Top trading cycles: from Shapley-Scarf to multiple-type housing markets

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Housing markets: from Shapley-Scarf to multiple-type

Shapley and Scarf's (1974) classical housing markets $(N, H, \succ)$:

- kidney exchange (Roth, Sönmez, and Unver, 2004).

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Moulin (1995) multiple-type housing markets $(N, H \times C, \succ)$:

- dynamic allocations (Monte and Tumennasan, 2015);
- multi-facet tasks (Mackin and Xia, 2016).

Example: On-call weekend exchange

A medical practice is run by four doctors (1, 2, 3, and 4).
Each weekend, one doctor has to be on-call.
On-call weekend plans are often made months in advance.

For instance, the initially planned schedule for October is:

- 1: O1 (“week 1 in October”); 2: O2; 3: O3; 4: O4.

Example: On-call weekend exchange

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Each weekend, one doctor has to be on-call.
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For instance, the initially planned schedule for October is:

- 1: O1 (“week 1 in October”); 2: O2; 3: O3; 4: O4.

However, suppose that a few days before October the doctors' weekend plans have changed and that the following allocation would be better:

- 1: O2; 2: O3; 3: O1; 4: O4.

This is a classical Shapley-Scarf housing market situation and we know how to deal with it.

Example: On-call weekend exchange (continued)

Now suppose that the plan for the two months October and November is made before summer:

- 1: (O1,N1); 2: (O2,N2); 3: (O3,N3); 4: (O4,N4).
- each doctor has to be on-call once per month.

Example: On-call weekend exchange (continued)

Now suppose that the plan for the two months October and November is made before summer:

- 1: (O1,N1); 2: (O2,N2); 3: (O3,N3); 4: (O4,N4).
- each doctor has to be on-call once per month.

After the summer it may turn out that the following allocation is better:

- 1: (O2,N1); 2: (O1,N4); 3: (O3,N3); 4: (O4,N2).

This is a *multiple-type* housing market situation.

Example: "multi-facet" tasks

A group of students has to give presentations. The pre-assigned schedule is:

- A: (micro, morning),
- B: (macro, lunch time),
- C: (metric, afternoon).

Example: “multi-facet” tasks

A group of students has to give presentations. The pre-assigned schedule is:

- A: (micro, morning),
- B: (macro, lunch time),
- C: (metric, afternoon).

Students are free to re-organize the schedule. Suppose A does not like mornings, B really likes (econo)metric(s) and mornings, and C does not like metric. Then, a better allocation would be:

- A: (micro, lunch time),
- B: (metric, morning),
- C: (macro, afternoon).

This is a *multiple-type* housing market situation.

Possible applications of multiple-type housing markets

- surgeons' schedule at a hospital, e.g., surgery staff, operating rooms, equipment, and dates (Huh et al., 2013);
- students' enrollment at universities where courses are taught in small groups and in multiple sessions;
- college students that require course slots across various disciplines, such as computer science, math, and social sciences (Mackin and Xia, 2016);
- cloud computing (Ghodsi et al., 2011, 2012);
- 5G network slicing (Peng et al., 2015; Bag et al., 2019; Han et al., 2019).

Positive results for classical Shapley-Scarf housing markets

Throughout the paper we focus on **strict** preferences.

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The so-called top trading cycles (TTC) mechanism is a remarkable mechanism for classical Shapley-Scarf housing markets because it satisfies many desirable properties.

In particular, TTC is the only mechanism that satisfies

(1) *individual rationality*, *Pareto efficiency*, and *strategy-proofness* (Ma, 1994; Svensson, 1999);

(2) *individual rationality*, *ontoness*, *strategy-proofness*, and *non-bossiness* (Takamiya, 2001).

Also, only the no-trade and TTC mechanisms satisfy

(3) *individual rationality*, *anonymity*, and *group strategy-proofness* (Miyagawa, 2002).

The TTC mechanism is great, but ...

