Online Stochastic Matching Under the First-Come-First-Matched Policy

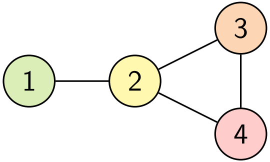
Céline Comte

"Online Stochastic Matching" Workshop
Toulouse – September 27, 2024

Compatibility graph

Graph G undirected, connected, without self-loop, non-bipartite.

- Set $\mathcal{V}$ of nodes $\rightarrow$ item classes.
- Set $\mathcal{V}_i$ of neighbors of node i $\rightarrow$ possible matches.



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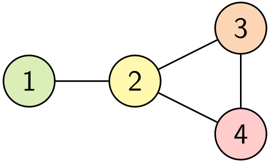

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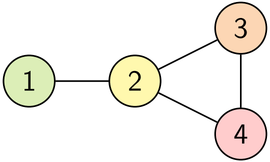

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Items are matched according to the **first-come-first-matched (FCFM)** policy.

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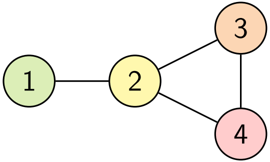

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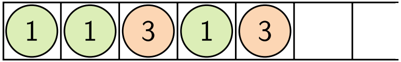

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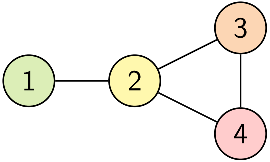

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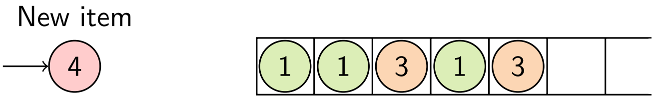

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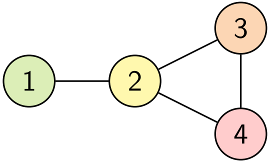

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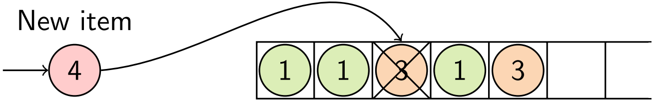

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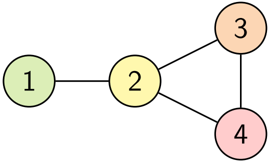

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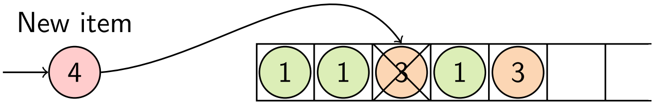
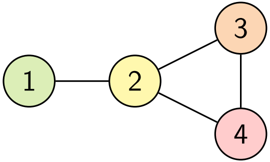


The evolution of the sequence of unmatched item classes defines a **Markov process** whose transition rates depend on the graph G and the arrival rates μ_i, $i \in \mathcal{V}$.

Neighbors and independent sets



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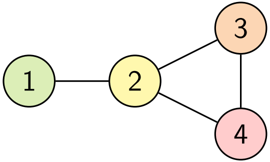

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- The neighbor set of a class $i \in \mathcal{V}$ is $\mathcal{V}_i$.
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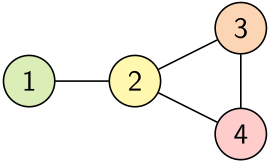

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Independent sets

- A node set $\mathcal{I} \subseteq \mathcal{V}$ is an **independent set** if its elements are (pairwise) nonadjacent.

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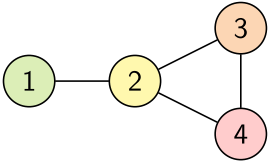

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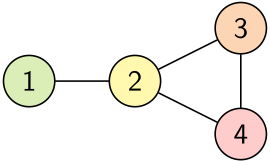

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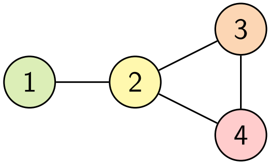

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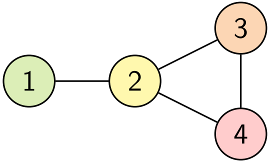

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- We let $\mathbb{I}^+ = \mathbb{I} \setminus \{\emptyset\}$ denote the family of nonempty independent sets.

Outline

1. Literature review
2. Basic results
   - Stability condition
   - Definition of the discrete-time Markov chain
   - Product-form stationary distribution and partial balance
3. Performance analysis
   - Normalization constant
   - Performance metrics
   - Heavy-traffic analysis
4. Concluding remarks

Outline

Early works

- René Caldentey, Edward H. Kaplan, and Gideon Weiss. "FCFS Infinite Bipartite Matching of Servers and Customers". *Advances in Applied Probability* 41.3 (Sept. 2009), pp. 695–730. DOI: 10.1239/aap/1253281061
- Ivo Adan and Gideon Weiss. "Exact FCFS Matching Rates for Two Infinite Multitype Sequences". *Operations Research* 60.2 (Apr. 1, 2012), pp. 475–489. DOI: 10.1287/opre.1110.1027

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Stability (i.e., positive recurrence) condition

- Ana Bušić, Varun Gupta, and Jean Mairesse. "Stability of the Bipartite Matching Model". *Advances in Applied Probability* 45.2 (June 2013), pp. 351–378. DOI: 10.1239/aap/1370870122

Performance evaluation

- Ivo Adan et al. "Reversibility and Further Properties of FCFS Infinite Bipartite Matching". *Mathematics of Operations Research* 43.2 (Dec. 12, 2017), pp. 598–621. DOI: 10.1287/moor.2017.0874
- Céline Comte and Jan-Pieter Dorsman. "Performance Evaluation of Stochastic Bipartite Matching Models". *Performance Engineering and Stochastic Modeling*. Ed. by Paolo Ballarini et al. Lecture Notes in Computer Science. Cham: Springer International Publishing, 2021, pp. 425–440. DOI: 10.1007/978-3-030-91825-5_26

Stability

- Jean Mairesse and Pascal Moyal. "Stability of the Stochastic Matching Model". *Journal of Applied Probability* 53.4 (Dec. 2016), pp. 1064–1077. DOI: 10.1017/jpr.2016.65

