Reduced order model approach for imaging with waves

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Motivation: Sensor array imaging

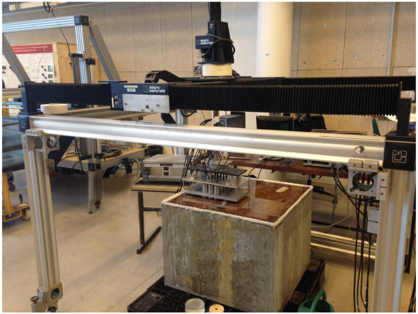
- Sensor array imaging (echography in medical imaging, sonar, non-destructive testing, seismic exploration, etc) has two steps:
  - data acquisition: an unknown medium is probed with waves; waves are emitted by a source (or a source array) and recorded by a receiver array.
  - data processing: the recorded signals are processed to identify the quantities of interest (reflector locations, etc).

Example:
Ultrasound echography

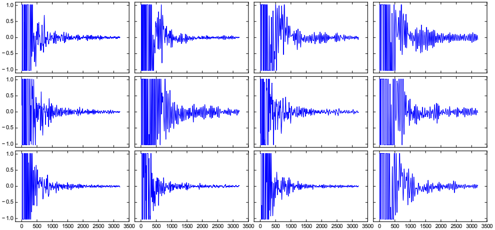
$\longrightarrow$

- Mathematically: Ill-posed inverse problems.

Example: Ultrasound in concrete

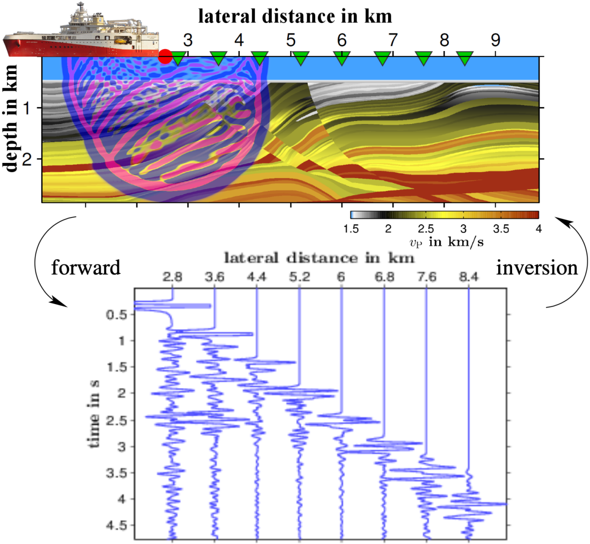


Experience: nondestructive testing



Data: recorded signals

Example: Seismology



Velocity estimation problem

- *Direct problem:* Given the velocity map $c = (c(x))_{x \in \Omega}$ compute the wavefield solution of the wave equation

$$[\partial_t^2 - c^2(x)\Delta]p^{(s)}(t,x) = f(t)\delta(x - x_s), \qquad t \in \mathbb{R}, \quad x \in \Omega \subset \mathbb{R}^d,$$

starting from $p^{(s)}(t,x) = 0$, $t \ll 0$, $+$ boundary conditions at $\partial\Omega$.
At the locations of the receivers:

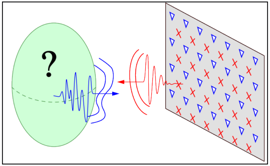
$$d_{r,s}(t) = p^{(s)}(t, x_r), \quad r, s = 1, .., N$$

$\hookrightarrow$ forward map

$$\mathcal{D} : c \mapsto \mathbf{d}$$

where $\mathbf{d} = ((d_{r,s}(t))_{r,s=1}^N)_{t \in [t_{\min}, t_{\max}]}$, is the array response matrix.

- *Inverse problem:*
Given the time-resolved measurements $\mathbf{d}$,
determine the velocity map c.