For classical Shapley-Scarf housing markets, we have strong positive results.

However, there are (at least) two important assumptions:

- **Selfishness** versus externalities
  e.g., Klaus and Meo (2022); Klaus (2024);
- **Unit-demand** versus multi-demand
  This presentation: we relax the unit-demand assumption and consider multiple-type housing markets.

Complications for multiple-type housing markets

Konishi, Quint, and Wako (2001):
For "separable" multiple-type housing markets, *individual rationality*, *Pareto efficiency*, and *strategy-proofness* are incompatible.

In particular, a characterization à la Ma **(1)** does not hold anymore.

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In particular, a characterization à la Ma **(1)** does not hold anymore.

Moreover, there are various ways to extend the TTC mechanism to multiple-type housing markets!

Research question I

Can a characterization à la Takamiya (2001) **(2)**, i.e.,

TTC $\iff$ *individual rationality*, *ontoness*, *strategy-proofness*, and *non-bossiness*

be established for multiple-type housing markets?

Research question I

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be established for multiple-type housing markets?

The answer is affirmative:

We characterize the typewise TTC mechanism by *individual rationality*, *ontoness*, *strategy-proofness*, and *non-bossiness*.

Research question II

Similarly, can a characterization à la Miyagawa (2002) **(3)**, i.e.,

no-trade or TTC $\iff$ *individual rationality*, *anonymity*, and *group strategy-proofness*

be established for multiple-type housing markets?

Research question II

Similarly, can a characterization à la Miyagawa (2002) **(3)**, i.e.,

$$\text{no-trade or TTC} \iff \begin{array}{l}\textit{individual rationality}, \textit{anonymity}, \text{and} \\ \quad \textit{group strategy-proofness}\end{array}$$

be established for multiple-type housing markets?

The answer is affirmative:

We characterize the new class of hybrid no-trade–bundled TTC mechanisms by *individual rationality*, *anonymity*, and *group strategy-proofness*.

Classical Shapley-Scarf housing markets

- agents: $N = \{1, \ldots, n\}$;
- objects (houses): $H = \{H_1, \ldots, H_n\}$;
- endowments: each $i \in N$ owns one house, H_i;
- preferences are linear orders over houses;
- market $(N, H, \succ = (\succ_i)_{i \in N})$;
- no monetary transfer.

Allocations

- allocations X: $x \in X$ is an (re)allocation of houses among agents.

  (Note: each agent $i \in N$ ends up with one house $x_i \in H$. Each house is assigned to exactly one agent.)

- x is *individually rational* if for each $i \in N$, $x_i \succeq_i H_i$.

- x is *Pareto efficient* if there is no Pareto improvement.

- x is *unanimously best* if for each $i \in N$ and each $y \in X$, $x_i \succeq_i y_i$.

Mechanisms

A mechanism f maps markets to allocations, i.e., for each market $(N, H, \succ)$, $f(N, H, \succ) \in X$.

A mechanism f is

- *individually rational* if it only assigns *individually rational* allocations;
- *anonymous* if it is independent of the names of the agents;
- *Pareto efficient* if it only assigns *Pareto efficient* allocations;
- *unanimous* if for each market, it assigns the *unanimously best* allocation whenever it exists;
- *onto* if each allocation can be obtained at some market.

Note: *Pareto efficiency* $\Rightarrow$ *unanimity* $\Rightarrow$ *ontoness*.

