Stable solutions for kidney exchanges

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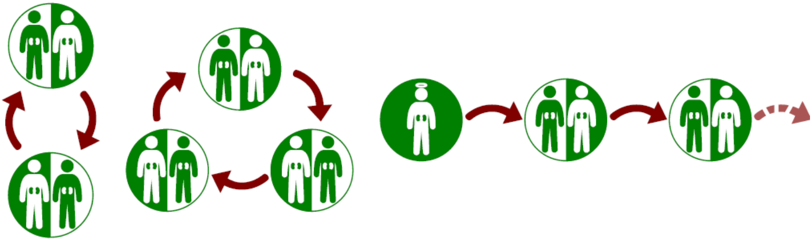
Workshop on Online Stochastic Matching
Toulouse
24 September 2024

Kidney exchange programme (KEP)

Patients with end-stage renal kidney disease exchange their willing, but immunologically incompatible donors with each other...



pairwise, three-way exchanges, altruistic chains

KEPs are operating in South Korea (1991-), USA (2004-), Canada (2009-), Australia (2007-) and in many European countries...

UK KEP: a pairwise kidney exchange from 2007

Daily Mail, Thursday, December 6, 2007

The transplant pact

Two saved as families exchange kidneys

By Luke Salkeld

THEY were both in desperate need of a kidney donor, and both had relatives who were willing to sacrifice an organ.

But without a family match, strangers Donald Planner and Margaret Wearn instead entered into an extraordinary pact.

Mr Planner's daughter donated her kidney to Mrs Wearn, whose husband gave his kidney to Mr Planner.

The operations took place 170 miles apart in synchronised procedures with the organs transported by ambulances travelling in opposite directions between the two hospitals.

Suzanne Wills (left) donated kidney to Margaret Wearn

Margaret's husband Roger (right) donated a kidney to Suzanne's father, Donald Planner

'Completely amazing': Donald Planner with his daughter Suzanne

Margaret and Roger Wearn: 'No different to a direct donation'

UK KEP: solutions in early years

Table 1. Results arising from matching runs from April 2008 to October 2009.

Matching run		2008			2009			
		Apr	Jul	Oct	Jan	Apr	Jul	Oct
# pairs		76	85	123	126	122	95	97
# possible donations		287	235	704	576	760	1212	866
Total #	2-cycles	5	2	14	16	20	54	4
	3 cycles	5	0	109	65	68	164	4
Pairwise exchanges	#2-cycles	2	1	6	5	5	10	2
	size	4	2	12	10	10	20	4
	weight	91	6	499	264	388	739	222
$\leq$ 3-way exchanges	#2-cycles	2	1	2	1	2	2	0
	#3-cycles	4	0	7	5	5	9	2
	size	16	2	25	17	19	31	6
	weight	620	6	1122	633	757	1300	300
the exact algorithm	size of S	5	0	18	13	14	25	3
	$\#Y \subseteq S$	24	0	3480	588	1440	67824	6
Running time (sec)		0.3	0.0	66.0	7.5	19.2	1494.3	2.0
Unbounded exchanges	size	22	2	33	28	28	40	6
	weight	857	6	1546	1134	1275	1894	300
	longest c.	20	2	27	19	23	28	3
Chosen solution (NHSBT)	#2-cycles	2	1	6	5	5	4	1
	#3-cycles	4	0	3	1	2	7	1
	size	16	2	21	13	16	29	5
	weight	620	6	930	422	618	1168	288

- P. Biró, D.F. Manlove and R. Rizzi. Maximum weight cycle packing in directed graphs, with application to kidney exchange programs. Discrete Mathematics, Algorithms and Applications 1(4), pp:499-517, 2009.

2016-2021: COST Action on Kidney Exchanges

EUROPEAN COOPERATION IN SCIENCE AND TECHNOLOGY

CA COST Action CA15210

European Network for Collaboration on Kidney Exchange Programmes

About one per thousand European citizens suffers from end stage renal disease. Living donor kidney transplantation is often the most effective treatment and the alternative of deceased donor kidney transplantation is severely limited by availability. As approximately 40% of living donors are incompatible with their specified recipient, several European countries have independently developed kidney exchange programmes (KEPs).

KEPs aim to match donors optimally to recipients for organ exchange within the population of recipient-donor pairs. Recent research shows that KEPs may greatly improve survival probabilities and quality of life, especially for recipients that are difficult to match. These recipients are disadvantaged disproportionally by the small scale of many national (or local) KEPs in Europe.

KEPs vary regarding the solutions provided for the problems in (i) the policy domain (prioritisation, equity, and accessibility); (ii) the clinical domain (clinical practice and evidence); and (iii) the optimisation domain (methods to solve the hard dynamic multi-criteria matching problems which take clinical evidence and health policy into account). Knowledge

COST Association COST Action CA15210

- Description
- Parties
- Management Committee

General Information*

Chair of the Action:
Prof Joris VAN DE KLUNDERT (NL)

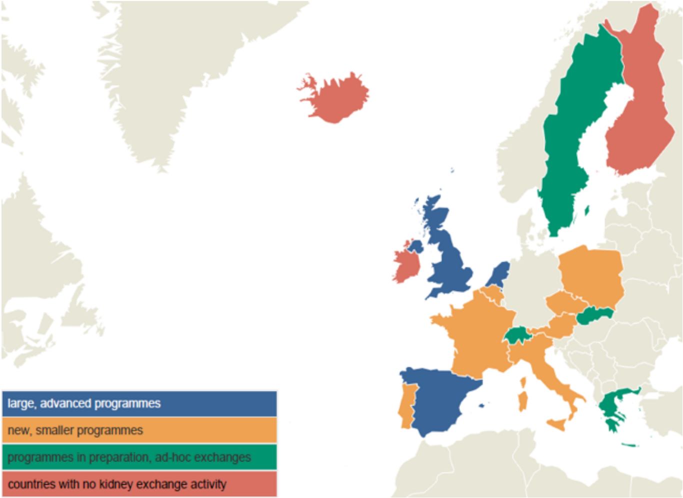
Vice Chair of the Action:
Dr David MANLOVE (UK)

Science officer of the Action:
Ms Estelle EMERIAU

Administrative officer of the Action:
Ms Carmencita MALIMBAN

Downloads*

Kidney exchange programmes in Europe



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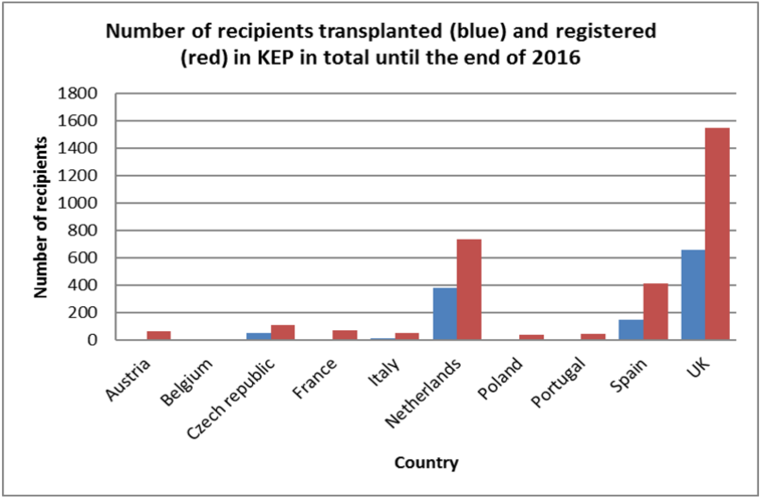
Kidney exchange programmes in Europe