Full Waveform Inversion (FWI)

- FWI fits data with the model prediction in L^2-norm:

$$\hat{c} = \underset{c}{\operatorname{argmin}} \left\{ \mathcal{O}_{FWI}[c] + \mathrm{Reg}[c] \right\},$$

$$\mathcal{O}_{FWI}[c] = \|\mathbf{d}_{meas} - \mathcal{D}[c]\|_2^2 = \sum_{r,s=1}^{N} \int_{t_{\min}}^{t_{\max}} |d_{meas}(t)_{r,s} - \mathcal{D}[c](t)_{r,s}|^2 dt$$

  Cf [Virieux and Operto 2009].
- Bayesian interpretation (Maximum A Posteriori).
- The objective function $\mathcal{O}_{FWI}[c]$ is not convex in c.
  $\hookrightarrow$ optimization needs hard to get good initial guess.

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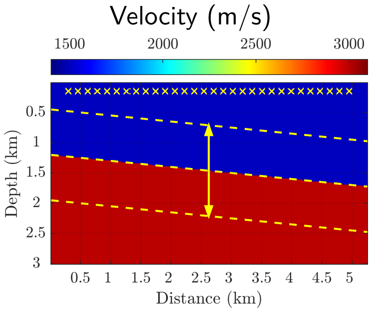
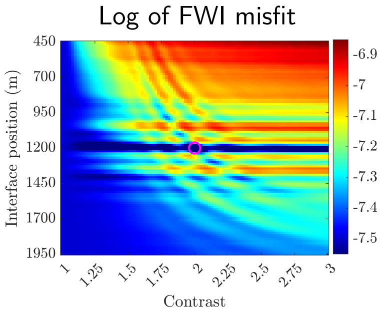
- The objective function $\mathcal{O}_{FWI}[c]$ is not convex in c.
  $\hookrightarrow$ optimization needs hard to get good initial guess.

- Layer stripping: Proceed hierarchically from the shallow part to the deep part [Wang et al. 2009]
- Frequency hopping: Successive inversion of subdata sets of increasing high-frequency content [Bunks et al. 1995]
- Optimal transport: Wasserstein distance instead of L^2-norm [Engquist et al. 2016, Yang et al. 2018]

Topography of the FWI objective function





- Probing pulse is a modulated Gaussian pulse with central frequency $6Hz$ and bandwidth $4Hz$ ($\lambda \simeq 300m$ at $10Hz$).
- $N = 30$ sensors; $N_{\mathrm{t}} = 39$ time samples at interval $\tau = 0.0435$s.
- Search velocity has two parameters: the bottom velocity and depth of the interface (the angle and top velocity are known).
- Objective function:

$$\mathcal{O}_{FWI}[c] = \|\mathbf{d}_{meas} - \mathcal{D}[c]\|_2^2$$

$\hookrightarrow$ Many local minima (cycle skipping issues).

Objective

- Find a convex formulation of FWI.
- Proposed approach: find a nonlinear mapping $\mathcal{R}$: data $\mathbf{d} \mapsto$ reduced order model (ROM) of wave operator $\mathbf{A}^{rom}$ (matrix) such that:
  - ROM can be computed with efficient numerical linear algebra tools in non-iterative fashion.
  - Minimization of ROM misfit is better for velocity estimation.

We can think of the data to ROM mapping $\mathcal{R}$ as a nonlinear preconditioner of the forward mapping $\mathcal{D}$:

$$c \overset{\mathcal{D}}{\mapsto} \mathbf{d} \overset{\mathcal{R}}{\mapsto} \mathbf{A}^{rom}$$

because the composition $\mathcal{R} \circ \mathcal{D}$, which gives $\mathbf{A}^{rom} = \mathcal{R} \circ \mathcal{D}[c]$, is easier to "invert".

Towards the ROM objective function

- *Ideal objective function 1:*

$$\mathcal{O}[c] = \|c - c^{meas}\|^2$$

but c^{meas} is not observed (i.e., cannot be extracted from $\mathbf{d}_{meas}$) !

Towards the ROM objective function

- Let us consider the wave operator

$$\mathcal{A}[c] = -c(x)\Delta\big[c(x)\cdot\big]$$

- Galerkin method to approximate the operator $\mathcal{A}$ by a matrix:

- consider a space of (piecewise polynomial) functions with basis $(\Psi_l(x))_{l=1}^L$,
- consider the row vector field $\boldsymbol{\Psi}(x) = (\Psi_1(x), \dots, \Psi_L(x))$ and define:

$$\mathbf{A}^{\Psi} = \int_{\Omega} dx\, \boldsymbol{\Psi}(x)^T \mathcal{A} \boldsymbol{\Psi}(x) \in \mathbb{R}^{L\times L}$$

- *Ideal objective function 2:*

$$\mathcal{O}[c] = \|\mathbf{A}^{\Psi}[c] - \mathbf{A}^{\Psi,meas}\|_2^2$$

but $\mathbf{A}^{\Psi,meas}$ is not observed !