Incentive properties (individual)

A mechanism f is

- *strategy-proof* if truth-telling is a weakly dominant strategy, i.e., for each market $(N, H, \succ)$, each agent i, and each individual misreport $\succ_i'$,

$$f_i(N, H, \succ) \succeq_i f_i(N, H, \succ_i', \succ_{-i}).$$

- *non-bossy* if whenever an agent changes his preferences but still gets the same allotment, the allotments of the other agents also remain unchanged, i.e., for each market $(N, H, \succ)$, each agent i, and each individual misreport $\succ_i'$,

$$f_i(N, H, \succ_i', \succ_{-i}) = f_i(N, H, \succ) \Rightarrow f(N, H, \succ_i', \succ_{-i}) = f(N, H, \succ).$$

Incentive properties (coalitional)

A mechanism f is

- *group strategy-proof* if no subset of agents can benefit from a joint misreport, i.e., for each market $(N, H, \succ)$ and each subset of agents $S \subseteq N$, there is no joint misreport $\succ'_S$ such that

$$\text{for each } i \in S,\ f_i(N, H, \succ'_S, \succ_{-S}) \succeq_i f_i(N, H, \succ),\ \text{and}$$

$$\text{for some } j \in S,\ f_j(N, H, \succ'_S, \succ_{-S}) \succ_j f_j(N, H, \succ).$$

Incentive properties (coalitional)

We next consider a weak version of group strategy-proofness via "self-enforcement" as follows.

- A coalition $S \subseteq N$ can manipulate a mechanism f *in a self-enforcing manner* at a market $(N, H, \succ)$ if there is some beneficial joint misreport $\succ'_S$ such that for each $V \subsetneq S$ and each $\ell \in V$,

$$f_\ell(\succ'_S, \succ_{N \setminus S}) \succeq_\ell f_\ell(\succ_V, \succ'_{S \setminus V}, \succ_{N \setminus S}).$$

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$$f_\ell(\succ'_S, \succ_{N \setminus S}) \succeq_\ell f_\ell(\succ_V, \succ'_{S \setminus V}, \succ_{N \setminus S}).$$

- A mechanism f is *self-enforcing group strategy-proof* if no coalition can manipulate f in a self-enforcing manner at any market.
- A mechanism f is *self-enforcing pairwise strategy-proof* if no coalition S with $S \leq |2|$ can manipulate f in a self-enforcing manner at any market.

Incentive properties (relations)

For Shapley-Scarf housing markets the following equivalences hold:

group strategy-proofness

$\Leftrightarrow$ *self-enforcing group/pairwise strategy-proofness*

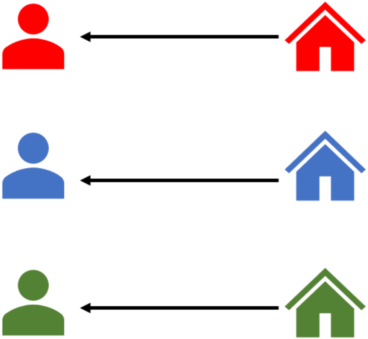
$\Leftrightarrow$ *strategy-proofness* + *non-bossiness*.

Top trading cycles (TTC), due to David Gale

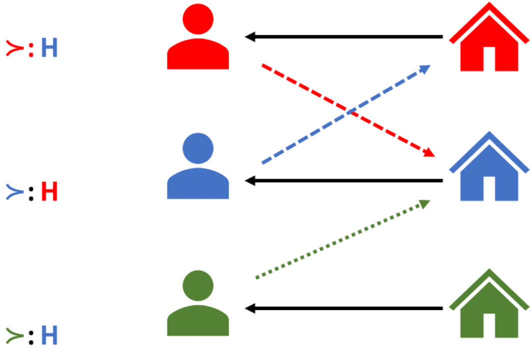
The TTC algorithm / mechanism is defined as follows:

- let each house point to its owner;
- let each agent point to his most preferred (top) house;
- execute the top trading cycles that form, and remove all the involved agents / houses;
- repeat this process until no agent / house is left.