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Performance evaluation

- Pascal Moyal, Ana Bušić, and Jean Mairesse. “A Product Form for the General Stochastic Matching Model”. *Journal of Applied Probability* 58.2 (June 2021), pp. 449–468. DOI: 10.1017/jpr.2020.100
- Céline Comte. “Stochastic Non-Bipartite Matching Models and Order-Independent Loss Queues”. *Stochastic Models* 38.1 (Jan. 2, 2022), pp. 1–36. DOI: 10.1080/15326349.2021.1962352

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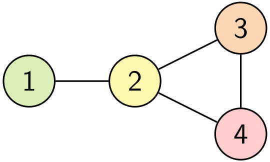
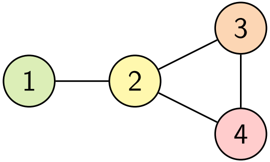
Stability condition

- Studied in [BGM13], [MM16], and [MBM21]



- Studied in [BGM13], [MM16], and [MBM21]
- The Markov process is positive recurrent if and only if

$$\rho(\mathcal{I}) \triangleq \frac{\sum_{i \in \mathcal{I}} \mu_i}{\sum_{i \in \mathcal{V}(\mathcal{I})} \mu_i} < 1, \quad \mathcal{I} \in \mathbb{I}^+.$$



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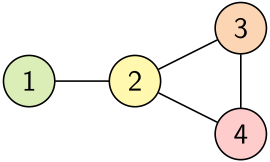

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In our toy example:

$$\begin{cases} \rho(\{1\}) = \dfrac{\mu_1}{\mu_2}, & \rho(\{2\}) = \dfrac{\mu_2}{\mu_1 + \mu_3 + \mu_4}, & \rho(\{3\}) = \dfrac{\mu_3}{\mu_2 + \mu_4}, \\ \rho(\{4\}) = \dfrac{\mu_4}{\mu_2 + \mu_3}, & \rho(\{1,3\}) = \dfrac{\mu_1 + \mu_3}{\mu_2 + \mu_4}, & \rho(\{1,4\}) = \dfrac{\mu_1 + \mu_4}{\mu_2 + \mu_3}. \end{cases}$$

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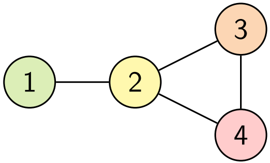

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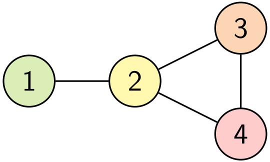
- This condition can be satisfied only if the graph is non-bipartite.

State and state space

- **Sequence of unmatched items ordered by arrival times**:

$$c = (c_1, c_2, \dots, c_\ell) \in \mathcal{V}^*,$$

where c_1 is the class of the oldest unmatched item.



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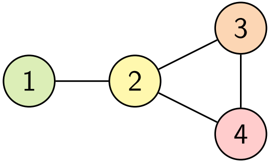

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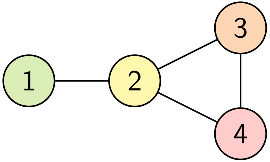
- The set of unmatched items is an **independent set**



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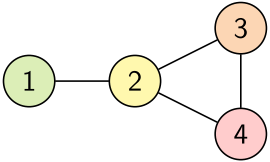

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$$\mathcal{C} = \bigcup_{\mathcal{I} \in \mathbb{I}} \mathcal{C}_{\mathcal{I}},$$

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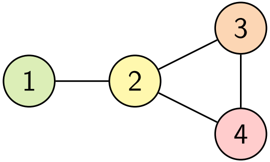

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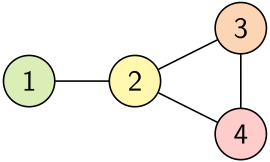
Product-form stationary distribution

- Stationary distribution [MM16]: for $(c_1, c_2, \ldots, c_\ell) \in \mathcal{C} \setminus \{\varnothing\}$,

$$\pi(c_1, c_2, \ldots, c_\ell) = \pi(\varnothing) \prod_{p=1}^{\ell} \frac{\mu_{c_p}}{\sum_{i \in \mathcal{V}(\{c_1, c_2, \ldots, c_p\})} \mu_i}.$$



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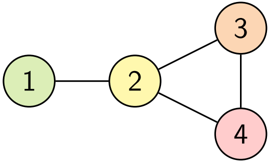

- In our toy example:

$$\pi\left(\textcircled{1}, \textcircled{1}, \textcircled{3}, \textcircled{1}, \textcircled{3}\right)$$

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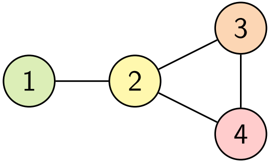

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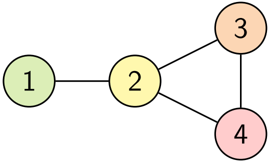

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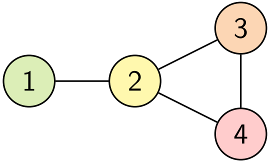

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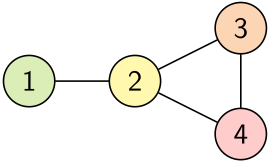

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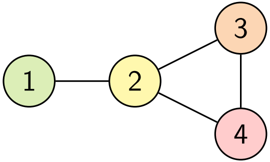

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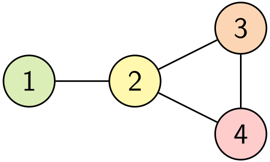

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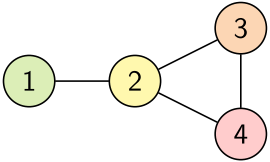

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- In other words, for each $\mathcal{I} \in \mathbb{I}^+$ and $(c_1, c_2, \dots, c_{\ell-1}, c_\ell) \in \mathcal{C}_\mathcal{I}$,

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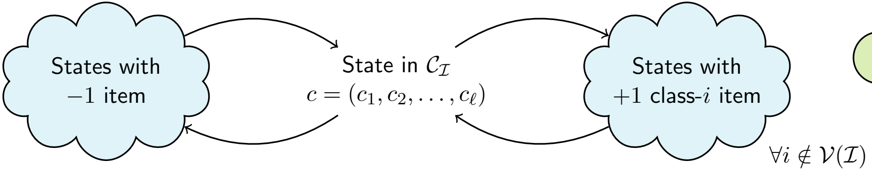
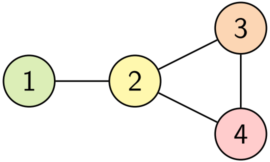
Direct proof using partial balance