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Details of the European KEPs

	Austria	Belgium	Czech Republic	France	Italy	Netherlands	Poland	Portugal	Spain	UK	Sweden	Switzerland
First exchange in KEP: 20XX	13	14	11	14	07	04	15	13	09	07		
Altruistic donor chains possible?	✓	✗	✓	✗	✓	✓	✗	✓	✓	✓	✗	✗
Compatible pairs/ couples participate?	✗	✗	✓	✗	✓	✓	✓	✗	✓	✓	✓	✓
Multiple donors register for one patient?	✓	✗	✓	✗	✓	✗	✓	✓	✓	✓	✓	✓
Incompatible transplants allowed within KEP?	✓	✗	✓	✗	✗	✗	✗	✗	✓	✓	✓	✓
Single lab carries out cross matching after virtual matching?	✓	✓	✓	✗	✗	✓	✗	✗	✗	✗	✗	✗
Simultaneous surgery required for an exchange in KEP?	✓	✓	✗	✓	✓	✓	✗	✓	✓	✓	✓	✓
Organs usually travel (O) or donors (D)?	D	O	-	O	O	D	O	O	O	O	O	D
Matching process every x months (NR=not regular)	NR	NR	3	3	NR	3	1	3	4	3	na	3
Longest exchange already conducted	3	3	7	2	2	4	3	3	3	3	na	na
Longest chain already conducted	na	na	6	na	6	2	na	na	6	3	na	na

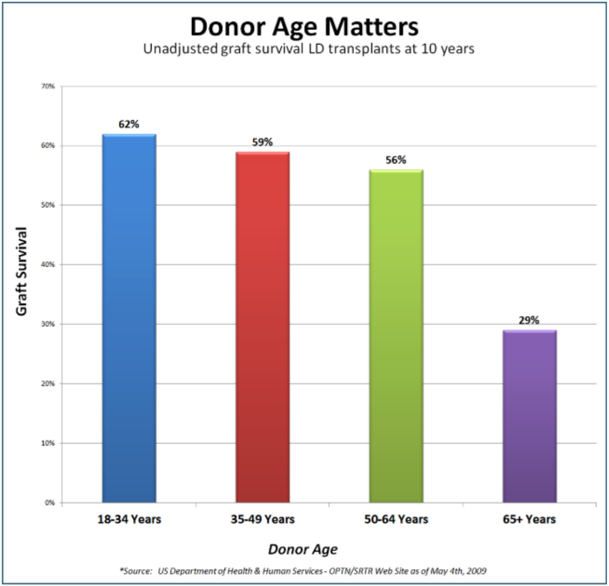
- P. Biró, Bernadette Haase, and et al.: Building kidney exchange programmes in Europe – an overview of exchange practice and activities. Transplantation, 103 (7): 1514-1522, 2019.

Optimisation in Europe: size vs quality

	Belgium	Czech Republic	Netherlands	Poland	Portugal	Spain	Sweden	United Kingdom
max size of solution	1	1	1	1	1	1	1	2
min lengths of the cycles	-	-	4	-	-	-	-	-
max # cycles selected	-	2	-	-	-	2	-	3
max # back-arcs	-	-	-	-	-	3	-	4
max # 2-cycles and 3-cycles with embedded 2-cycles	-	-	-	-	-	-	-	1
min# desensitisations	-	w	-	-	-	-	3	-
max HLA-matching	-	w	-	w	-	-	-	w
max DR-antigen matching in particular	-	w	-	-	-	-	-	-
min age-differences between the donors and patients	5	-	-	w	-	w	-	-
priority for paediatric patient	-	-	-	-	-	w	-	-
priority for patients not yet on dialyses	4	-	-	w	-	-	-	-
priority for highly sensitive patients	-	-	-	w	-	4	-	w
priority for O patients	-	-	-	w	-	-	-	-
priority for hard-to-match patients	3	-	3	w	w	w	2	-
priority for waiting time in KEP	-	-	6	-	-	w	-	w
priority for waiting time on the deceased WL	-	-	-	-	-	-	-	-
priority for time on dialyses	4	-	-	-	w	w	-	-
priority for same blood-group transplants	2	-	2	-	w	w	-	-
priority for O-to-O transplants	-	-	-	-	-	-	-	-
priority for pairs with AB-donors	-	-	-	-	-	w	-	-
max # of transplant centres in (long) cycles	-	-	5	-	-	-	-	-
priority for donor-patients in the same region	-	-	-	-	-	w	-	-
min the donor-donor age differences	-	-	-	w	w	-	-	w

- P. Biró, J. van de Klundert, D. Manlove, and et al.: Modelling and optimisation in European Kidney Exchange Programmes. European Journal of Operational Research, 291:447-456, 2021

NKR (US): quality incentives for compatible pairs



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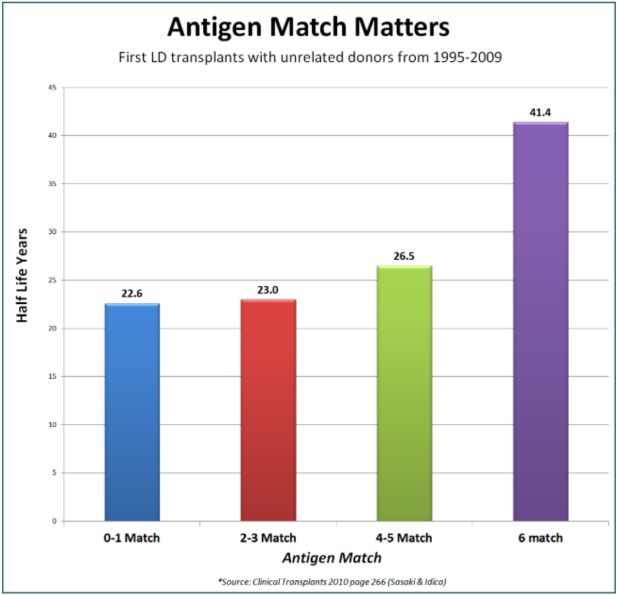
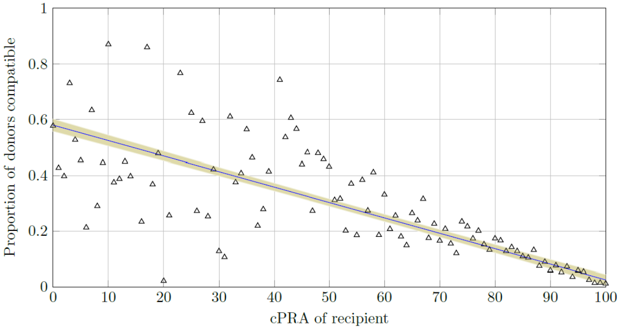


Quality factors: acceptability thresholds in the UK

Significant differences in expected graft survival times based on:

- age of living donor
- HLA-matching between donor and patient
- whether patient needs desensitisation for ABOi transplants

UK practice: acceptability thresholds can be set for each of the above parameters by the individual patients/doctors!