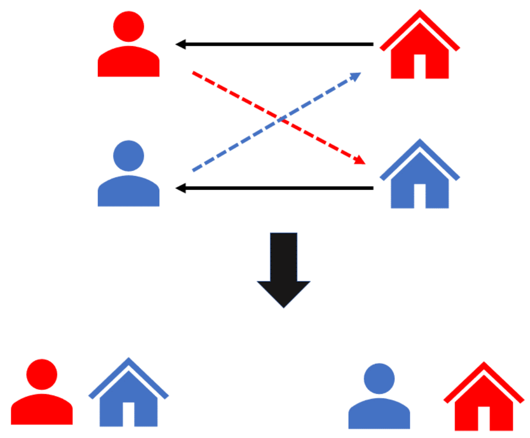
Example TTC, 1/4



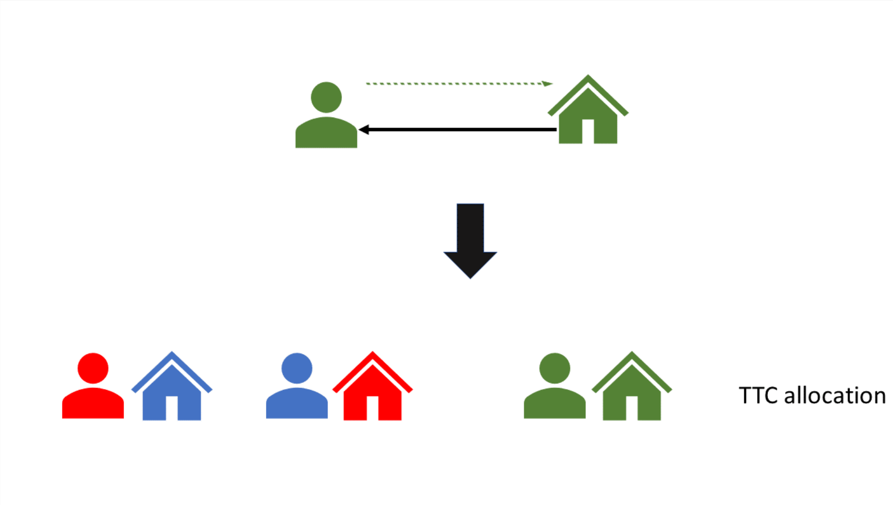
Example TTC, 2/4



Example TTC, 3/4



Example TTC, 4/4



Characterizations of Takamiya (2001) and Miyagawa (2002)

Recall the characterization **(2)** of Takamiya (2001) for housing markets:

TTC $\iff$ *individual rationality*, *ontoness*, *strategy-proofness*, and *non-bossiness*

and the characterization **(3)** of Miyagawa (2002) for housing markets:

no-trade or TTC $\iff$ *individual rationality*, *anonymity*, and *group strategy-proofness*.

Multiple-type housing markets

- agents: $N = \{1, \ldots, n\}$;
- types: $T = \{house, car\}$ ($|T|$ could be larger);
- houses: $H = \{H_1, \ldots, H_n\}$;
- cars: $C = \{C_1, \ldots, C_n\}$;
- objects: $O = H \cup C$;
- endowments: each $i \in N$ owns a house-car pair, (H_i, C_i);
- preferences are (strict) linear orders over house-car pairs;
- market $(N, O, \succ = (\succ_i)_N)$;
- no monetary transfer.

Allocations, mechanisms, and properties

Allocation: a (re)allocation of houses among agents together with a (re)allocation of cars.

- each agent ends up with a pair (H, C) that consists of one house H and one car C;
- each house is assigned to exactly one agent;
- each car is assigned to exactly one agent.

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- each car is assigned to exactly one agent.

The definition of mechanism is straightforwardly extended.

The definitions of the properties are straightforwardly extended.

However, regarding incentive properties ...

Recall, for Shapley-Scarf housing markets the following equivalences hold:

$\quad$ *group strategy-proofness*

$\Leftrightarrow$ *strategy-proofness* + *non-bossiness*

$\Leftrightarrow$ *self-enforcing group/pairwise strategy-proofness*.

However, regarding incentive properties ...