Transitions out of and into a state





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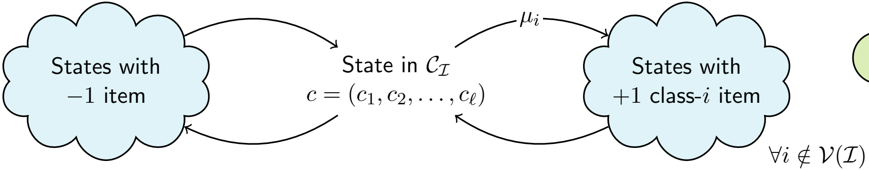

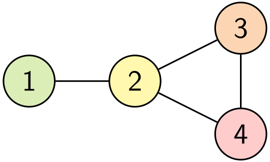

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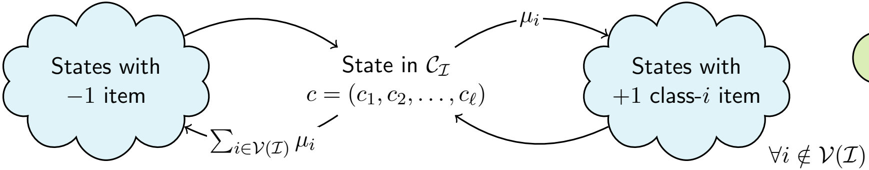

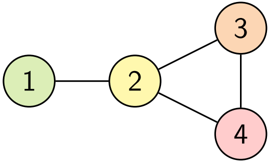

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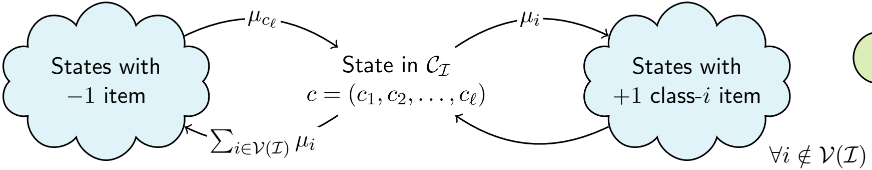

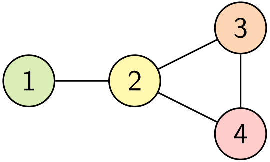

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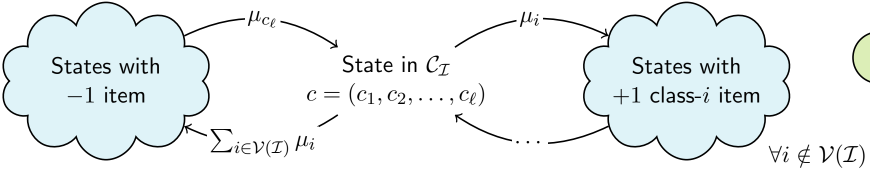

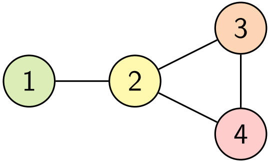

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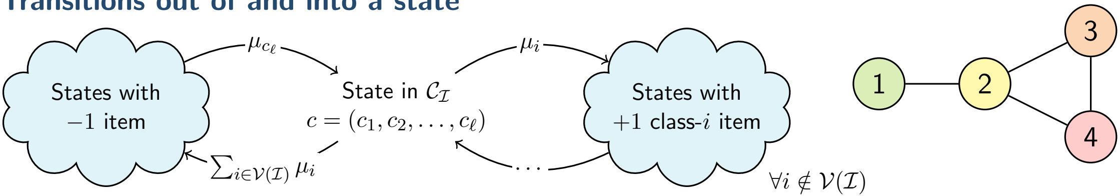


"Partial" balance equations

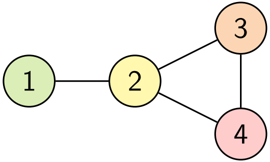
- Balance between state c and states with -1 item:

$$\pi(c_1,\ldots,c_\ell)\left(\sum_{i\in\mathcal{V}(\mathcal{I})}\mu_i\right) = \pi(c_1,\ldots,c_{\ell-1})\mu_{c_\ell},$$

- Balance between state c and states with $+1$ class-i item:

$$\pi(c_1,\ldots,c_\ell)\mu_i = \sum_{p=1}^{\ell+1}\pi(c_1,\ldots,c_{p-1},i,c_p,\ldots,c_\ell)\left(\sum_{j\in\mathcal{V}_i\setminus\mathcal{V}(\{c_1,\ldots,c_{p-1}\})}\mu_j\right), \quad i\notin\mathcal{V}(\mathcal{I}).$$

The bigger picture



What are product-form stationary distributions useful for?

- Compute performance metrics → in the rest of this talk
- Analyze scaling regimes → in the rest of this talk
- Optimization and learning → preprint + ongoing work

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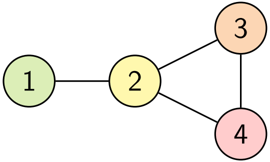

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Variants of the model

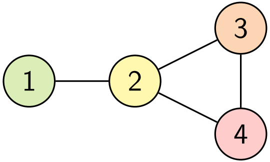
- Bipartite vs. non-bipartite graph
- Abandonment (a.k.a. renegging)
- Admission control

Outline

Normalization constant

- Stationary distribution of the **set of unmatched item classes**:

$$\pi(\mathcal{I}) = \sum_{c \in \mathcal{C}_{\mathcal{I}}} \pi(c), \quad \mathcal{I} \in \mathbb{I}.$$



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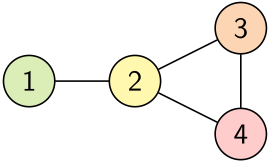
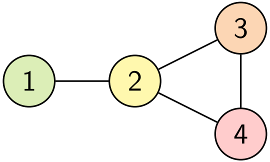


- We prove that this distribution satisfies the recursion

$$\pi(\mathcal{I}) = \frac{\rho(\mathcal{I})}{1 - \rho(\mathcal{I})} \left(\sum_{i \in \mathcal{I}} \frac{\mu_i}{\sum_{j \in \mathcal{I}} \mu_j} \pi(\mathcal{I} \setminus \{i\}) \right), \quad \mathcal{I} \in \mathbb{I}^+.$$

Normalization constant



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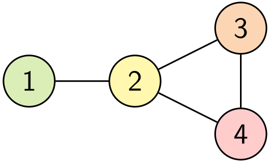

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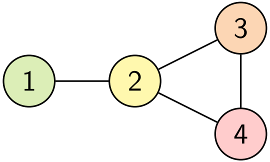
- Key step: $\pi(c_1, \ldots, c_\ell, i) = \pi(c_1, \ldots, c_\ell) \frac{\mu_i}{\sum_{j \in \mathcal{V}(\mathcal{I})} \mu_j}$, $\mathcal{I} \in \mathbb{I}^+$, $i \in \mathcal{I}$, $(c_1, \ldots, c_\ell) \in \mathcal{C}_\mathcal{I} \cup \mathcal{C}_{\mathcal{I} \setminus \{i\}}$.