Online stochastic matching in KEP with patients' decisions

Choices of patients with end-stage kidney disease:

1. stay on dialysis
2. register for the deceased kidney waiting list, and wait 2-10 years* for a deceased kidney
3. find a compatible willing donor and get a direct living transplant
4. find a half-compatible (ABOi) donor, and get a direct living transplant after desensitisation treatment (UK: 3%, France: 18%, Germany: 25%, one treatment costs 100k EUR)
5. find some willing donor(s) and register for a KEP and wait 3-36 months** to get an exchange living donor

* depending on the country, age, sensitivity (PRA)
** depending on the characteristics of donor(s), recipients (blood types, ages, sensitivity of the recipient), **their acceptability thresholds, and the richness of the KEP pool**!

UK: expected waiting times in deceased WL vs KEP

Incompatible Pairs Living Donor Kidney Application

NHS Blood and Transplant

Variable	Select
Recipient Blood Group	A
Calculated Reaction Frequency	85-94
Donor Blood Group	O
ABOi TX with willing Donor†	Select
HLAi TX with willing Donor†	Select
Recipient Age	51-60

Reset

Estimated Chance of Transplant

	Deceased Donor	NLDKSS	ABOi	HLAi
6 Months	<10%	41-50%	-	-
1 Year	11-20%	71-80%	-	-
3 Years	41-50%	>90%	-	-

-

Transplant Survival Rates

	Deceased Donor	NLDKSS	ABOi	HLAi
6 Months	93%	97%	-	-
1 Year	91%	96%	-	-
3 Years	85%	93%	-	-

UK: expected waiting times in deceased WL vs KEP

Incompatible Pairs Living Donor Kidney Application

Variable	Select
Recipient Blood Group	O
Calculated Reaction Frequency	85-94
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6 Months	<10%	<10%	-	-
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3 Years	41-50%	31-40%	-	-

-

Transplant Survival Rates

	Deceased Donor	NLDKSS	ABOi	HLAi
6 Months	93%	97%	-	-
1 Year	91%	96%	-	-
3 Years	85%	93%	-	-

Online stochastic matching in KEP with patients' decisions

The design of the KEP policy also matters!

- legal/design constrains (e.g. France: pairwise exchanges only, no altruistic donation, no multiple registered donors, no compatible donors)
- optimisation criteria used in matching runs
- allowing patients/doctors to express their quality thresholds

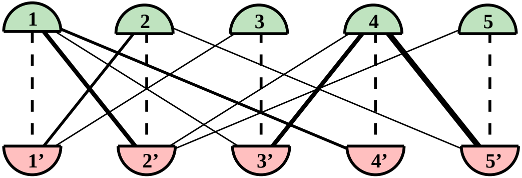
Number of living transplants in five Western European countries:

2022	UK	Spain	Netherlands	France	Germany
total living	848	350	516	514	535
# in KEP	216	24	30	4	-
# direct ABOi	24	40	39	95	119

The performance of national/international KEP should be analysed by agent-based simulations, and besides the optimisation policies, the decisions of the patients should also be taken into account!

Individual fairness vs social welfare (#transplants)

Patients needing transplant exchange their incompatible donors



Qualities of transplants are determined by age and HLA-matching
$\sim$ expected lifetime gains $\sim$ graph survival times

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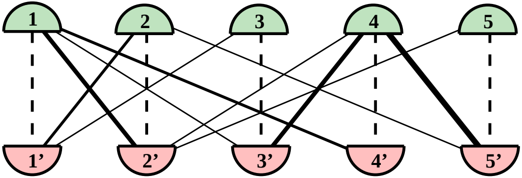

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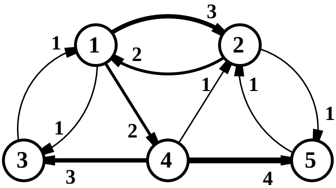

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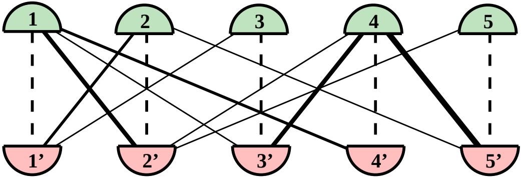

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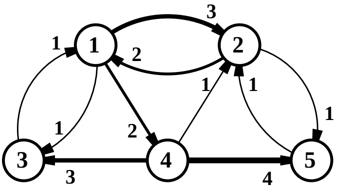

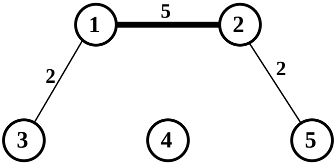

set of 2-way exchanges $\iff$ matching in an undirected graph

Individual fairness vs social welfare (#transplants)

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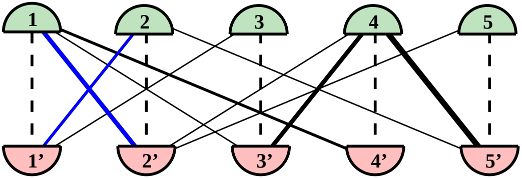

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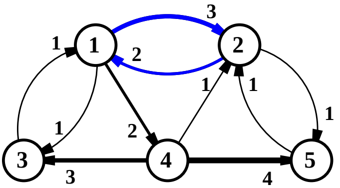
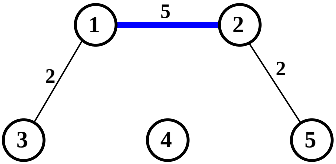


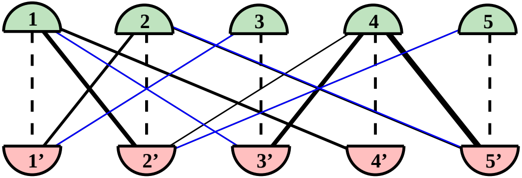


set of 2-way exchanges $\iff$ matching in an undirected graph

quality transplants for some vs kidneys for more patients

Individual fairness vs social welfare (#transplants)

Patients needing transplant exchange their incompatible donors



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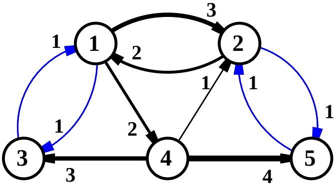

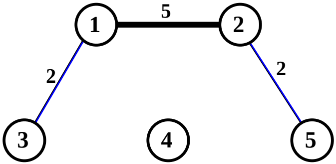

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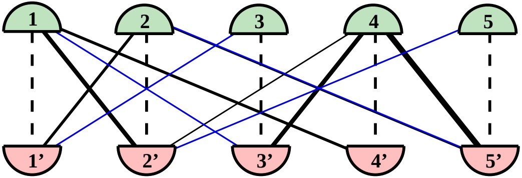