Recall, for Shapley-Scarf housing markets the following equivalences hold:

$$\begin{aligned} & \textit{group strategy-proofness} \\ \Leftrightarrow\ & \textit{strategy-proofness} + \textit{non-bossiness} \\ \Leftrightarrow\ & \textit{self-enforcing group/pairwise strategy-proofness}. \end{aligned}$$

For multiple-type housing markets, depending on the preference domain, *group strategy-proofness* can be stronger than the combination of *strategy-proofness* + *non-bossiness*:

$$\begin{aligned} & \textit{group strategy-proofness} \\ \overset{\not\Leftarrow}{\Rightarrow}\ & \textit{strategy-proofness} + \textit{non-bossiness} \\ \Leftrightarrow\ & \textit{self-enforcing group/pairwise strategy-proofness}. \end{aligned}$$

Preference domains

We consider three domains of **strict** preferences:

Full preference domain P

- any strict ranking over house-car pairs is allowed

Separable preference domain P_s

- agents have marginal type-preferences $(\succ_i^H, \succ_i^C)$
- rankings over pairs are responsive to marginal preferences
- a general domain in economic theory

Lexicographic preference domain P_l

- for each agent, either "houses are more important" or "cars are more important"
- a small preference domain (but useful as stepping stone!)

$$P_l \subsetneq P_s \subsetneq P$$

Example: Lexicographic preferences

$$\succ_i: (H_1, C_1), (H_1, C_2), (H_1, C_3), (H_2, C_1), (H_2, C_2), (H_2, C_3), (H_3, C_1), (H_3, C_2), (H_3, C_3)$$

$$\Updownarrow$$

$$\succ_i: H_1,\ H_2,\ H_3,\ C_1,\ C_2,\ C_3$$

TTC extensions

- **typewise TTC mechanism (tTTC)**: operating sub-markets independently (Wako, 2005; Feng, Klaus, and Klijn, 2024a);
- **no-trade mechanism (ntTTC)**: agents keep their endowments;

  bundle TTC mechanism (bTTC): exchanging only full bundles (Feng, 2023);

  hybrid no-trade–bundled TTC mechanism (nt-bTTC): some types are bundle-traded, the remaining types are not traded (Feng, Klaus, and Klijn, 2024b);
- **mTTC mechanism**: "full flexibility" (Sikdar, Adalı, and Xia, 2017).

Here is an example for tTTC, bTTC, and nt-bTTC.

TTC extensions: comparison w.r.t. preference domains

The TTC extensions are well-defined on different domains of preferences:

mechanisms \ domain	P_l	P_s	P
bTTC and nt-bTTC	✓	✓	✓
tTTC	✓	✓	
mTTC	✓		

Recall: $P_l \subsetneq P_s \subsetneq P$ where
P_l is the domain of lexicographic preferences,
P_s is the domain of separable preferences, and
P is the full domain.

Note: The mTTC mechanism is only defined for (generalized) lexicographic preferences (and it is not *strategy-proof*);
we do not further consider it in this presentation.

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	IR^{2}_{3}	PE	SP^{2}	$\mathsf{UN/ONTO}^{2}$	AN_{3}	GSP_{3}	NB^{2}
tTTC	✓		✓	✓	✓		✓
nt-bTTC	✓		✓		✓	✓	✓

2 refers to the four "Takamiya-properties"

3 refers to the three "Miyagawa-properties"

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Both mechanisms (tTTC and nt-bTTC) are *individually rational*, *strategy-proof*, *anonymous*, *non-bossy*, and not *Pareto efficient*.

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Difference (i): tTTC is *unanimous/onto*, but nt-bTTC is not.

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Both mechanisms (tTTC and nt-bTTC) are *individually rational*, *strategy-proof*, *anonymous*, *non-bossy*, and not *Pareto efficient*.

Difference (i): tTTC is *unanimous/onto*, but nt-bTTC is not.

Difference (ii): nt-bTTC is *group strategy-proof*, but tTTC is not.

