - \zeta^i) \prod_{j \neq p-i} (\zeta^j - \zeta^i) \\ &= (-1)^{\frac{p-1}{2}} i^{\frac{p-1}{2}} \sqrt{p} \cdot p^{\frac{p-3}{2}} \end{aligned}$$

V is conjugate (choosing Dirichlet characters as basis) to a matrix decomposing into 2×2 principal block matrices thus yielding:

$$\begin{aligned} \det(V) &= (-1)^{\frac{p-3}{2}} \prod_{\chi} \tau_{\chi} = (-1)^{\frac{p-1}{2}} \tau_{\rho} (\rho\chi_2(-1)) \cdots (\rho\chi_{\frac{p-1}{2}}(-1)) \\ &= (-1)^{\frac{p-1}{2}} \tau_{\rho} \cdot \left(\frac{\left(\frac{p-1}{2}\right)!}{p} \right) \cdot p^{\frac{p-3}{2}} \\ &= (-1)^{\frac{p-1}{2}} \tau_{\rho} \cdot (-1)^{\frac{(p-1)(p-3)}{8}} \cdot p^{\frac{p-3}{2}} \end{aligned}$$



Proof.

$$\begin{aligned} \det(V) &= \zeta^{\binom{p-1}{2}} \prod_{j>i} (\zeta^j - \zeta^i) = \prod_{j=p-i} (\zeta^j - \zeta^i) \prod_{j \neq p-i} (\zeta^j - \zeta^i) \\ &= (-1)^{\frac{p-1}{2}} i^{\frac{p-1}{2}} \sqrt{p} \cdot p^{\frac{p-3}{2}} \end{aligned}$$

V is conjugate (choosing Dirichlet characters as basis) to a matrix decomposing into 2×2 principal block matrices thus yielding:

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