Game Connectivity and Adaptive Dynamics

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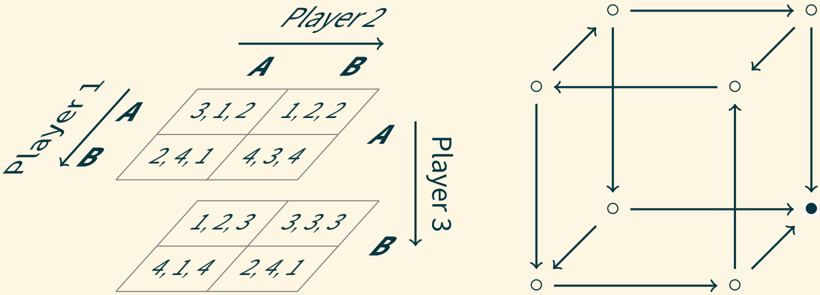
What is game connectivity?

Connectivity property of a game's **best-response graph**.



Example above:

- ∘ One sink, no sources
- ∘ Some cycles (front, top)
- ∘ From each vertex there is a path to the sink

Looking ahead to our main result

- We classify games according to the connectivity properties of their best-response graphs.
- We quantify the relative sizes of the game classes.

Theorem

Almost every game that

is generic (i.e. no ties)

has a pure Nash equilibrium

has a large number of players

has a best-response graph that is **connected**.
i.e. every action profile that is not a pure Nash equilibrium can reach every pure Nash equilibrium profile via best-response paths.

The possible and the impossible in adaptive dynamics

- The behaviors of many game dynamics are determined by the connectivity properties of a game's best-response graph.
- Connectedness is conducive to convergent dynamics.

Theorem ([Hart and Mas-Colell, 2003],[2006])

There is no simple adaptive dynamic that leads to a pure Nash equilibrium in every game that has one.

Theorem

There is a simple adaptive dynamic that leads to a pure Nash equilibrium in almost every large generic game that has one.

Games: standard notions

- A **game** consists of

  A set of players $[n] := \{1, \ldots, n\}$

  A set of actions $[k_i] := \{1, \ldots, k_i\}$ for each player $i \in [n]$

  A preference relation $\succsim_i$ over the set of action profiles $A := \times_{i \in [n]} [k_i]$ for each $i \in [n]$

- a_i is a **best-response** to a_{-i} if $(a_i, a_{-i}) \succsim_i (x, a_{-i})$ for each $x \in [k_i]$

- $a \in A$ is a **pure Nash equilibrium** if a_i is a best-response to a_{-i} for each $i \in [n]$

- The **best-response graph** of a game is the directed graph $(A, \to)$ where for $a, b \in A$, there is $a \to b$ iff there exists $i \in [n]$ such that

  $a_{-i} = b_{-i}$, $b \succ_i a$, and b_i is a best-response to a_{-i}

Notions of connectivity

- A game is **acyclic** if its best-response graph has no cycles.

  Potential games [Monderer and Shapley, 1996] are acyclic

- A game is **weakly acyclic** if its best-response graph has the property that every vertex can reach a sink.

  [Young, 1993] and many others

We introduce the following notions

- A game is **connected** if its best-response graph has at least one sink and the property that every non-sink can reach every sink.
- A game is **super-connected** if its best-response graph has at least one sink and the property that every non-sink can reach every non-source.