The ROM matrix

- Our Galerkin approximation space:

- consider a time discretization $\{t_j = j\tau\}_{0\leq j\leq N_{\rm t}}$ with uniform stepping τ,
- gather the waves $p^{(s)}(t,x)$ evaluated at $t = t_j$ for all the N sources:

$$\boldsymbol{p}_j(x) = \left(p^{(1)}(t_j,x),\dots,p^{(N)}(t_j,x)\right), \quad x \in \Omega.$$

(note: apply first a linear preprocessing).

- organize the first $N_{\rm t}$ snapshots in the $NN_{\rm t}$ dimensional row vector field:

$$\boldsymbol{U}(x) = \left(\boldsymbol{p}_0(x),\dots,\boldsymbol{p}_{N_{\rm t}-1}(x)\right), \quad x \in \Omega.$$

- apply Gram-Schmidt orthogonalization onto $\boldsymbol{U}(x) = \boldsymbol{V}(x)\mathbf{R}$.

- Define ROM matrix:

$$\mathbf{A}^{rom} = \int_\Omega dx\, \boldsymbol{V}(x)^T \mathcal{A} \boldsymbol{V}(x) \in \mathbb{R}^{NN_{\rm t}\times NN_{\rm t}}.$$

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- Define ROM matrix:

$$\mathbf{A}^{rom} = \int_\Omega dx\, \boldsymbol{V}(x)^T \mathcal{A}\boldsymbol{V}(x) \in \mathbb{R}^{NN_{\rm t}\times NN_{\rm t}}.$$

- *Ideal objective function 3:*

$$\mathcal{O}[c] = \|\mathbf{A}^{rom}[c] - \mathbf{A}^{rom,meas}\|_2^2$$

but $\mathbf{A}^{rom,meas}$ is not observed (neither $\mathcal{A}$ nor $\boldsymbol{V}(x)$ is observed) !

The ROM matrix

- Our Galerkin approximation space:

- consider a time discretization $\{t_j = j\tau\}_{0 \le j \le N_{\rm t}}$ with uniform stepping τ,
- gather the waves $p^{(s)}(t,x)$ evaluated at $t = t_j$ for all the N sources:

$$\boldsymbol{p}_j(x) = \left(p^{(1)}(t_j,x), \dots, p^{(N)}(t_j,x)\right), \quad x \in \Omega.$$

(note: apply first a linear preprocessing).

- organize the first $N_{\rm t}$ snapshots in the $NN_{\rm t}$ dimensional row vector field:

$$\boldsymbol{U}(x) = (\boldsymbol{p}_0(x), \dots, \boldsymbol{p}_{N_{\rm t}-1}(x)), \quad x \in \Omega.$$

- apply Gram-Schmidt orthogonalization onto $\boldsymbol{U}(x) = \boldsymbol{V}(x)\mathbf{R}$.

- Define ROM matrix:

$$\mathbf{A}^{rom} = \int_\Omega dx\, \boldsymbol{V}(x)^T \mathcal{A} \boldsymbol{V}(x) \in \mathbb{R}^{NN_{\rm t} \times NN_{\rm t}}.$$

- **Proposition**: *The ROM matrix $\mathbf{A}^{rom}$ can be extracted from the measurements $\mathbf{d}$, without knowing $\mathcal{A}$ nor $\boldsymbol{V}(x)$.*
$\hookrightarrow \mathcal{O}_{ROM}[c] = \|\mathbf{A}^{rom}[c] - \mathbf{A}^{rom,meas}\|_2^2$ *is a legitimate objective function.*

First step: *Linear preprocessing*.

