Why Does Q-learning Work?

The Projected Bellman Equation in Reinforcement Learning

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COGNITION & CONTROL IN COMPLEX SYSTEMS

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Resources

Admittedly self-centered

ODE Method

(using different meaning than in the 1970s)

Goal: *find solution to* $\bar{f}(\theta^*) = 0$

$$\text{ODE algorithm:} \quad \frac{d}{dt}\vartheta_t = \bar{f}(\vartheta_t)$$

design for stability

$$\text{Euler approximation:} \quad \theta_{n+1} = \theta_n + \alpha_{n+1}\bar{f}(\theta_n)$$

- CS&RL, Chapters 4 and 8
- The ODE Method for Asymptotic Statistics in Stochastic Approximation and Reinforcement Learning [56, 57]
- And of course *Borkar's manifesto*

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Stochastic Approximation: $\theta_{n+1} = \theta_n + \alpha_{n+1} f(\theta_n, \Xi_{n+1})$

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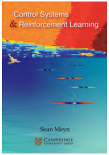
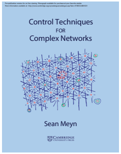
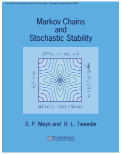
TD Methods CS&RL:

- Chapter 5 (purely deterministic setting)
- Chapters 9 & 10 (traditional MDP)







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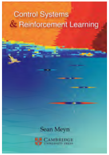
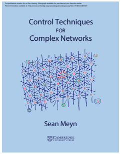
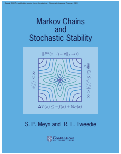






New material in this lecture:

[2] *The projected Bellman equation in reinforcement learning.* *IEEE Transactions on Automatic Control* (to appear).

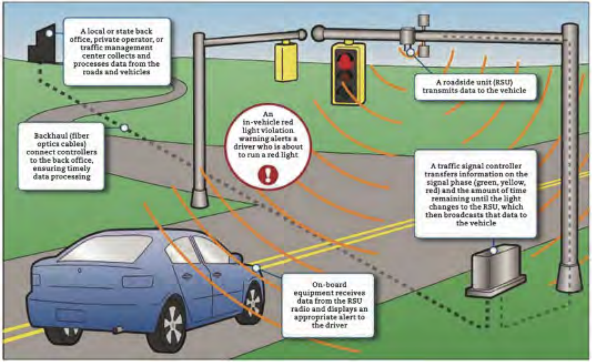
[3] *Stability of Q-learning through design and optimism.* *arXiv 2307.02632*, 2023.

Too many resources to list

Sadly, I am leaving out all of the fun **zero-variance** theory with Caio Lauand

Introducing Dr. Lauand in May, 2025

Stick around for tutorial next Thursday at LAAS CNRS


A local or state back office, private operator, or traffic management center collects and processes data from the roads and vehicles
Backhaul (fiber optics cables) connect controllers to the back office, ensuring timely data processing
An in-vehicle red light violation warning alerts a driver who is about to run a red light
A roadside unit (RSU) transmits data to the vehicle
A traffic signal controller transfers information on the signal phase (green, yellow, red) and the amount of time remaining until the light changes to the RSU, which then broadcasts that data to the vehicle
On-board equipment receives data from the RSU radio and displays an appropriate alert to the driver

Q Learning

Stochastic Optimal Control (Review)

MDP Model

$\boldsymbol{X}$ is a stationary controlled Markov chain, with input $\boldsymbol{U}$

- For all states x and sets A,

$$\mathsf{P}\{X_{n+1} \in A \mid X_n = x,\ U_n = u, \text{and prior history}\} = P_u(x, A)$$

- $c\colon \mathsf{X} \times \mathsf{U} \to \mathbb{R}$ is a cost function
- $\gamma < 1$ a discount factor

Q function:

$$Q^*(x,u) = \min_{\boldsymbol{U}} \sum_{n=0}^{\infty} \gamma^n \mathsf{E}[c(X_n, U_n) \mid X(0) = x, U(0) = u]$$

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Bellman equation:

$$Q^*(x,u) = c(x,u) + \gamma \mathsf{E}\big[\min_{u'} Q^*(X_{n+1}, u') \mid X_n = x,\ U_n = u\big]$$