Performance metrics

- **Waiting probability** of class i:

$$\omega_i = \sum_{\mathcal{I} \in \mathbb{I} \colon i \notin \mathcal{V}(\mathcal{I})} \pi(\mathcal{I}),$$



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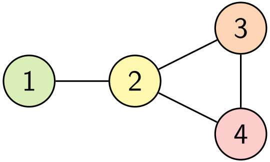

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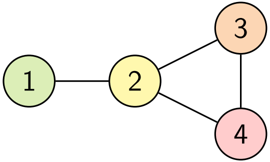

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$$L = \sum_{\mathcal{I} \in \mathbb{I}^+} \ell(\mathcal{I}), \quad \text{where} \quad \ell(\mathcal{I}) = \frac{\pi(\mathcal{I})}{1 - \rho(\mathcal{I})} + \frac{\rho(\mathcal{I})}{1 - \rho(\mathcal{I})} \left(\sum_{i \in \mathcal{I}} \frac{\mu_i}{\sum_{j \in \mathcal{I}} \mu_j} \ell(\mathcal{I} \setminus \{i\}) \right).$$

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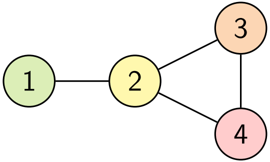

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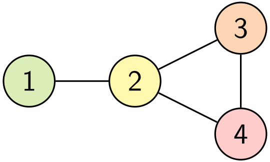
Similar formulas for the per-class performance.

Performance metrics



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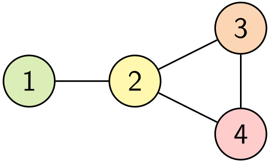

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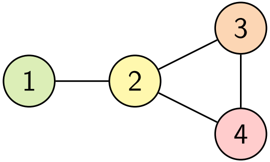
- The **mean waiting time** follows by applying Little's law.
- Similar results for the bipartite variant of the model [CD21].

Heavy-traffic analysis

- Independent set $\mathcal{I} \in \mathbb{I}$ that is **maximal** for the inclusion, meaning that $\mathcal{V}(\mathcal{I}) = \mathcal{V} \setminus \mathcal{I}$.



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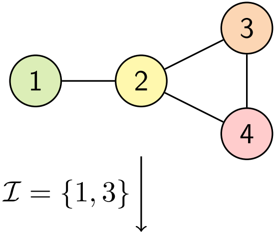

with $\mathcal{I} = \{1, 3\}$

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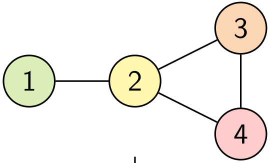

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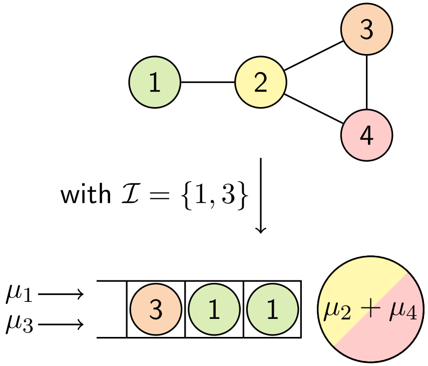

$\simeq$ M/M/1 multi-class queue

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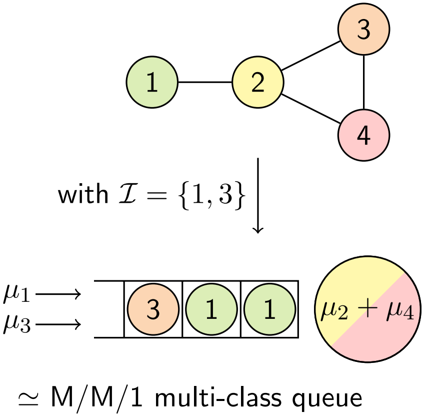

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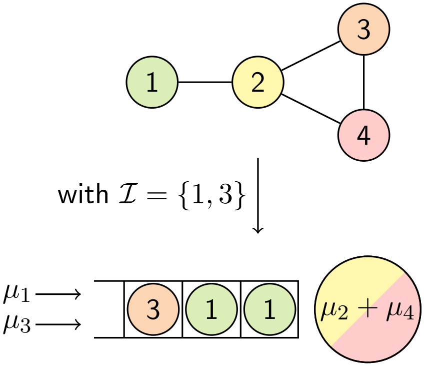

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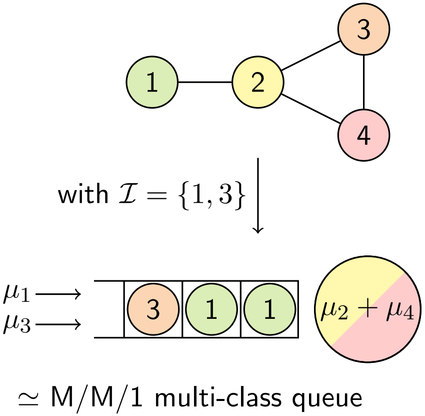
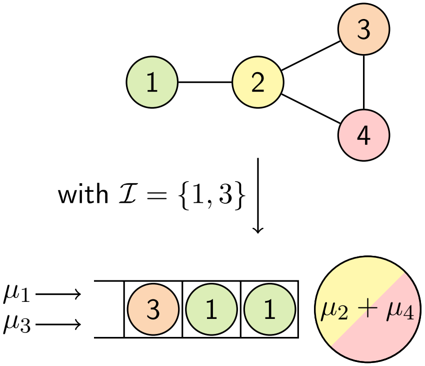
  - items of classes in $\mathcal{I}$ wait with probability 1, while items of classes in $\mathcal{V} \setminus \mathcal{I}$ wait with probability 0,
  - the mean number of unmatched items is $\sim \frac{\rho(\mathcal{I})}{1-\rho(\mathcal{I})}$.