- Define the new data matrix $\mathbf{D}(t)$:

$$\mathbf{D}(t) = \mathbf{d}^f(t) + \mathbf{d}^f(-t), \quad \text{with } \mathbf{d}^f(t) = -f'(-t) *_t \mathbf{d}(t).$$

Second step: *Expression of the new data entries as wave correlations*.

- Introduce the solution $u^{(s)}(t,x)$ of the homogeneous wave equation

$$(\partial_t^2 + \mathcal{A})u^{(s)}(t,x) = 0, \qquad t > 0, \quad x \in \Omega,$$

with boundary conditions on $\partial\Omega$, with initial state

$$u^{(s)}(0,x) = u_0^{(s)}(x) = \left|\hat{f}(\sqrt{\mathcal{A}})\right| \delta(x - x_s), \qquad \partial_t u^{(s)}(0,x) = 0.$$

It has the form

$$u^{(s)}(t,x) = \cos\left(t\sqrt{\mathcal{A}}\right) u_0^{(s)}(x).$$

$\rightarrow$ The entries of $\mathbf{D}(t)$ can be expressed as wave correlations:

$$D_{r,s}(t) = \int_\Omega dx \, u_0^{(r)}(x) u^{(s)}(t,x).$$

Third step: *Definition of the ROM.*

Let $\tau > 0$ be fixed.

- Gather the snapshots for all the N sources in the row vector fields

$$\boldsymbol{u}_j(x) = \Big(u^{(1)}(j\tau, x), \dots, u^{(N)}(j\tau, x) \Big), \qquad 0 \le j \le N_{\rm t}.$$

- Organize the first $N_{\rm t}$ snapshots in the $NN_{\rm t}$ dimensional row vector field:

$$\boldsymbol{U}(x) = \left(\boldsymbol{u}_0(x), \dots, \boldsymbol{u}_{N_{\rm t}-1}(x) \right), \quad x \in \Omega.$$

- Apply Gram-Schmidt orthogonalization onto $\boldsymbol{U}(x) = \boldsymbol{V}(x)\mathbf{R}$.

(note: we have $\int_\Omega dx\, \boldsymbol{V}(x)^T \boldsymbol{V}(x) = \mathbf{I}_{NN_{\rm t}}$).

- Define

$$\mathbf{A}^{rom} = \int_\Omega dx\, \boldsymbol{V}(x)^T \mathcal{A} \boldsymbol{V}(x)$$

Fourth step: *Expression of the ROM in terms of mass and stiffness.*

- Define the $NN_{\rm t} \times NN_{\rm t}$ "mass" and "stiffness" matrices:

$$\mathbf{M} = \int_\Omega dx \, \boldsymbol{U}^T(x)\boldsymbol{U}(x), \qquad \mathbf{S} = \int_\Omega dx \, \boldsymbol{U}^T(x)\mathcal{A}\boldsymbol{U}(x)$$

- Since $\boldsymbol{U}(x) = \boldsymbol{V}(x)\mathbf{R}$, we get

$$\begin{aligned}\mathbf{M} &= \mathbf{R}^T \int_\Omega dx \, \boldsymbol{V}^T(x)\boldsymbol{V}(x)\mathbf{R} \\ &= \mathbf{R}^T\mathbf{R}\end{aligned}$$

and

$$\begin{aligned}\mathbf{A}^{rom} &= \int_\Omega dx \, \boldsymbol{V}(x)^T\mathcal{A}\boldsymbol{V}(x) = \mathbf{R}^{-T}\int_\Omega dx \, \boldsymbol{U}(x)^T\mathcal{A}\boldsymbol{U}(x)\mathbf{R} \\ &= \mathbf{R}^{-T}\mathbf{S}\mathbf{R}\end{aligned}$$

$\hookrightarrow$ $\mathbf{A}^{rom}$ can be expressed in terms of $\mathbf{M}$ and $\mathbf{S}$.