Notions of connectivity: picture

acyclic $\Longrightarrow$ weakly acyclic

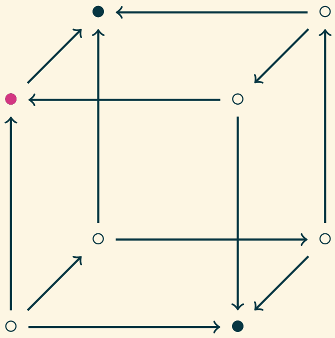
$\Uparrow$

connected

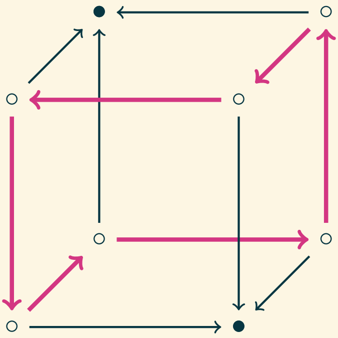
$\Uparrow$

super-connected

Examples



acyclic, not super-connected



super-connected, not acyclic

$\mathcal{G}(n, \mathbf{k})$

- A game is **generic** if for each i, each a_{-i}, and all distinct actions a_i and a_i', either $(a_i, a_{-i}) \succ_i (a_i', a_{-i})$ or $(a_i', a_{-i}) \succ_i (a_i, a_{-i})$.
- For $n \geqslant 2$ and $\mathbf{k} = (k_1, \ldots, k_n) \in \{2, 3, \ldots\}^n$,

let $\mathcal{G}(n, \mathbf{k})$ denote **the set of all generic games**

with player set $[n]$ in which each player $i \in [n]$ has action set $[k_i]$.

Main result 1: Connectedness

Theorem (connectedness)

There exists $c > 0$ such that for all $n \geqslant 2$ and all $\mathbf{k} \in \{2, 3, \ldots\}^n$, if n is sufficiently large relative to $\max_i k_i$ then

$$\frac{|\{g \in \mathcal{G}(n, \mathbf{k}) : g \text{ is connected}\}|}{|\{g \in \mathcal{G}(n, \mathbf{k}) : g \text{ has a pure Nash equilibrium}\}|} \geqslant 1 - e^{-cn}$$

'Sufficiently large' means

$$\max_i k_i \leqslant \delta \sqrt{n / \log n}$$

for a suitable constant $\delta > 0$.

NB. Actions can grow too, provided the above continues to hold.

Main result 2: Acyclicity

Proposition (acyclicity)

There exists $c > 0$ such that for all $n \geqslant 2$ and all $\mathbf{k} \in \{2, 3, \ldots\}^n$,

$$\frac{|\{g \in \mathcal{G}(n, \mathbf{k}) \colon g \text{ is acyclic}\}|}{|\{g \in \mathcal{G}(n, \mathbf{k}) \colon g \text{ has a pure Nash equilibrium}\}|} \leqslant e^{-cn2^n}$$

Main result 3: Super-connectedness

Proposition (super-connectedness)

For $\mathbf{k} = \mathbf{2}$ or $\mathbf{k} = \mathbf{3}$ there exists $c > 0$ such that for all integers $n \geqslant 2$,

$$\frac{|\{g \in \mathcal{G}(n, \mathbf{k})\colon g \text{ is super-connected}\}|}{|\{g \in \mathcal{G}(n, \mathbf{k})\colon g \text{ has a pure Nash equilibrium}\}|} \geqslant 1 - e^{-cn}$$

However, for each $\mathbf{k} \geqslant \mathbf{4}$, the fraction above tends to 0 as $n \to \infty$.

NB. Properties for small $\mathbf{k}$ doesn't necessarily extend to large $\mathbf{k}$.

Next slides: further implications for connectivity in games

I. Connectivity of better-response graphs
II. Non-zero measure classes of games
III. Existing work: [Amiet et al., 2021]

I. Better-response graphs

- a_i is a **better-response** to a_{-i} than a_i' if $(a_i, a_{-i}) \succ_i (a_i', a_{-i})$
- The **better-response graph** of a game is the directed graph $(A, \rightarrow)$ where for $a, b \in A$, there is $a \rightarrow b$ iff there exists $i \in [n]$ such that

  $a_{-i} = b_{-i}$ and $b \succ_i a$

- For each connectivity property

$$P \in \{\text{acyclic, weakly acyclic, connected}\}$$

we say that the game is **globally** P if the property holds for its better-response graph.