Example tTTC, 1/2

Example (*tTTC* is not *Pareto efficient*)

Consider the market with $N = \{1, 2\}$, $T = \{H(ouse), C(ar)\}$, $O = \{H_1, H_2, C_1, C_2\}$, and where each agent i's endowment is (H_i, C_i). Lexicographic preferences:

$$\succ_1 : H_2, H_1, C_1, C_2,$$

$$\succ_2 : C_1, C_2, H_2, H_1.$$

One easily verifies that $tTTC(\succ) = \big((H_1, C_1), (H_2, C_2)\big)$, the no-trade allocation. However, note that since preferences are lexicographic, both agents would be strictly better off if they traded both cars and houses. Thus, allocation $\big((H_2, C_2), (H_1, C_1)\big)$ Pareto dominates $tTTC(\succ)$. Hence, $tTTC$ is not *Pareto efficient*. ⋄

Example tTTC, 2/2

Example (*tTTC* is not *group strategy-proof*)

Furthermore, assume that both agents (mis)report their preferences as follows:

$$\succ_1' \colon H_2, H_1, \underline{C_2, C_1},$$

$$\succ_2' \colon C_1, C_2, \underline{H_1, H_2}.$$

Then, $tTTC(\succ') = \big((H_2, C_2), (H_1, C_1)\big)$, making both agents better off compared to $tTTC(\succ)$. Hence, $tTTC$ is not *group strategy-proof*.

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By the way, note that

$$tTTC_1(\succ_1, \succ_2') = (H_2, C_1) \succ_1 (H_2, C_2) = tTTC_1(\succ') \text{ and}$$
$$tTTC_2(\succ_1', \succ_2) = (H_2, C_1) \succ_2 (H_1, C_1) = tTTC_2(\succ').$$

(This is because the tTTC mechanism satisfies *self-enforcing group (pairwise) strategy-proofness*!) $\diamond$

A characterization à la Takamiya (2) for multiple-type housing markets.

Main result for lexicographic preferences

Theorem 1

tTTC is the unique mechanism satisfying *individual rationality*, *ontoness* (or *unanimity*), *strategy-proofness*, and *non-bossiness*.

Corollary 1

tTTC is the unique mechanism satisfying *individual rationality*, *ontoness* (or *unanimity*), and *self-enforcing group/pairwise strategy-proofness*.

Extension to separable preferences

Theorem 2

tTTC is the unique mechanism satisfying *individual rationality*, *ontoness* (or *unanimity*), *strategy-proofness*, and *non-bossiness*.

Corollary 2

tTTC is the unique mechanism satisfying *individual rationality*, *ontoness* (or *unanimity*), and *self-enforcing group/pairwise strategy-proofness*.

Logical independence of properties, 1/4

Example (*Ontoness* and *unanimity*)

The no-trade mechanism that always assigns the endowment allocation to each market is *individually rational*, *(group) strategy-proof*, and *non-bossy*, but neither *onto* nor *unanimous*. $\diamond$

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Example (*Individual rationality*)

By ignoring property rights that are established via the endowments, we can easily adjust the well-known mechanism of serial dictatorship to our setting: based on an ordering of agents, we let agents sequentially choose their allotments.

Serial dictatorship mechanisms satisfy *Pareto efficiency* (and hence *ontoness* and *unanimity*), *strategy-proofness*, and *non-bossiness*; since property rights are ignored, they violate *individual rationality*. ⋄

Example (*Strategy-proofness*)

We adapt so-called Multiple-Serial-IR mechanisms introduced by Biró, Klijn, and Pápai (2022) for their circulation model to our multiple-type housing markets model. A Multiple-Serial-IR mechanism is determined by a fixed order of the agents. At any preference profile and following the order, the mechanism lets each agent pick his most preferred allotment from the available objects such that this choice together with previous agents' choices is compatible with an *individually rational* allocation.