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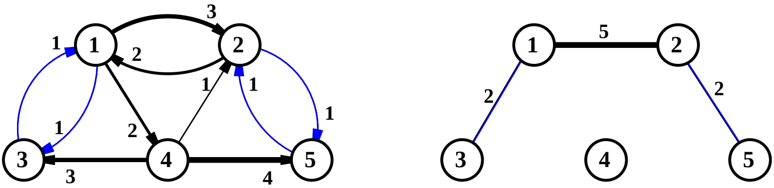


set of 2-way exchanges $\iff$ matching in an undirected graph

quality transplants for some vs kidneys for more patients

stable exchange (= core solution): no blocking cycle

Complexity of exchange problems

		exchanges		
		pairwise		
maximum size/weight	does exist?	**yes**		
	hard to find?			
stable	does exist?			
	hard to find?			

Complexity of exchange problems

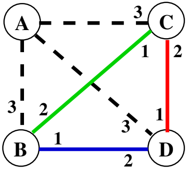
		exchanges		
		pairwise		
maximum size/weight	does exist?	yes		
	hard to find?	**P**		
stable	does exist?			
	hard to find?			

Edmonds (1967): Polynomial time algorithms for maximum size / maximum weight matching problem.

Complexity of exchange problems

		exchanges		
		pairwise		
maximum size/weight	does exist?	yes		
	hard to find?	P		
stable	does exist?	**may not**		
	hard to find?			

stable pairwise exchange = stable roommates



Gale and Shapley (1962):

Stable matching may not exist!

Complexity of exchange problems

		exchanges		
		pairwise		
maximum size/weight	does exist?	yes		
	hard to find?	P		
stable	does exist?	may not		
	hard to find?	**P**		

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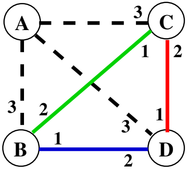

Gale and Shapley (1962):
Stable matching may not exist!
Irving (1985): A stable matching can be found in linear time, if one exists.

Complexity of exchange problems

		exchanges		
		pairwise		
maximum size/weight	does exist?	yes		
	hard to find?	P		
stable	does exist?	may not		
	hard to find?	P		

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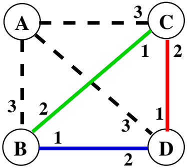

Gale and Shapley (1962):
Stable matching may not exist!
Irving (1985): A stable matching can be found in linear time, if one exists.

Abraham-Biró-Manlove (2006): The problem of minimising the number of blocking pairs is NP-hard.

Complexity of exchange problems

		exchanges		
		pairwise	**2-3-way**	
maximum size/weight	does exist?	yes	**yes**	
	hard to find?	P		
stable	does exist?	may not		
	hard to find?	P		

Complexity of exchange problems

		exchanges		
		pairwise	2-3-way	
maximum size/weight	does exist?	yes	yes	
	hard to find?	P	**NP-hard**	
stable	does exist?	may not		
	hard to find?	P		

Biró-Manlove-Rizzi (2009): Finding a maximum size/weight 2-3-way exchange is NP-hard, but there is a $O(2^{\frac{m}{2}})$-time exact algorithm. This was implemented for NHS Blood and Transplant in 2007 and used to compute optimal solutions subsequently.

- P. Biró, D.F. Manlove and Romeo Rizzi. Maximum weight cycle packing in directed graphs, with application to kidney exchange programs. Discrete Mathematics, Algorithms and Applications, 1 (4) : 499-517, 2009.
- P. Biró and E. McDermid. Three-sided stable matchings with cyclic preferences. Algorithmica, 58: 5-18, 2010. (COMSOC 2008)

Complexity of exchange problems

		exchanges		
		pairwise	2-3-way	
maximum size/weight	does exist?	yes	yes	
	hard to find?	P	NP-hard	
stable	does exist?	may not	**may not**	
	hard to find?	P	**NPc**	

Biró-Manlove-Rizzi (2009): Finding a maximum size/weight 2-3-way exchange is NP-hard, but there is a $O(2^{\frac{m}{2}})$-time exact algorithm. This was implemented for NHS Blood and Transplant in 2007 and used to compute optimal solutions subsequently.

Biró-McDermid (2010): Stable 2-3-way exchange may not exist, and the related problem is NP-complete, even for tripartite graphs.

- P. Biró, D.F. Manlove and Romeo Rizzi. Maximum weight cycle packing in directed graphs, with application to kidney exchange programs. Discrete Mathematics, Algorithms and Applications, 1 (4) : 499-517, 2009.
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Complexity of exchange problems: unbounded case

		exchanges		
		pairwise	2-3-way	**unbounded**
maximum size/weight	does exist?	yes	yes	**yes**
	hard to find?	P	NPc	
stable	does exist?	may not	may not	
	hard to find?	P	NPc	

Complexity of exchange problems: unbounded case

		exchanges		
		pairwise	2-3-way	unbounded
maximum size/weight	does exist?	yes	yes	yes
	hard to find?	P	NPc	**P**
stable	does exist?	may not	may not	
	hard to find?	P	NPc	

Graph Theory folklore: The problem of finding a maximum size/weight (unbounded) exchange is P-time solvable.

Complexity of exchange problems: unbounded case

		exchanges		
		pairwise	2-3-way	unbounded
maximum size/weight	does exist?	yes	yes	yes
	hard to find?	P	NPc	P
stable	does exist?	may not	may not	**yes**
	hard to find?	P	NPc	**P**

Graph Theory folklore: The problem of finding a maximum size/weight (unbounded) exchange is P-time solvable.

Shapley-Scarf (1972): Stable exchange always exists, one can be found by the Top Trading Cycle algorithm of Gale.
This was the original model in the seminal paper on Kidney exchange by Roth-Sönmez-Ünver (QJE 2005)!

Overviews on **European KEPs** by the ENCKEP COST Action

- P. Biró, Bernadette Haase, and et al.: Building kidney exchange programmes in Europe – an overview of exchange practice and activities. Transplantation, 103 (7): 1514-1522, 2019.
- P. Biró, J. van de Klundert, D. Manlove, and et al.: Modelling and optimisation in European Kidney Exchange Programmes. EJOR, 2021.