Q-Learning and Galerkin Relaxation

Dynamic programming

Find function Q^* that solves ($\mathcal{F}_n$ means history)

$$\mathsf{E}\Big[c(X_n,U_n) + \gamma\underline{Q}^*(X_{n+1}) - Q^*(X_n,U_n) \mid \mathcal{F}_n\Big] = 0$$

$$\underline{H}(x) = \min_u H(x,u)$$

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Goal of Q-Learning

Given $\{Q^\theta : \theta \in \mathbb{R}^d\}$, find θ^* that solves $\bar{f}(\theta^*) = 0$,

$$\bar{f}(\theta) \stackrel{\text{def}}{=} \mathsf{E}\big[\{c(X_n, U_n) + \gamma \underline{Q}^\theta(X_{n+1}) - Q^\theta(X_n, U_n)\}\zeta_n\big]$$

The family $\{Q^\theta\}$ and eligibility vectors $\{\zeta_n\}$ are part of algorithm design.

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Projected Bellman Equation: $\bar{f}(\theta^*)=0$

Q(0)-Learning Goal $\bar{f}(\theta^*) = 0$

$$\bar{f}(\theta) = \mathsf{E}\Big[\big\{c(X_n, U_n) + \gamma \underline{Q}^\theta(X_{n+1}) - Q^\theta(X_n, U_n)\big\}\zeta_n\Big]$$

Prototypical choice $\zeta_n = \nabla_\theta Q^\theta(X_n, U_n)\big|_{\theta=\theta_n}$

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$\Longrightarrow$ prototypical Q-learning algorithm

Q(0) Learning Algorithm

Estimates obtained using SA

$$\theta_{n+1} = \theta_n + \alpha_{n+1} f_{n+1} \qquad f_{n+1} = \{c_n + \gamma \underline{Q}^\theta_{n+1} - Q^\theta_n\}\zeta_n\Big|_{\theta=\theta_n}$$

$$\underline{Q}^\theta_{n+1} = Q^\theta(X_{n+1}, \phi^\theta(X_{n+1}))$$

- $\phi^\theta(x) = \arg\min_u Q^\theta(x, u)$ [Q^θ-greedy policy]
- Input $\{U_n\}$ chosen for *exploration*.

Q(0)-Learning   Goal $\bar{f}(\theta^*) = 0$

$Q(0)$-learning with linear function approximation

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$\bar{f}(\theta) = \overline{A}(\theta)\theta - \bar{b}$ p.w. constant if $\boldsymbol{U}$ is oblivious

$$\overline{A}(\theta) = \mathsf{E}\big[\zeta_n \big[\gamma\psi(X_{n+1}, \phi^{\theta}(X_{n+1})) - \psi(X_n, U_n)\big]^{\intercal}\big]$$

$$\bar{b} \stackrel{\text{def}}{=} \mathsf{E}\big[\zeta_n c(X_n, U_n)\big]$$

Watkins' Q-learning

$$\mathsf{E}\Big[\big\{c(X_n, U_n) + \gamma \underline{Q}^{\theta^*}(X_{n+1}) - Q^{\theta^*}(X_n, U_n)\big\}\zeta_n\Big] = 0$$

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Watkin's algorithm *A special case of Q(0)-learning*

The family $\{Q^\theta\}$ and *eligibility vectors* $\{\zeta_n\}$ in this design:

- Linearly parameterized family of functions: $Q^\theta(x,u) = \theta^{\intercal}\psi(x,u)$
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Convergence of Q^{θ_n} to Q^* holds under mild conditions

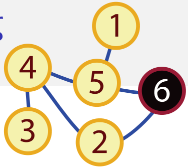
Asymptotic covariance is infinite for $\gamma\geq 1/2$ [5]

$$\sigma^2=\lim_{n\to\infty} n\mathsf{E}[\|\theta_n-\theta^*\|^2]=\infty$$

Using the standard step-size rule $\alpha_n=1/n(x,u)$

Asymptotic Covariance of Watkins' Q-Learning

This is what infinite variance looks like



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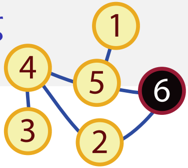

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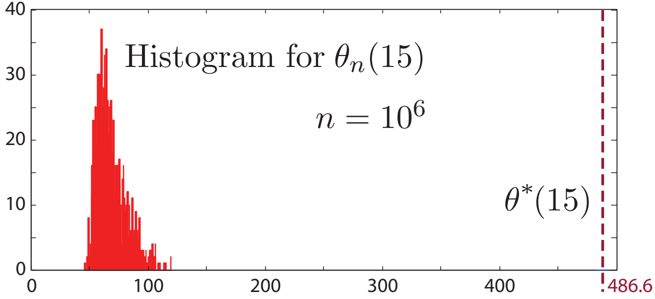
Wild oscillations?