Biró, Klijn, and Pápai (2022) showed that Multiple-Serial-IR mechanisms are *individually rational* and *Pareto efficient*. It is easy to show that Multiple-Serial-IR mechanisms are *non-bossy*. $\diamond$

Logical independence of properties, 3/4

Note that if $n = 2$, then any mechanism is *non-bossy*. Thus, for our last independence example, we assume $n > 2$.

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Example (*Non-bossiness*)

Let $N = \{1, 2, 3\}$ and $T = \{H(ouse)\}$. Let $R \in \mathcal{R}^N$.
We say that agents 1 and 2 are *in conflict* if H_3 is the most preferred object for both R_1 and R_2.
We say that agents 1 and 3 are *in conflict* if H_2 is the most preferred object for both R_1 and R_3.
Let mechanism f be defined as follows: for each $R \in \mathcal{R}^N$,

(a) if agents 1 and 2 are in conflict, then
(i) transform R_2 to $\bar{R}_2$ by dropping H_3 to the bottom, i.e., $\bar{R}_2 : \ldots, H_3$, while keeping the relative order of H_1 and H_2, and
(ii) set $f(R) \equiv TTC(R_1, \bar{R}_2, R_3)$;

Example (*Non-bossiness, cont'd*)

(b) if agents 1 and 3 are in conflict, then
(i) transform R_3 to $\bar{R}_3$ by dropping H_2 to the bottom, i.e., $\bar{R}_3 : \ldots, H_2$, while keeping the relative order of H_1 and H_3, and
(ii) set $f(R) \equiv TTC(R_1, R_2, \bar{R}_3)$;

(c) if agent 1 is not in conflict with either agent 2 or agent 3, then $f(R) \equiv TTC(R)$.

It is easy to see that f is *individually rational* and *unanimous*.
The proof that f is *strategy-proof* is easy but cumbersome. $\diamond$

Example (*Non-bossiness, cont'd*)

(b) if agents 1 and 3 are in conflict, then
(i) transform R_3 to $\bar{R}_3$ by dropping H_2 to the bottom, i.e., $\bar{R}_3 : \ldots, H_2$, while keeping the relative order of H_1 and H_3, and
(ii) set $f(R) \equiv TTC(R_1, R_2, \bar{R}_3)$;

(c) if agent 1 is not in conflict with either agent 2 or agent 3, then $f(R) \equiv TTC(R)$.

It is easy to see that f is *individually rational* and *unanimous*. The proof that f is *strategy-proof* is easy but cumbersome. $\diamond$

Next, we extend mechanism f from $n = 3$ to any $n > 3$.
Finally, we extend it from classical housing markets to multiple-type housing markets with lexicographic (or separable) preferences by applying it typewise to all object types.

Open question

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Note that there are economically interesting domains that contain non-separable preferences and on which there exists a mechanism that satisfies all properties.

For instance, consider a market with $T = \{H(ouse), C(ar)\}$ and where each agent i's endowment is (H_i, C_i). Let agents have (and report) preferences where they primarily care about houses. However, each agent's preferences over cars is allowed to *depend* on the house he receives. Sikdar, Adalı, and Xia (2017) showed that on this domain their mechanism satisfies all four properties.

Impossibility of extension to strict preferences

Note that for $m > 1$, the tTTC mechanism is not well-defined for strict preferences since for non-separable preferences, marginal type preferences cannot be derived.

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Then, a natural question is if there exists an extension of the tTTC mechanism to the domain of strict preference profiles that satisfies our properties.

Theorem 5

There is no mechanism satisfying *individual rationality*, *ontoness*, *strategy-proofness*, and *non-bossiness*.

Theorem 6

There is no mechanism satisfying *individual rationality*, *unanimity*, and *strategy-proofness*.

A characterization à la Miyagawa (3) for multiple-type housing markets.

Main result for lexicographic preferences

Theorem 1

Hybrid no-trade–bundled TTC mechanisms are the unique mechanisms satisfying *individual rationality*, *anonymity*, and *group strategy-proofness*.