Stable exchanges: individual fairness, respecting improvement property

- Klimentova-Biró-Costa-Viana-Pedroso: Novel IP formulations for the stable kidney exchange problem. EJOR 2022
- Biró-Klijn-Klimentova-Viana: Shapley-Scarf Housing Markets: Respecting Improvement, Integer Programming, and Kidney Exchange. MOR 2023
- Schlotter-Biró-Fleiner: The core of housing markets from an agent's perspective: Is it worth sprucing up your home? MOR 2024

Compensation schemes for international KEPs: fairness for countries

- Biró-Kern-Paulusma-Pálvölgyi: Generalized Matching Games for International Kidney Exchange. AAMAS-2019
- Benedek-Biró-Kern-Paulusma: Computing Balanced Solutions for Large International Kidney Exchange Schemes. AAMAS-2022
- Benedek-Biró-Csáji-Johnson-Paulusma-Ye: Computing Balanced Solutions for Large International Kidney Exchange Schemes When Cycle Length Is Unbounded. AAMAS-2024

New publications on stable exchanges

- Klimentova-Biró-Costa-Viana-Pedroso: Novel IP formulations for the stable kidney exchange problem (2022-EJOR)

-Computation of bounded length stable exchanges by IP techniques
-Measuring size vs stability tradeoffs

- Biró-Klijn-Klimentova-Viana: Shapley-Scarf Housing Markets: Respecting Improvement, Integer Programming, and Kidney Exchange (2021, 2023-MOR)

-Respecting improvement property for strong core: if a patient brings a better donor (e.g., younger or with a better blood type: $0 > A, B > AB$), or an additional donor, then in the TTC solution she must receive an exchange donor at least as good as before.

- Schlotter-Biró-Fleiner: The core of housing markets from an agent's perspective: Is it worth sprucing up your home? (2021-WINE, 2024-MOR)

-Follow-up results for the core solutions under partial orders

Main results of Biró-Klijn-Klimentova-Viana (2023 MOR)

1. Proving the respecting improvement property for strong core and CE for unbounded exchanges, and examples for violations
2. New IP models for computing the strong core, CE, core
3. Simulations for measuring the price of fairness and the amount of respecting improvement violations for kidney exchange instances

Shapley-Scarf 1974 housing market model

A housing market (N, R) consists of
set of agents $N = \{1, \ldots, n\}$ with one house each, where
each agent $i \in N$ has complete and transitive (weak) preferences
R_i over the houses, where P_i denotes the strict relation.

An allocation x is a one-to-one re-assignment of the houses to agents, where x_i is the allotment of i.

A coalition $S \subseteq N$ blocks x if there is an allocation z s.t.
(1) $\{z_i : i \in S\} = S$ and
(2) for each $i \in S$, $z_i P_i x_i$.
x is in the core of the market if there is no blocking coalition.

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An allocation x is a one-to-one re-assignment of the houses to agents, where x_i is the allotment of i.

A coalition $S \subseteq N$ weakly blocks x if there is an allocation z s.t.
(1) $\{z_i : i \in S\} = S$,
(2) for each $i \in S$, $z_i R_i x_i$, and
(3) for some $j \in S$, $z_j P_j x_j$.
x is in the strong core of the market if there is no weakly blocking coalition.

Shapley-Scarf 1974 housing market model

A housing market (N, R) consists of
set of agents $N = \{1, \ldots, n\}$ with one house each, where
each agent $i \in N$ has complete and transitive (weak) preferences
R_i over the houses, where P_i denotes the strict relation.

An allocation x is a one-to-one re-assignment of the houses to agents, where x_i is the allotment of i.

For price-vector p let p_i denote the price of object i. A competitive equilibrium is a pair (x, p) s.t.
(1) for each agent $i \in N$, object x_i is affordable, i.e., $p_{x_i} \leq p_i$ and
(2) for each agent $i \in N$, each object she prefers to x_i is not affordable, i.e., jP_ix_i implies $p_j > p_i$.
An allocation is a competitive allocation if it is part of some competitive equilibrium.

Shapley-Scarf 1974 housing market model

A housing market (N, R) consists of
set of agents $N = \{1, \ldots, n\}$ with one house each, where
each agent $i \in N$ has complete and transitive (weak) preferences
R_i over the houses, where P_i denotes the strict relation.

An allocation x is a one-to-one re-assignment of the houses to agents, where x_i is the allotment of i.

Wako (1999) showed that a competitive allocation can be characterised by the lack of antisymmetrically weakly blocking coalitions, that is a coalition $S \subseteq N$ s.t.
(1) $\{z_i : i \in S\} = S$,
(2) for each $i \in S$, either $z_i = x_i$ or $z_j P_j x_j$, and
(3) for some $j \in S$, $z_j P_j x_j$.
We also call the set of competitive allocation as Wako-core.

Shapley-Scarf (1974): housing market

Gale's Top Trading Cycles algorithm (TTC)

- Everybody points to the best house in the market (or one of the best houses if we have ties), we get at least one TTC
- Agents in a TTC exchange and then they leave the market
- We repeat the process in the remaining market...
  see an example at http://www.matchu.ai/

Shapley-Scarf (1974): housing market

Gale's Top Trading Cycles algorithm (TTC)

- Everybody points to the best house in the market (or one of the best houses if we have ties), we get at least one TTC
- Agents in a TTC exchange and then they leave the market
- We repeat the process in the remaining market...
  see an example at `http://www.matchu.ai/`
- The resulting solution is a competitive allocation.

  Proof: We set the prices of the houses decreasingly according to their removal order in the TTC...

  An allocation is competitive $\iff$ it can be obtained by TTC

Further theoretical results

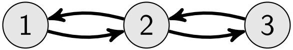
Roth-Postlewaite (1977): For strict preferences, the TTC algorithm returns the unique competitive allocation, which is also the unique strong core allocation.

Roth (1982): For strict preferences, the TTC algorithm is strategy-proof.

Ma (1994): For strict preferences, the TTC algorithm is the unique mechanism which is individually rational, Pareto-efficient and strategy-proof.

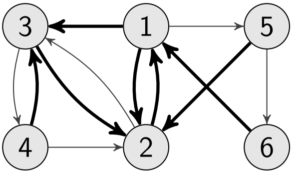
Quint-Wako (2004): For weak preferences, it is possible to find a strong core allocation efficiently, if there exists one.

An example for empty strong core:



An example for core, CE, and strong core

Acceptability graph:



Preferences:

1	2	3	4	5	6
2,3	1	2	3	2	1
5	3	4	2	6	

Allocations:

$x^a = \{(1,3,2)\}$
$x^b = \{(1,2),(3,4)\}$
$x^c = \{(1,5,2),(3,4)\}$
$x^d = \{(1,3,4,2)\}$
$x^e = \{(1,5,6),(2,3,4)\}$

Solution sets:

strong core: $\{x^a\}$
CE/Wako-core: $\{x^a, x^b\}$
core: $\{x^a, x^b, x^c, x^d\}$
max size allocations: $\{x^e\}$

By definition strong core $\subseteq$ CE/Wako-core $\subseteq$ core

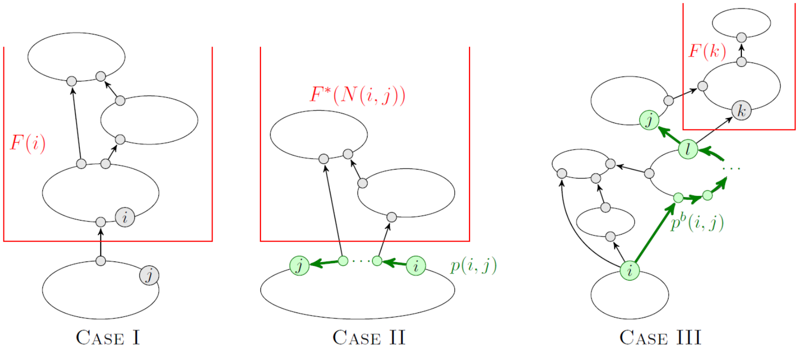
New results on the respecting improvement property