Not at all, the sample paths appear frozen

Histogram of parameter estimates after 10^6 iterations.



Example from [5] 2017

Asymptotic Covariance of Watkins' Q-Learning

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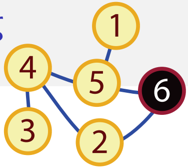

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Sample paths using a higher gain, or relative Q-learning [8]

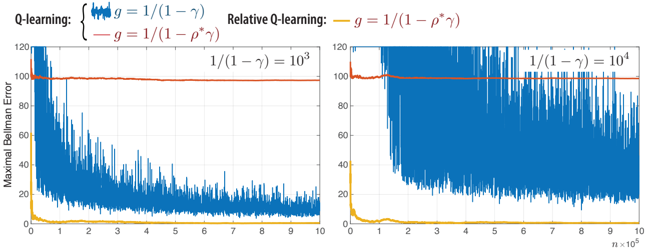
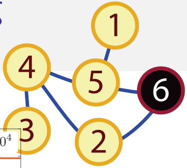


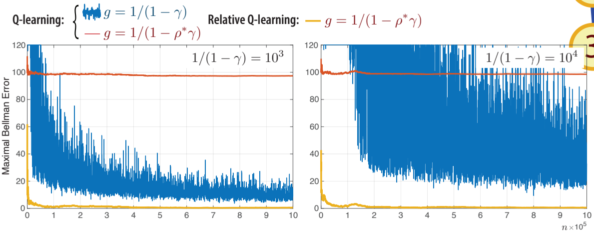
Figure 1: Comparison of Q-learning and Relative Q-learning algorithms for the stochastic shortest path problem of [4]. The relative Q-learning algorithm is unaffected by large discounting.

Example from [5] 2017, and [8], CS&RL, 2021

Asymptotic Covariance of Watkins' Q-Learning

Can we do better?





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Relative Q-learning: estimate *relative Q-function*,

$$H^*(x,u) = Q^*(x,u) - \delta\langle \nu, Q^* \rangle$$

And don't use step-size $\alpha_n = g/n$ [Recall Eric's Tuesday plenary]

Asymptotic Covariance of Watkins' Q-Learning

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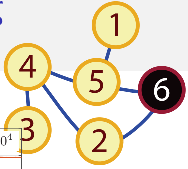

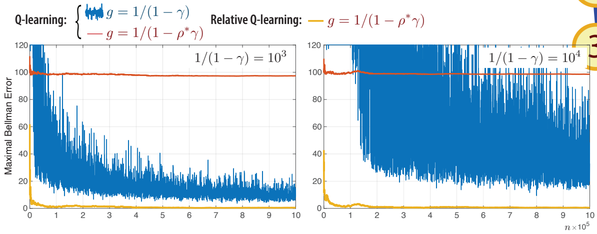

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An intelligent mouse might offer other clues

Asymptotic Covariance of Watkins' Q-Learning

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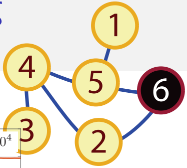