I. Better-response graphs: picture

$$
\begin{array}{ccccccc}
\text{globally acyclic} & \Longrightarrow & \text{acyclic} & \Longrightarrow & \text{weakly acyclic} & \Longrightarrow & \text{globally weakly acyclic} \\
 & & & & \Uparrow & & \Uparrow \\
 & & & & \text{connected} & \Longrightarrow & \text{globally connected}
\end{array}
$$

II. Non-zero measure

- Consider any set of games

$$\mathcal{X}(n, \mathbf{k}) \subseteq \{g \in \mathcal{G}(n, \mathbf{k}) \colon g \text{ has a pure Nash equilibrium}\}$$

that has non-zero measure. i.e there is $p \in (0, 1]$ such that

$$\lim_{n \to \infty} \frac{|\mathcal{X}(n, \mathbf{k})|}{|\{g \in \mathcal{G}(n, \mathbf{k}) \colon g \text{ has a pure Nash equilibrium} |} = p$$

Corollary

For n large enough, almost every game in $\mathcal{X}(n, \mathbf{k})$ is connected.

II. Non-zero measure: example

- [Rinott and Scarsini, 2000] show that for $z \geqslant 0$,

$$\lim_{n \to \infty} \frac{|\{g \in \mathcal{G}(n, \mathbf{k}) : g \text{ has exaclty } z \text{ pure Nash equilibria}\}|}{|\mathcal{G}(n, \mathbf{k})|} = \frac{e^{-1}}{z!}$$

E.g. the fraction of large generic games that have at least 2 pure Nash equilibria is $1 - 2/e \approx 26\%$.

Corollary

For any $z \geqslant 1$, almost every large generic game that has at least z pure Nash equilibria is connected.

III. [Amiet et al., 2021]

• A game is *x*-**connected** if its best-response graph has a sink and the property that if x is a non-sink then it can reach every sink.

NB. A game is connected if it is x-connected for each vertex x.

• A game is *x*-**super-connected** if its best-response graph has a sink and the property that if x is a non-sink then it can reach every non-source.

NB. A game is super-connected if it is x-super-connected for each vertex x.

III. [Amiet et al., 2021]

For any game and vertex x we have:

$$\begin{array}{ccc} \text{x-super-connected} & \Longrightarrow & \text{x-connected} \\ \Uparrow & & \Uparrow \\ \text{super-connected} & \Longrightarrow & \text{connected} \end{array}$$

• The arguments of [Amiet et al., 2021] imply that for any vertex x there is a $c > 0$ such that for all $n \geqslant 2$,

$$\frac{|\{g \in \mathcal{G}(n, \mathbf{2}) : g \text{ is } x\text{-super-connected}\}|}{|\{g \in \mathcal{G}(n, \mathbf{2}) : g \text{ has a pure Nash equilibrium}\}|} \geqslant 1 - e^{-cn}$$

Next slides: Insights for adaptive dynamics in games

I. The possible vs the impossible for adaptive dynamics
II. Extension of existing results for adaptive dynamics
III. Equilibrium selection in large games

Adaptive dynamics: standard notions

- A **strategy** for a player with action set $[k]$ is a function $f\colon O_k \to \Delta([k])$ where O_k is the set of all possible observation sets.

- A **dynamic** on $\mathcal{G}(n, \mathbf{k})$ consists of a specification for what information enters into each player's observation set at each time, and a strategy f_i with action set $[k_i]$ for each player i.

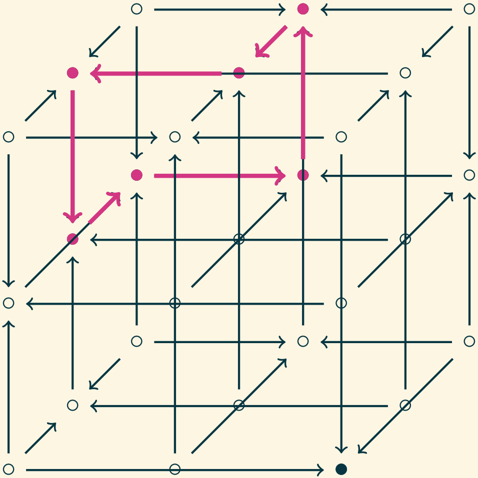
- A dynamic is **simple** if it is <u>uncoupled</u>, <u>1-recall</u> and <u>stationary</u>. i.e. at each time t each player i's observation set consists of their own preference relation $\succsim_i$ and of last period's play a^{t-1}.