Biro-Klijn-Klimentova-Viana (2021-WP, 2023-MOR): Suppose that the house of an agent i becomes better to another agent j, then

- for strict preferences, then the TTC solution in the new market can only be better for i than the TTC solution in the old market (RI-property)
- for weak preferences, if strong core solutions exist for both the old and new markets then the above respecting improvement property holds (conditional RI-property)
- for weak preferences, the best/worst competitive allocation can only get better for the improving agent (RI-best/worst property), moreover, for TTC with uniform random tie-breakings the new probabilistic allocation for i stochastically dominates the old one

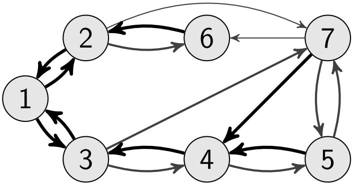
Schlotter-Biro-Fleiner (2021-WINE, 2024-MOR): the RI-best property holds for core allocations even for partial orders, but the RI-worst property is violated even for strict preferences

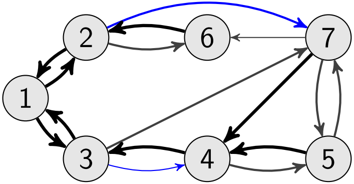
Sketch proof for respecting improvement property of TTC



- Case I: agent *i* has left the market earlier - no effect
- Case II: they left the market at the same time - TTC cycle may get shorter, but *i* gets the same house
- Case III: *j* left the market earlier - *j* will be involved in a TTC earlier, she may get a better house

An example for stochastic dominance for CE





Preferences:

1	2	3	4	5	6	7
2,3	1	1	3	4	2	4
	6	4,7	5	7		5
	7					6

New preferences:

1	2	3	4	5	6	7
2,3	1	1	3	4	2	4
	6,7	7	5	7		5
		4				6

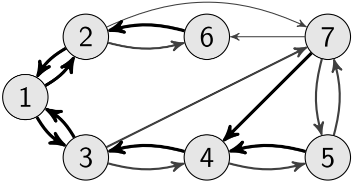
Competitive allocations:

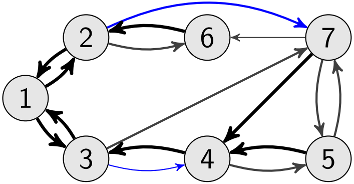
$$x^a = \{(1,3),(2,6),(4,5)\}$$
$$x^c = \{(1,2),(3,4),(5,7)\}$$
$$x^d = \{(1,2),(3,7,4)\}$$

New competitive allocations:

$$x^a = \{(1,3),(2,6),(4,5)\}$$
$$x^b = \{(1,3),(2,7,6),(4,5)\}$$
$$x^d = \{(1,2),(3,7,4)\}$$

An example for stochastic dominance for CE





Preferences:

1	2	3	4	5	6	7
2,3	1	1	3	4	2	4
	6	4,7	5	7		5
	7					6

New preferences:

1	2	3	4	5	6	7
2,3	1	1	3	4	2	4
	6,7	7	5	7		5
		4				6

Competitive allocations:

1/2 $x^a = \{(1,3),(2,6),(4,5)\}$

1/4 $x^c = \{(1,2),(3,4),(5,7)\}$

1/4 $x^d = \{(1,2),(3,7,4)\}$

TTC probabilities!

New competitive allocations:

1/4 $x^a = \{(1,3),(2,6),(4,5)\}$

1/4 $x^b = \{(1,3),(2,7,6),(4,5)\}$

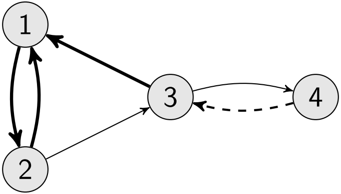
1/2 $x^d = \{(1,2),(3,7,4)\}$

TTC probabilities!

Violations of the respecting improvement property

Example for **max size unbounded exchanges**:

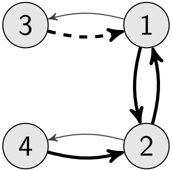
If house 3 becomes acceptable for agent 4, then 3 receives a worse house in the max size solution.



Violations of the respecting improvement property

Example for **max size pairwise exchanges**:

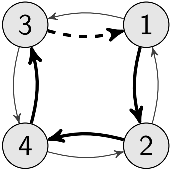
If donor 1 becomes acceptable for recipient 3, e.g., because recipient 1 brings a second donor, then she receives a worse kidney in the max size/weight solution.



Violations of the RI-worst property

Example for (strong) core pairwise exchanges, strict preferences:

If student 1 becomes acceptable for school 3, e.g., because she improves her score, then she receives a worse school seat in the school-optimal stable matching.



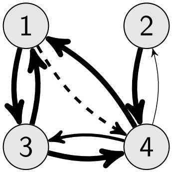
Balinski-Sönmez (1999): The student-optimal stable matching by the Gale-Shapley algorithm respects improvements for students.

Schlotter-Biro-Fleiner (2021): for strict preferences, the core/CE/strong core solutions satisfy the RI-best property (generalisation for the roommates problem)

Violations of the RI-best property

Example for (strong) core pairwise exchanges, weak preferences:

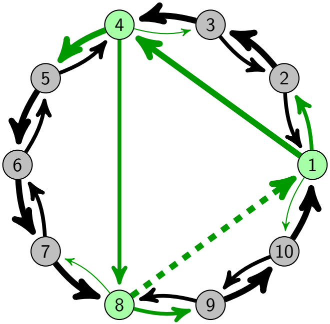
If agent 1 becomes acceptable for possible roommate 4, then she receives a strictly worse roommate in the unique stable matching.



Violations of the RI-best property

Example for (strong) core 3-way exchanges, strict preferences:

If house 1 becomes acceptable for agent 8, then 1 receives a strictly worse allotment in the best core allocation than before.