Heavy-traffic analysis

- Independent set $\mathcal{I} \in \mathbb{I}$ that is **maximal** for the inclusion, meaning that $\mathcal{V}(\mathcal{I}) = \mathcal{V} \setminus \mathcal{I}$.
- When the load $\rho(\mathcal{I}) \triangleq \frac{\sum_{i \in \mathcal{I}} \mu_i}{\sum_{i \in \mathcal{V}(\mathcal{I})} \mu_i}$ tends to 1,
  - the set of unmatched classes is $\mathcal{I}$ with probability 1,
  - items of classes in $\mathcal{I}$ wait with probability 1, while items of classes in $\mathcal{V} \setminus \mathcal{I}$ wait with probability 0,
  - the mean number of unmatched items is $\sim \frac{\rho(\mathcal{I})}{1-\rho(\mathcal{I})}$.
- Take-away: **minimizing the maximum load** is likely a good heuristic to optimize performance.



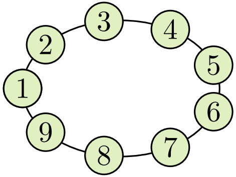
$\simeq$ M/M/1 multi-class queue

Numerical results: Cycle

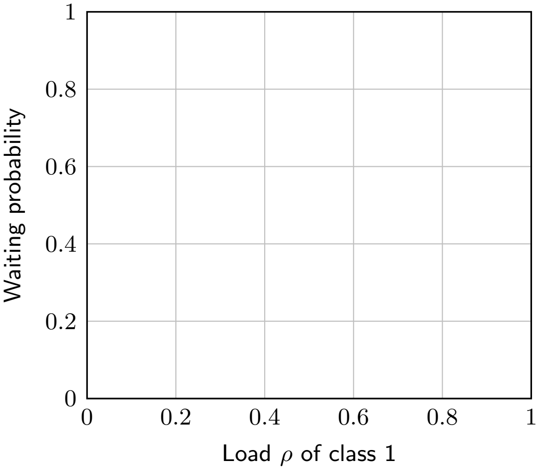
$\mu_1 \propto \rho$

$\mu_2 = \mu_3 = \ldots = \mu_9 \propto \frac{1}{2}$

$\rho = \frac{\mu_1}{\mu_2+\mu_9} = \rho(\{1\})$



Numerical results: Cycle



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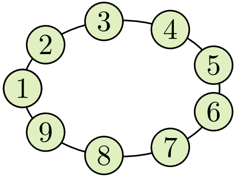
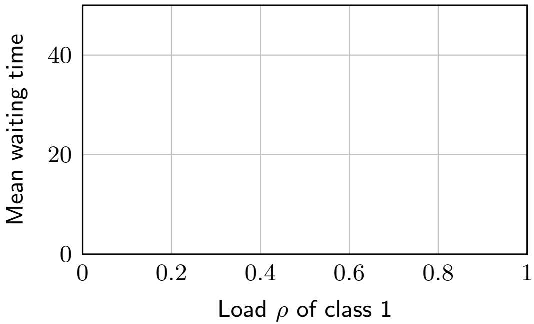
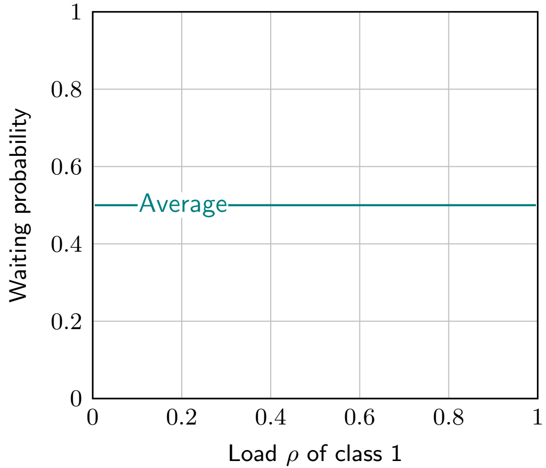




Numerical results: Cycle



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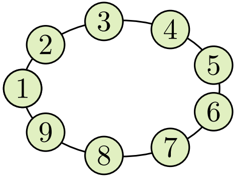
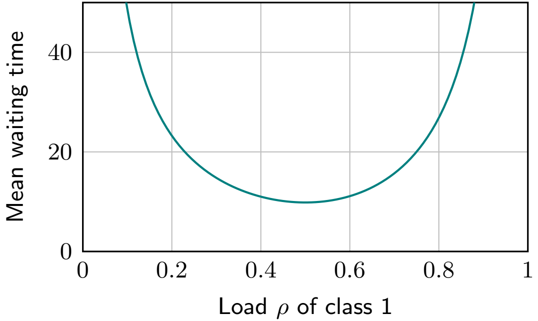
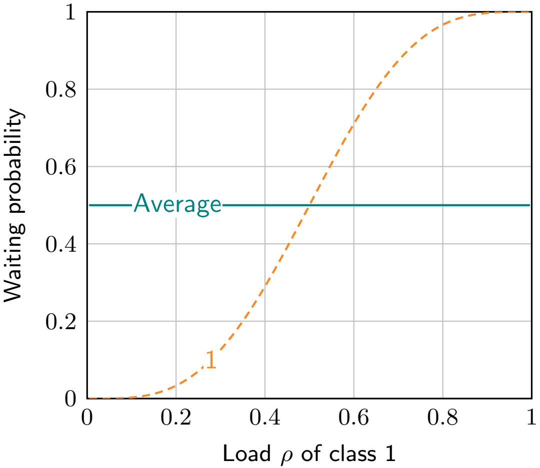