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Corollary 1

The no-trade and the bundle TTC mechanisms are the unique mechanisms satisfying *individual rationality*, *anonymity*, *group strategy-proofness*, and *type-neutrality*.

Extension to separable preferences

Theorem 2

Hybrid no-trade–bundled TTC mechanisms are the unique mechanisms satisfying *individual rationality*, *anonymity*, and *group strategy-proofness*.

Corollary 2

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Extension to strict preferences

Theorem 3

Hybrid no-trade–bundled TTC mechanisms are the unique mechanisms satisfying *individual rationality*, *anonymity*, and *group strategy-proofness*.

Corollary 3

The no-trade and the bundle TTC mechanisms are the unique mechanisms satisfying *individual rationality*, *anonymity*, *group strategy-proofness*, and *type-neutrality*.

Logical independence of properties

The original independence-of-properties examples of Miyagawa (2002, Section 5) can be adjusted as follows:

- allow agents to only trade object type 1, i.e., they keep their endowments of all other types;
- preferences over these restricted allotments correspond to Shapley-Scarf housing market preferences over objects of type 1;
- properties of the mechanism correspond to Shapley-Scarf housing market mechanism properties.

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- properties of the mechanism correspond to Shapley-Scarf housing market mechanism properties.

Example (*Group strategy-proofness*)

For lexicographic and separable preferences, the tTTC mechanism satisfies *individual rationality*, *anonymity*, *strategy-proofness*, and *non-bossiness* but it does not satisfy *group strategy-proofness*. ◇

Thank you!

If you're interested, this is your chance to download
the open access paper that covers part I (Feng, Klaus, and Klijn, 2024a)

...

... part II (Feng, Klaus, and Klijn, 2024b) will be available soon!

Example of tTTC

Preferences are separable such that

$$\succ_1^H: \boldsymbol{H_3}, H_1, H_2 \quad \text{and} \quad \succ_1^C: \boldsymbol{C_2}, C_1, C_3$$

$$\succ_2^H: H_1, \boldsymbol{H_2}, H_3 \quad \text{and} \quad \succ_2^C: \boldsymbol{C_1}, C_2, C_3$$

$$\succ_3^H: \boldsymbol{H_1}, H_2, H_3 \quad \text{and} \quad \succ_3^C: C_2, C_1, \boldsymbol{C_3}$$

Then, the tTTC allocation is

$$(1, (H_3, C_2)); (2, (H_2, C_1)); (3, (H_1, C_3)),$$

i.e., agents 1 and 3 swap houses and agents 1 and 2 swap cars.

Example of bTTC

Preferences are lexicographic such that

$$\succ_1: \boldsymbol{C_2}, C_1, C_3, H_3, H_1, \boldsymbol{H_2},$$

$$\succ_2: \boldsymbol{H_1}, H_2, H_3, \boldsymbol{C_1}, C_2, C_3,$$

$$\succ_3: H_1, H_2, \boldsymbol{H_3}, C_2, C_1, \boldsymbol{C_3}.$$

Then, the bTTC allocation is

$$(1, (H_2, C_2)); (2, (H_1, C_1)); (3, (H_3, C_3)),$$

i.e., agents 1 and 2 swap their endowment bundles.

Example of nt-bTTC

Preferences are lexicographic such that

$$\succ_1: C_2, \boldsymbol{C_1}, C_3, \boldsymbol{H_3}, H_1, H_2,$$

$$\succ_2: H_1, \boldsymbol{H_2}, H_3, C_1, \boldsymbol{C_2}, C_3,$$

$$\succ_3: \boldsymbol{H_1}, H_2, H_3, C_2, C_1, \boldsymbol{C_3}.$$

Assume that houses can be traded, but not cars. Then, the nt-bTTC allocation is

$$(1, (H_3, C_1)); (2, (H_2, C_2)); (3, (H_1, C_3)),$$

i.e., agents 1 and 3 swap houses (while cars are not traded).

Go back