IP formulations with edge variables

$$y_{ij} = \begin{cases} 1 & \text{if agent } i \text{ receives object } j; \\ 0 & \text{otherwise.} \end{cases}$$

$$\sum_{j:(i,j)\in E} y_{ij} = 1 \qquad \forall i \in N \qquad (1)$$

$$\sum_{j:(j,i)\in E} y_{ji} = 1 \qquad \forall i \in N \qquad (2)$$

$$y_{ij} \in \{0,1\} \qquad \forall (i,j) \in E \qquad (3)$$

IP: no blocking constraints for core/Wako/strong core

Quint-Wako (2004): IP formulations for all permutations re-written for cycles (works for both bounded&unbounded)

core:

$$\sum_{(i,j)\in A(c)} \sum_{k:kR_ij} y_{ik} \geq 1 \qquad \forall c \in \mathcal{C} \qquad (4)$$

CE/Wako-core:

$$\sum_{(i,j)\in A(c)} y_{ij} + |c| \cdot \left[\sum_{(i,j)\in A(c)} \sum_{k:kR_ij,k\neq j} y_{ik} \right] \geq |c| \qquad \forall c \in \mathcal{C} \qquad (5)$$

strong core:

$$\sum_{(i,j)\in A(c)} \sum_{k:kI_ij} y_{ik} + |c| \cdot \left[\sum_{(i,j)\in A(c)} \sum_{k:kP_ij} y_{ik} \right] \geq |c| \qquad \forall c \in \mathcal{C} \qquad (6)$$

IP: new compact formulations for unbounded case

We introduce prices (corresponding to a topological order):

$$p_i \in \{1, \ldots, n\} \qquad \forall i \in N \qquad (7)$$

core:

$$p_i + 1 \leq p_j + n \cdot \sum_{k:kR_ij} y_{ik} \qquad \forall (i,j) \in E \qquad (8)$$

CE/Wako-core: in addition to the above constraints

$$p_i \leq p_j + n \cdot (1 - y_{ij}) \qquad \forall (i,j) \in E \qquad (9)$$

strong core: in addition to the above constraints

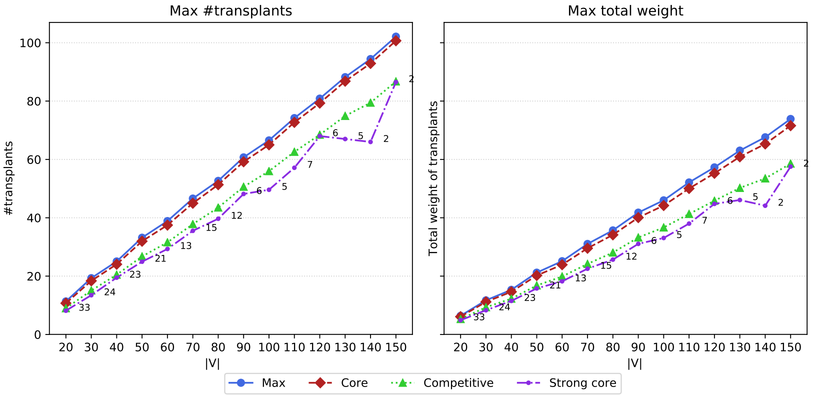
$$p_i \leq p_j + n \cdot \left(\sum_{k:kP_ij} y_{ik} \right) \qquad \forall (i,j) \in E \qquad (10)$$

Computer simulations for bounded/unbounded cases

Testing strong core/Wako-core/core/**max size/max weight** solutions on realistic kidney exchange instances

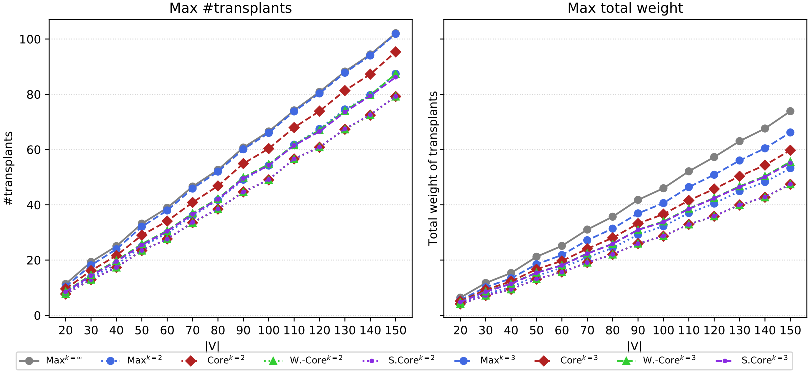
- efficiency of IP formulations
- price of fairness (i.e., size vs stability)
- counting the number of violations of the RI-best property for difference solution concepts

Optimality vs stability tradeoff (i.e., price of fairness)



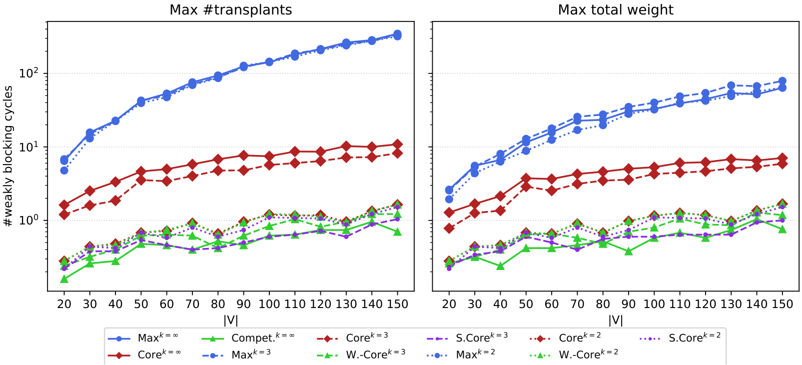
Number of transplants (left) and total weight of transplants (right) for unbounded length exchanges and weak preferences.

Optimality vs stability tradeoff (i.e., price of fairness)



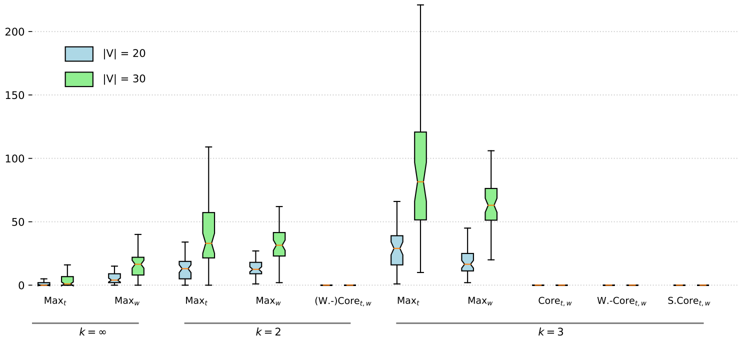
Comparison of the number of transplants (left) and the total weight of transplants (right) for bounded length exchange cycles ($k = 2, 3$) and weak preferences.

Optimality vs stability tradeoff (i.e., price of fairness)



Number of weakly blocking cycles of size $l = 2$ for solutions with maximum number of transplants (left) and maximum total weight of transplants (right), for unbounded exchange cycles and exchange cycles of size up to $k = 2$ and $k = 3$ for weak preferences.

Violations of the RI-best property



Number of violations of the RI-best property for instances of sizes $20, 30$, weak preferences.