Numerical results: Cycle



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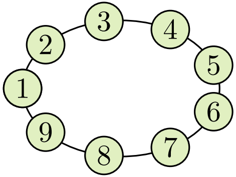
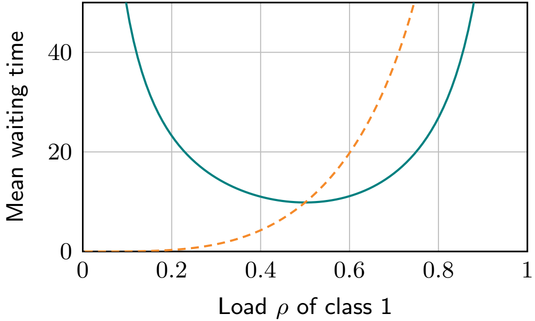
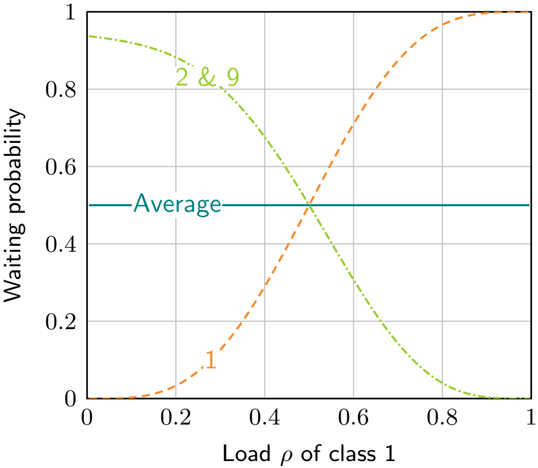




Numerical results: Cycle



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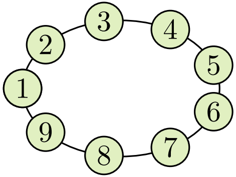
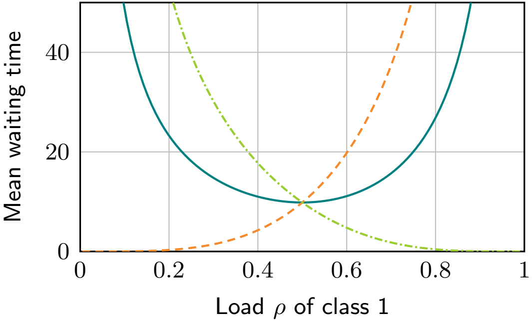
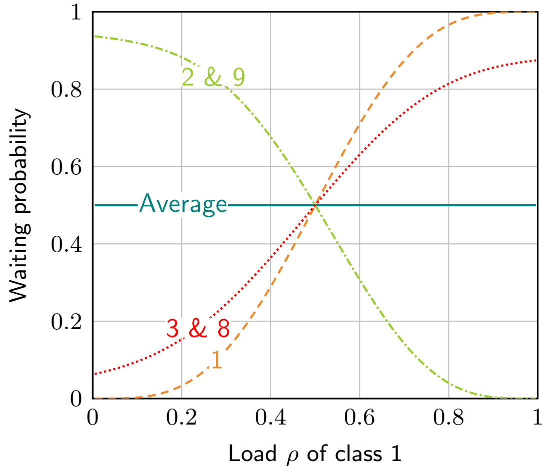




Numerical results: Cycle



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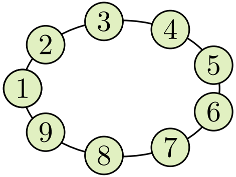
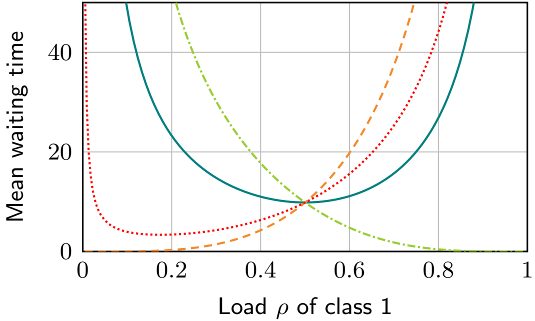
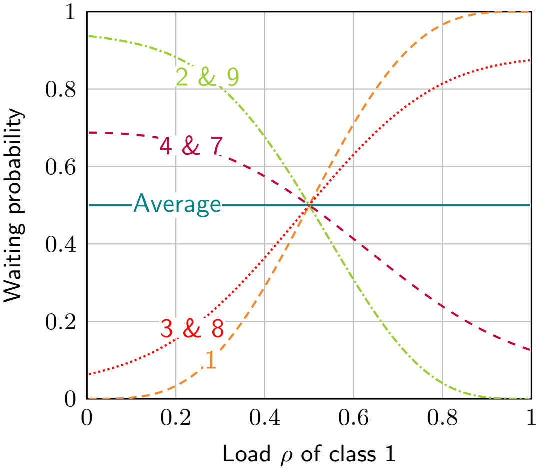




Numerical results: Cycle



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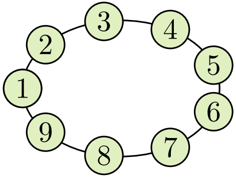
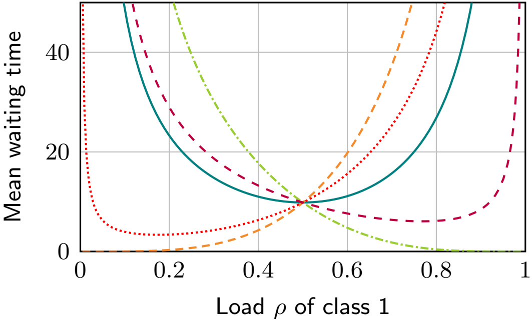
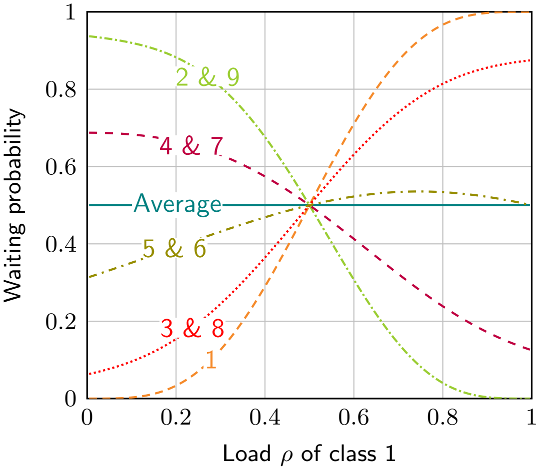