Fifth step: *Expression of the ROM in terms of data.*
The $N \times N$ blocks of the mass matrix $\mathbf{M}$ are

$$
\begin{aligned}
\mathbf{M}_{i,j} &= \langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle_{L^2(\Omega)} = \langle \cos\big(i\tau\sqrt{\mathcal{A}}\big)\boldsymbol{u}_0, \cos\big(j\tau\sqrt{\mathcal{A}}\big)\boldsymbol{u}_0 \rangle_{L^2(\Omega)} \\
&= \langle \boldsymbol{u}_0, \cos\big(i\tau\sqrt{\mathcal{A}}\big)\cos\big(j\tau\sqrt{\mathcal{A}}\big)\boldsymbol{u}_0 \rangle_{L^2(\Omega)} \\
&= \frac{1}{2}\langle \boldsymbol{u}_0, \big[\cos\big((i+j)\tau\sqrt{\mathcal{A}}\big) + \cos\big(|i-j|\tau\sqrt{\mathcal{A}}\big)\big]\boldsymbol{u}_0 \rangle_{L^2(\Omega)} \\
&= \frac{1}{2}\langle \boldsymbol{u}_0, \boldsymbol{u}_{i+j} + \boldsymbol{u}_{|i-j|} \rangle_{L^2(\Omega)} \\
&= \frac{1}{2}\Big(\mathbf{D}_{i+j} + \mathbf{D}_{|i-j|}\Big), \quad 0 \le i, j \le N_{\mathrm{t}} - 1.
\end{aligned}
$$

Idem for the stiffness matrix $\mathbf{S}$.
$\hookrightarrow$ $\mathbf{M}$ and $\mathbf{S}$ can be expressed in terms of the data $\mathbf{D}$.

Algorithm for data-driven ROM matrix

Input: The matrices $\mathbf{d}(t) = (d_{r,s}(t))_{r,s=1}^{N}$ of measurements.

1. Compute $d^f_{r,s}(t) = -f'(-t) *_t d_{r,s}(t)$ and

$$\mathbf{D}_j = \mathbf{d}^f(j\tau) + \mathbf{d}^f(-j\tau), \quad 0 \le j \le 2N_{\rm t} - 2.$$

2. Compute $\ddot{\mathbf{D}}_j = \ddot{\mathbf{d}}^f(j\tau) + \ddot{\mathbf{d}}^f(-j\tau)$, for $j = 0, \dots, 2N_{\rm t} - 2$ with $\ddot{d}^f_{r,s}(t) = \partial_t^2 d^f_{r,s}(t)$ using, e.g., the Fourier transform.
3. Calculate $\mathbf{M}, \mathbf{S} \in \mathbb{R}^{NN_{\rm t} \times NN_{\rm t}}$ with the block entries

$$\mathbf{M}_{i,j} = \frac{1}{2}(\mathbf{D}_{i+j} + \mathbf{D}_{|i-j|}) \in \mathbb{R}^{N \times N},$$
$$\mathbf{S}_{i,j} = -\frac{1}{2}(\ddot{\mathbf{D}}_{i+j} + \ddot{\mathbf{D}}_{|i-j|}) \in \mathbb{R}^{N \times N},$$

for $0 \le i, j \le N_{\rm t} - 1$.

4. Perform block Cholesky factorization $\mathbf{M} = \mathbf{R}^T \mathbf{R}$.

Output: $\mathbf{A}^{rom} = \mathbf{R}^{-T} \mathbf{S} \mathbf{R}^{-1}$.

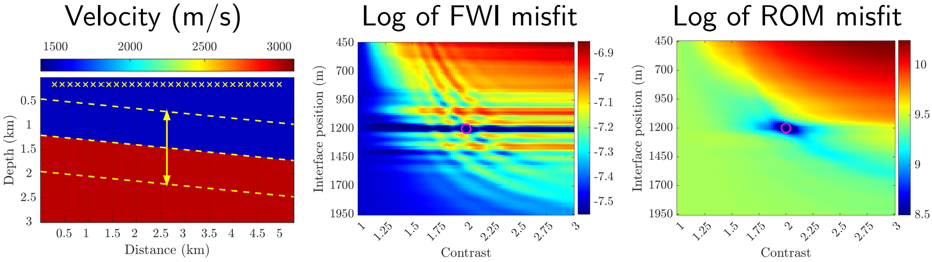
ROM objective function

- ROM misfit function:

$$\mathcal{O}_{ROM}[c] = \|\mathbf{A}^{rom}[c] - \mathbf{A}^{rom,meas}\|_2^2$$

  where $\mathbf{A}^{rom}[c]$ is computed from $\mathcal{D}[c]$ and $\mathbf{A}^{rom,meas}$ is computed from $\mathbf{d}_{meas}$.
- For a rich enough space of snapshots, the ROM matrix $\mathbf{A}^{rom}$ contains roughly the same information as $\mathcal{A} = -c(x)\Delta\big[c(x)\cdot\big]$.
  $\hookrightarrow$ The ROM misfit function should have nice convexity properties.
- Conjecture: "rich enough" means for sensors all around the domain of interest, separated by roughly half a wavelength, for time sampling satisfying the Nyquist criterium.
  $\hookrightarrow$ Conjecture proved only in special situations.

Topographies of the FWI and ROM objective functions



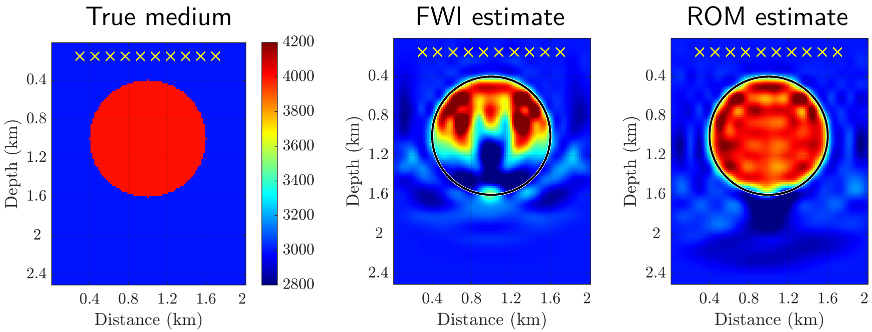
- Search velocity has two parameters: the contrast and the depth of the interface (the angle and top velocity are known).
- FWI objective function:

$$\mathcal{O}_{FWI}[c] = \|\mathcal{D}[c] - \mathbf{d}^{meas}\|_2^2$$

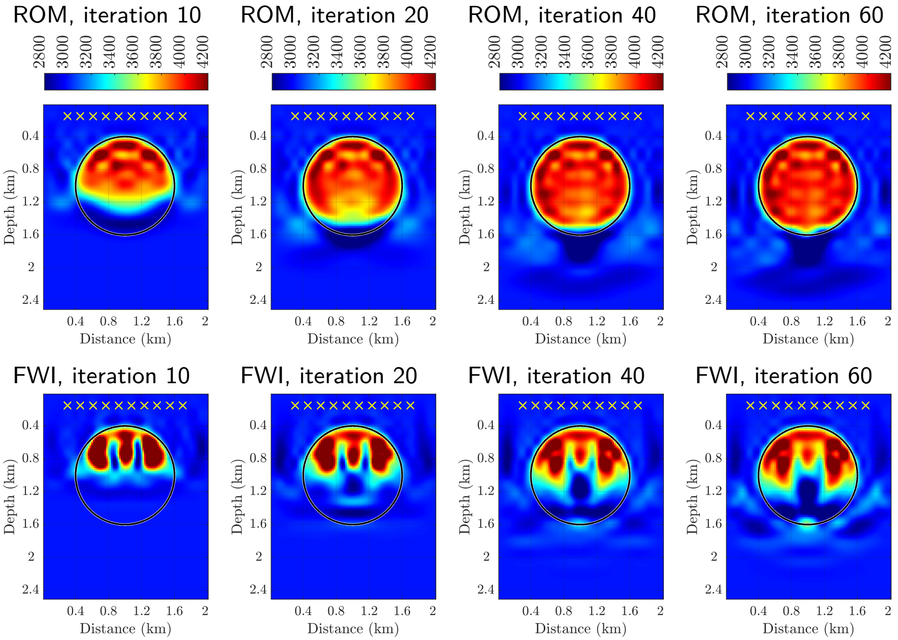
- ROM objective function:

$$\mathcal{O}_{ROM}[c] = \|\mathbf{A}^{rom}[c] - \mathbf{A}^{rom,meas}\|_2^2$$

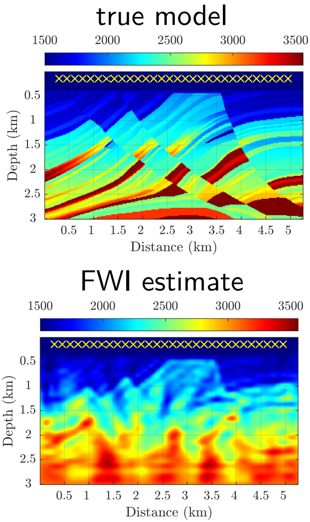
Camembert model



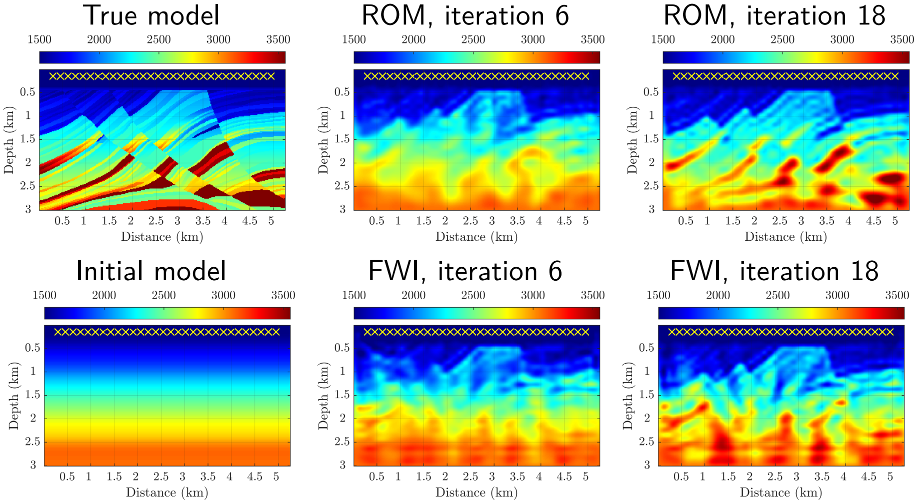
- Probing pulse is a modulated Gaussian pulse with central frequency $6Hz$ and bandwidth $4Hz$ ($\lambda = 300m$ at $10Hz$).
- Search velocity: $c(x, \boldsymbol{\eta}) = c_o + \sum_l \eta_l \phi_l(x)$, $\boldsymbol{\eta} = (\eta_l)_{l=1}^L$.
- $\phi_l(x)$ are Gaussian peaks with centers on a regular grid, $L = 400$, with width 60m (0.2λ).
- FWI minimizes $\mathcal{O}_{FWI}(\boldsymbol{\eta}) = \|\mathcal{D}[c(\boldsymbol{\eta})] - \mathbf{d}^{meas}\|_2^2 + \mu\|\boldsymbol{\eta}\|_2^2$
- ROM minimizes $\mathcal{O}_{ROM}(\boldsymbol{\eta}) = \|\mathbf{A}^{rom}[c(\boldsymbol{\eta})] - \mathbf{A}^{rom,meas}\|_2^2 + \mu\|\boldsymbol{\eta}\|_2^2$


ROM, iteration 10
ROM, iteration 20
ROM, iteration 40
ROM, iteration 60
2800 3000 3200 3400 3600 3800 4000 4200
Depth (km)
Distance (km)
FWI, iteration 10
FWI, iteration 20
FWI, iteration 40
FWI, iteration 60