Summary of the main results

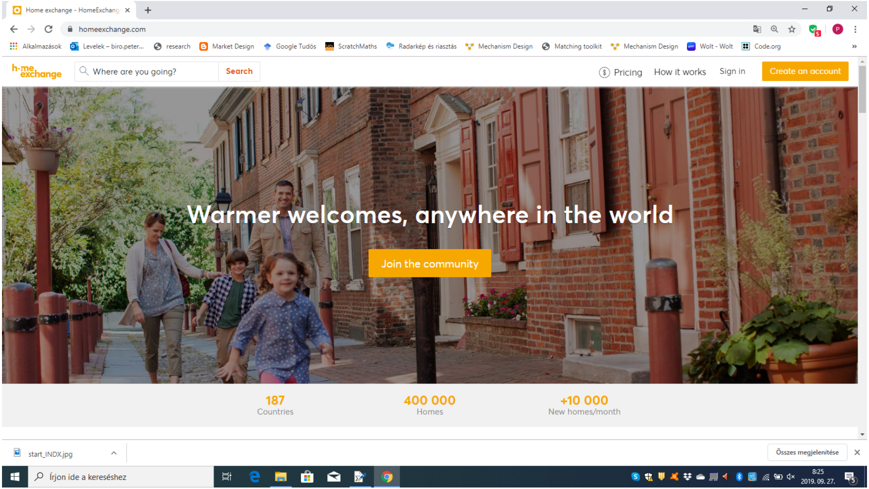
1. Proving the respecting improvement property for strong core and CE for unbounded exchanges, and examples for violations
2. New IP models for computing the strong core, CE, core
3. Simulations for measuring the price of fairness and the respecting improvement violations for kidney exchange instances

+Further plans: simulations on real KEP instances

+Open questions: Characterisation of the TTC mechanism with the RI-property? What about other settings/mechanisms?
A new paper: Ehlers, L. (2023). Respecting Improvement in Markets with Indivisible Goods. Available at SSRN 4581876.

+Follow-up paper: Schlotter-Biro-Fleiner (2021-WINE, 2024-MOR) on RI for core under partial orders, and further complexity results

An example for barter exchange: home-exchange



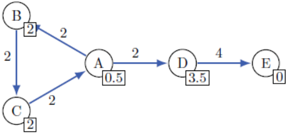
Besides reciprocal (pairwise) exchanges, now visits for Guest Points is also allowed, perhaps close to competitive allocations?! See a new paper by Julius Goedde: Pricing in markets without money: Theory and evidence from home exchanges

Examples for barter exchange: time banks

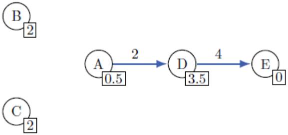
Examples for barter exchange: portfolio compression

Schuldenzucker-Seuken (2020):

Figure 1 Financial System where compressing a cycle decreases social welfare. Let $\alpha = \beta = 0.5$.



(a) Uncompressed Network

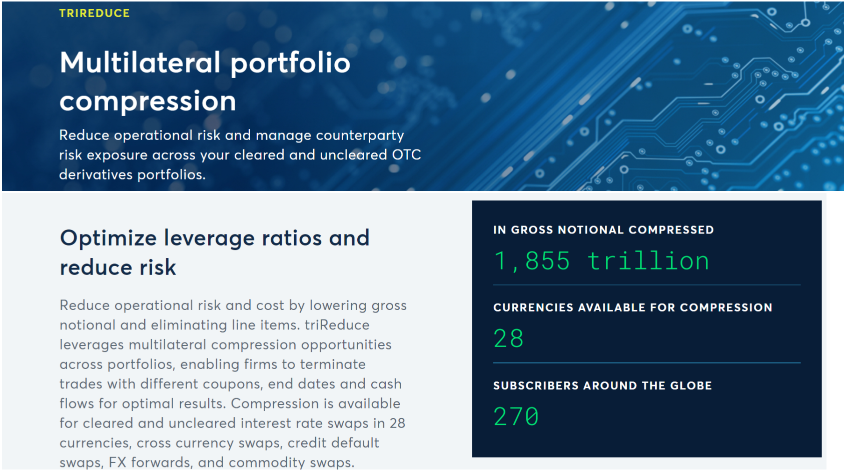


(b) Compressed Network

- Schuldenzucker, S., and Seuken, S. (2020) Portfolio compression in financial networks: Incentives and systemic risk. In Proceedings of EC-2020
- Veraart, L. A. M. (2020) When does portfolio compression reduce systemic risk? SSRN 3688495

Examples for barter exchange: portfolio compression

coordinated by companies: TriOptima, CLS Group, Markit, SwapClear, mainly on OTC markets



Examples for barter exchange: portfolio compression

by a Romanian ministry for companies:

NETTING SUMMARIES 2000 - 2017

Year	Number of circuits	Amounts (RON)	Average exchange rate EUR-RON	Amounts (billion euros)	GDP (billion euros)	% of GDP
2000	5,585	2,403,053,727	1.9955	1.2	40.3	2.99
2001	42,652	12,421,327,031	2.6026	4.8	44.9	10.63
2002	47,034	19,911,421,429	3.1255	6.4	48.5	13.14
2003	52,436	22,112,743,029	3.7555	5.9	52.6	11.19
2004	49,863	22,801,524,621	4.0532	5.6	60.8	9.25
2005	33,974	17,779,815,045	3.6234	4.9	79.5	6.17
2006	24,535	14,848,420,479	3.5245	4.2	97.7	4.31
2007	16,250	12,041,806,899	3.3373	3.6	123.7	2.92
2008	14,215	13,869,438,123	3.6827	3.8	139.7	2.70
2009	17,066	15,876,671,658	4.2373	3.7	118.3	3.17
2010	18,432	15,159,786,445	4.2099	3.6	124.1	2.90
2011	18,311	15,715,962,869	4.2379	3.7	131.5	2.82
2012	15,018	15,246,545,476	4.4560	3.4	133.9	2.56
2013	10,643	12,774,006,762	4.4190	2.9	144.7	2.00
2014	8,400	9,929,328,661	4.4446	2.2	150.8	1.48
2015	7,117	8,371,338,603	4.4450	1.9	159	1.18
2016	5,719	6,377,781,136	4.4908	1.4	185.7	0.76
2017	4,218	5,720,174,633	4.5682	1.3	182.1	0.74
	391,468	**243,361,146,626**		**64.6**		

Fig. 1. Netting amounts - Romanian Institute of Management and Informatics, 2000 - 2017.

- Gavrila, L. I., and Popa, A. (2021) A novel algorithm for clearing financial obligations between companies – an application within the Romanian Ministry of Economy. Algorithmic Finance, 9:49-60

New European projects on kidney exchanges

COST Innovators Grant (Nov 2021 - Oct 2022): KEPSOFT

We developed a new software tool, KEPsoft, building on the ENCKEP prototype and drawing on the European expertise, which includes clinicians, policy makers, optimisation experts, computer scientists, mathematicians and economists.

KEPsoft is now available as a common IT-platform to the European transplantation community, and to National Transplantation Organisations through a non-profit company KEPsoft Community established by Glasgow University to support the national and international KEPs in Europe.

EU4Health Programme (Oct 2024 - Sep 2027): EURO-KEP

Developing a EUROpean Kidney Exchange Program: Further development of the ENCKEP-simulator and KEPsoft software, and a new initiative for establishing international collaborations.