Numerical results: Cycle



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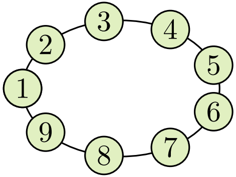
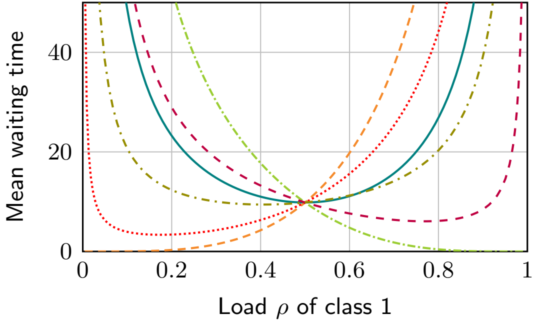


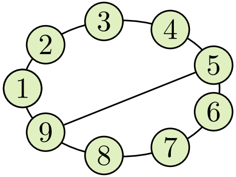


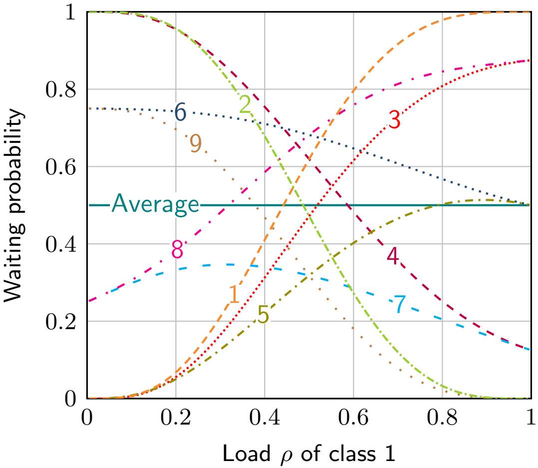
Numerical results: Cycle with a chord

$\mu_1 \propto \rho$

$\mu_2 = \mu_3 = \ldots = \mu_9 \propto \frac{1}{2}$

$\rho = \frac{\mu_1}{\mu_2+\mu_9} = \rho(\{1\})$





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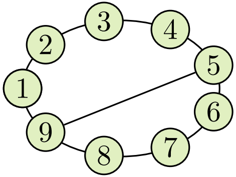
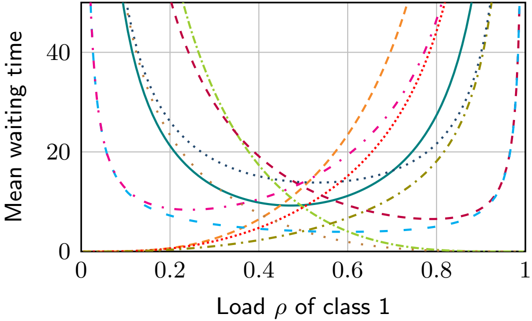
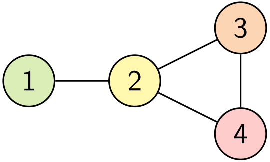




Remark: Augmented state descriptor of [MBM21]

- Independent copy $\overline{\mathcal{V}} = \{\bar{i}, i \in \mathcal{V}\}$ of the set of item classes.



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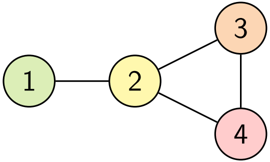

- Independent copy $\overline{\mathcal{V}} = \{\bar{i}, i \in \mathcal{V}\}$ of the set of item classes.
- Augmented state descriptor $s = (s_1, s_2, \ldots, s_k) \in (\mathcal{V} \cup \overline{\mathcal{V}})^*$:

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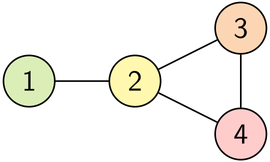

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$s = \emptyset$

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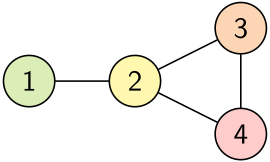

$s = (1)$

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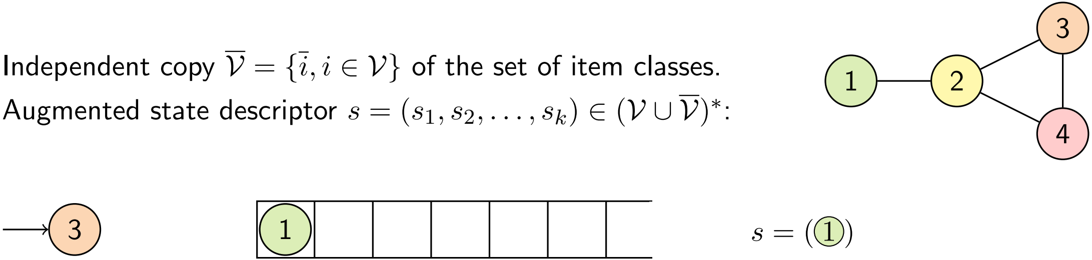
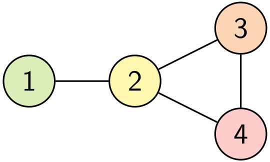


$s = (①)$

Remark: Augmented state descriptor of [MBM21]



- Independent copy $\overline{\mathcal{V}} = \{\bar{i}, i \in \mathcal{V}\}$ of the set of item classes.
- Augmented state descriptor $s = (s_1, s_2, \ldots, s_k) \in (\mathcal{V} \cup \overline{\mathcal{V}})^*$:

$s = (\textcircled{1}, \textcircled{3})$

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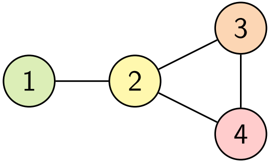

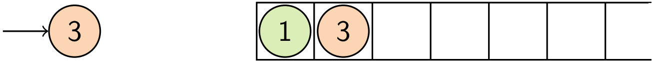

$s = (①, ③)$

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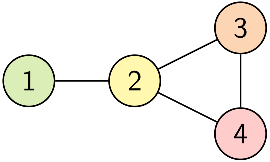
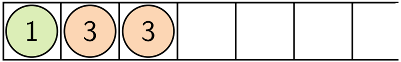
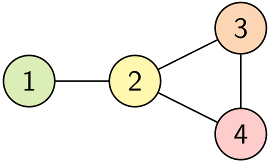




$$s = (\textcircled{1}, \textcircled{3}, \textcircled{3})$$

Remark: Augmented state descriptor of [MBM21]