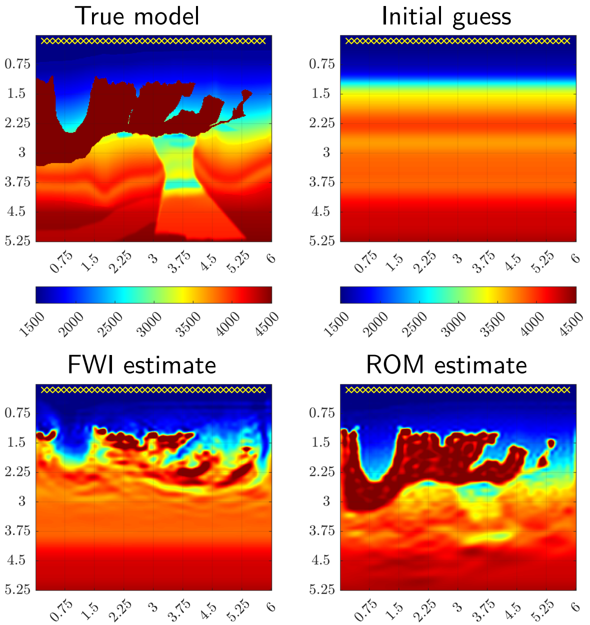
Marmousi model



Marmousi model



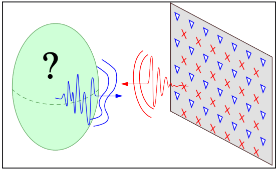
Salt body (BP - model)



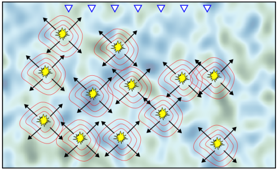
A limitation and an extension to passive imaging

- One limitation of the method:
  We need co-located sources and receivers.
- Extension to passive imaging:
  Consider a receiver array recording signals transmitted by noise sources (uncontrolled, opportunistic sources).
  Compute the cross correlation matrix of the recorded signals.
  $\rightarrow$ The cross correlation matrix is related to the Green's function (virtual active array) [Shapiro et al. 2005; Garnier et al. 2016].
  $\rightarrow$ The ROM procedure is natural in the passive framework, since the cross correlation matrix gives directly the data matrix $\mathbf{D}(t)$.
  - the virtual sources and receivers are naturally co-located,
  - the signals are even (because cross correlations are even).

Passive imaging



Active acquisition



Passive acquisition

- Consider the solution $p(t,x)$ of the wave equation

$$\partial_t^2 p - c^2(x)\Delta p = s(t,x), \qquad t \in \mathbb{R}, \quad x \in \Omega \subset \mathbb{R}^d,$$

where $s(t,x)$ is a zero-mean, stationary in time random process with

$$\langle s(t_1,y_1)s(t_2,y_2)\rangle = F(t_1-t_2)K(y_1)\delta(y_1-y_2)$$

- The empirical cross correlation of the recorded waves at x_r and $x_{r'}$ is

$$C_T(\tau,x_r,x_{r'}) = \frac{1}{T}\int_0^T dt\, p(t,x_r)p(t+\tau,x_{r'})$$

Passive imaging

- The statistical cross correlation

$$C^{(1)}(\tau, x_r, x_{r'}) = \big\langle C_T(\tau, x_r, x_{r'}) \big\rangle$$

is independent of T by stationarity of the noise sources.

- The statistical stability follows from the ergodicity of the noise sources:

$$C_T(\tau, x_r, x_{r'}) \overset{T\to+\infty}{\longrightarrow} C^{(1)}(\tau, x_r, x_{r'}),$$

in probability.

• **Proposition.** We have, for any $r, r' = 1, \dots, N$,

$$\partial_\tau^2 C^{(1)}(\tau, x_r, x_{r'}) = -\frac{1}{4} D_{r,r'}(\tau),$$

where $\mathbf{D}(t)$ is the active data matrix obtained with a source signal $f(t)$ that satisfies $|\hat{f}(\omega)| = \hat{F}(\omega)^{1/2}$.

• **Corollary.** *The passive data (cross correlation matrix) can be used directly in the ROM algorithm (no preprocessing).*

Conclusions

- The ROM is an approximation of the wave operator on a space defined by the snapshots of the wavefield.
- This space is not known and neither is the wave operator.
- Yet, we can compute the ROM from the data !
- We can then formulate a velocity estimation algorithm that minimizes the ROM misfit and that avoids cycle skipping and other problems.
- The method can be applied to active and passive imaging.

L Borcea, J Garnier, AV Mamonov, J Zimmerling, When data driven reduced order modeling meets waveform inversion, arXiv:2302.05988

L Borcea, J Garnier, AV Mamonov, J Zimmerling, Waveform inversion with a data driven estimate of the internal wave, SIAM Journal on Imaging Sciences 16 (1), 2023, 280-312.

L Borcea, J Garnier, AV Mamonov, J Zimmerling, Waveform inversion via reduced order modeling, Geophysics 88 (2), 2023, R175-R191.