- A dynamic on $\mathcal{G}(n, \mathbf{k})$ **converges almost surely to a pure Nash equilibrium** of a game $g \in \mathcal{G}(n, \mathbf{k})$ if when g is played according to the dynamic from any initial action profile, almost surely there exists $T < \infty$ and a pure Nash equilibrium a^* of g such that $a^t = a^*$ for all $t \geqslant T$.

I. Impossibility for adaptive dynamics

> ***Theorem ([Hart and Mas-Colell, 2006, Jaggard et al., 2014])***
>
> There is no simple dynamic for which play converges almost surely to a pure Nash equilibrium in every generic game that has one.

I. Example: this is a generic game with a pure Nash but...



...any simple dynamic starting in the cycle stays in it forever.
[Hart and Mas-Colell, 2006]

I. Possibility for adaptive dynamics

Theorem ([Young, 2004])

The **best-response dynamic with inertia** converges almost surely to a pure Nash equilibrium in every generic weakly acyclic game.

Corollary (Possibility for adaptive dynamics)

There exists $c > 0$ such that for integers $n \geqslant 2$ and $\mathbf{k} \in \{2, 3, \ldots\}^n$, if n is sufficiently large relative to $\max_i k_i$, the fraction of games in

$$\{g \in \mathcal{G}(n, \mathbf{k}) \colon g \text{ has a pure Nash equilibrium}\}$$

for which the best-response dynamic with inertia converges almost surely to a pure Nash equilibrium is at least $1 - e^{-cn}$.

There is a simple adaptive dynamic that leads to a pure Nash equilibrium in almost every large generic game that has one.

II. Extension of existing results

- [Young, 1993] shows that ‘adaptive play’ converges almost surely to a pure Nash equilibrium in all globally weakly acyclic games.
- [Friedman and Mezzetti, 2001] shows the same for ‘better-reply dynamics with sampling’.
- [Marden et al., 2007] and [Marden et al., 2009] describe, respectively, regret-based and payoff-based dynamics that lead to play that is at a pure Nash equilibrium in every weakly acyclic game ‘most of the time’.

> All of these results apply to almost every large generic game that has a pure Nash equilibrium.

III. Equilibrium selection in large games

- When there are multiple equilibria, it is natural to ask which of these equilibria will be played.

- One approach is to ask at which states a dynamic spends most of its time.

- Consider a perturbed version of the best-response dynamic with inertia in which, at each time, any updating player plays a best-response with probability $1-\epsilon$ and, with complementary probability $\epsilon > 0$, selects an action uniformly at random.

- The **stochastically stable** states of this dynamic are the action profiles that are assigned positive probability as $\epsilon \to 0$ in the invariant distribution of the Markov process induced by this dynamic.

III. Equilibrium selection in large games

- Commonly used methods for determining stochastic stability include the minimum-cost tree technique and the radius-coradius technique [Kandori and Rob, 1995, Young, 1993, Freidlin et al., 2012, Ellison, 2000].

- They require checking global properties: a stochastically stable state must be ‘hard’ to leave and ‘easy’ to enter.

- [Newton and Sawa, 2024] observe that, in connected games, the problem reduces to checking a local ‘one-shot’ property.

- [Newton and Sawa, 2024] are able to determine which Nash equilibria (according to their welfare properties) are selected by different evolutionary dynamics in large games.

> Requires our notion of connectedness.

Open questions

- Many actions case? [in preparation]
- Completely uncoupled dynamics? [Babichenko, 2012]
- Speed of convergence? [Hart and Mansour, 2010]
- Non-generic games? [Amiet et al., 2021]
- Efficiency? [Pradelski and Young, 2012]
- Different types of deviation?

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