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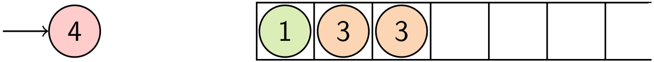
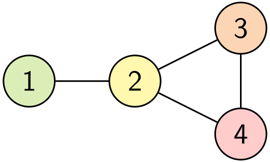


$s = (①, ③, ③)$

Remark: Augmented state descriptor of [MBM21]



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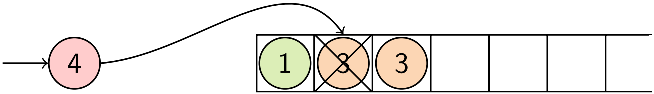

$s = (\textcircled{1}, \textcircled{3}, \textcircled{3})$

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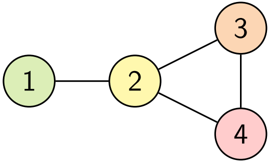

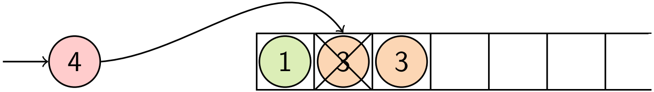

$s = (1, \overline{4}, 3)$

Remark: Augmented state descriptor of [MBM21]

- Independent copy $\overline{\mathcal{V}} = \{\bar{i}, i \in \mathcal{V}\}$ of the set of item classes.
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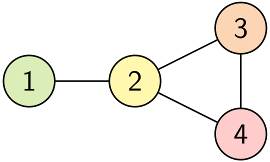

$$s = (①, \overline{④}, ③)$$

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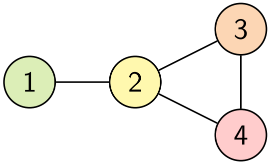

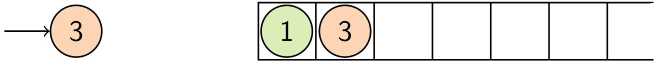

$s = (1, \overline{4}, 3)$

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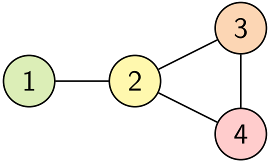
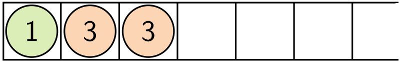
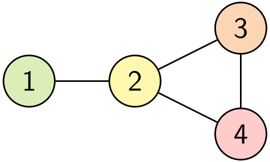




$$s = (1, \bar{4}, 3, 3)$$



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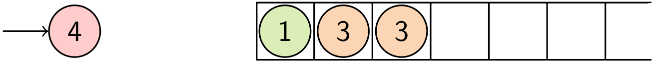
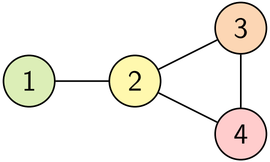


$$s = (\textcircled{1}, \overline{\textcircled{4}}, \textcircled{3}, \textcircled{3})$$



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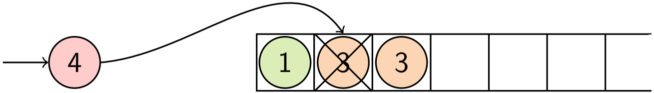

$s = (1, \overline{4}, \overline{4}, 3)$

Remark: Augmented state descriptor of [MBM21]

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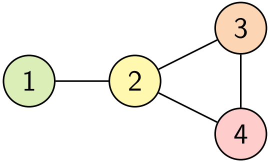

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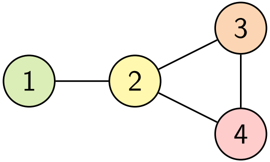

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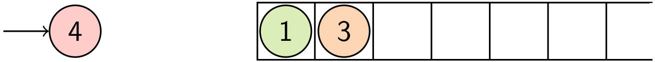

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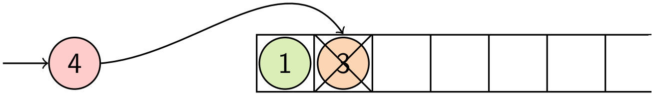

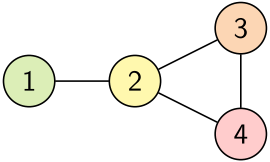

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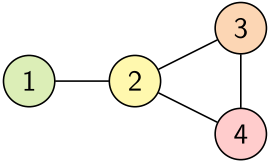

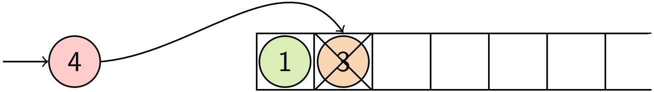

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- Stationary distribution [MBM21]: $\pi(s_1, s_2, \dots, s_k) \propto \prod_{p=1}^{k} \mu_{s_p}.$

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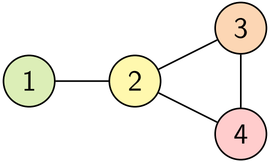

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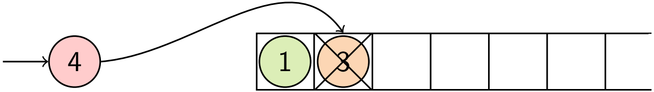

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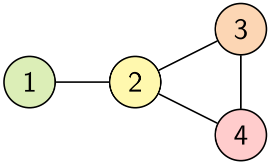
- Stationary distribution [MBM21]: $$\pi(s_1, s_2, \ldots, s_k) \propto \prod_{p=1}^{k} \mu_{s_p}.$$
- The complexity is "hidden" in the description of the state space and normalization constant.

Outline

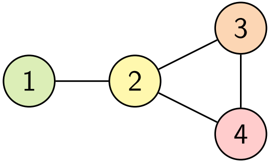
Concluding remarks



Take-away

- Product-form stationary distribution
- Closed-form expressions for performance metrics
- Heavy-traffic analysis

Concluding remarks



Take-away

- Product-form stationary distribution
- Closed-form expressions for performance metrics
- Heavy-traffic analysis

Future works

- Extensions to state-dependent arrival rates? hypergraphs? other policies?
- More fundamental relationship between balance, reversibility, and insensitivity?