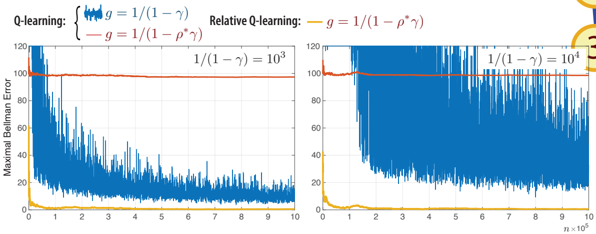

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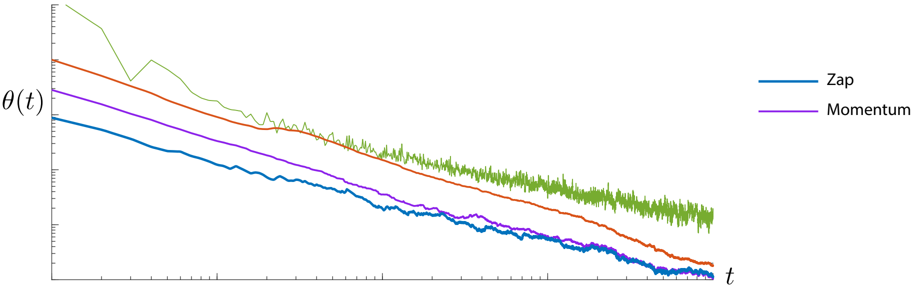
First consider second order methods



Zap

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Challenges we have faced with Q-learning:

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  1. Stability few results outside of Watkins' tabular setting
  2. $\bar{f}(\theta^*) = 0$ solves a relevant problem or has a solution

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- How can we better manage problems introduced by $1/(1-\gamma)$?

Relative Q-Learning is one approach

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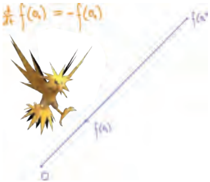
- How can we better manage problems introduced by $1/(1-\gamma)$?

Relative Q-Learning is one approach

Assuming we have solved 2, forget 1 and
approximate Newton-Raphson flow:

$$\frac{d}{dt}\bar{f}(\vartheta_t) = -\bar{f}(\vartheta_t) \qquad \textit{giving} \quad \bar{f}(\vartheta_t) = \bar{f}(\vartheta_0)e^{-t}$$



Zap Algorithm

Designed to emulate Newton-Raphson flow

$\frac{d}{dt}\vartheta_t = -[A(\vartheta_t)]^{-1}\bar{f}(\vartheta_t), \quad A(\theta) = \partial_\theta \bar{f}\,(\theta)$

Zap-SA

$$\theta_{n+1} = \theta_n + \alpha_{n+1} G_{n+1} f(\theta_n, \xi_{n+1}) \qquad G_{n+1} = -[\widehat{A}_{n+1}]^{-1}$$

$$\widehat{A}_{n+1} = \widehat{A}_n + \beta_{n+1}(A_{n+1} - \widehat{A}_n) \qquad A_{n+1} = \partial_\theta f(\theta_n, \xi_{n+1})$$

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Numerics that follow: $\alpha_n = 1/n$, $\beta_n = (1/n)^\rho$, $\rho \in (0.5, 1)$

Zap Q-Learning: $f(\theta_n, \xi_{n+1}) = \big\{c(X_n, U_n) + \gamma \underline{Q}^{\theta^*}(X_{n+1}) - Q^{\theta^*}(X_n, U_n)\big\}\zeta_n$

$$\zeta_n = \nabla_\theta Q^\theta(X_n, U_n)\big|_{\theta=\theta_n}$$

$$A_{n+1} = \zeta_n\big[\gamma\psi(X_{n+1}, \phi^\theta(X_{n+1})) - \psi(X_n, U_n)\big]^T$$

$$\phi^\theta(x) = \arg\min_u Q^\theta(x, u)$$

Challenges

Q-learning: $\{Q^\theta(x,u) : \theta \in \mathbb{R}^d\,,\ u \in \mathsf{U}\,,\ x \in \mathsf{X}\}$

Find θ^* such that $\bar{f}(\theta^*) = 0$, with

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What makes theory difficult:

1. Does $\bar{f}$ have a root?
2. Does the inverse of A exist?
3. SA theory is weak for a discontinuous ODE

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$\Longrightarrow$ 3 resolved for Zap by exploiting special structure, even for NN function approximation [6, 1]

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What makes theory difficult:

1. Does $\bar{f}$ have a root?
2. Does the inverse of A exist?
3. SA theory is weak for a discontinuous ODE

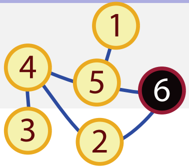
$\Longrightarrow$ 3 resolved for Zap by exploiting special structure, even for NN function approximation [6, 1]

Conclusions for Zap: Stability and optimal asymptotic covariance Σ^*

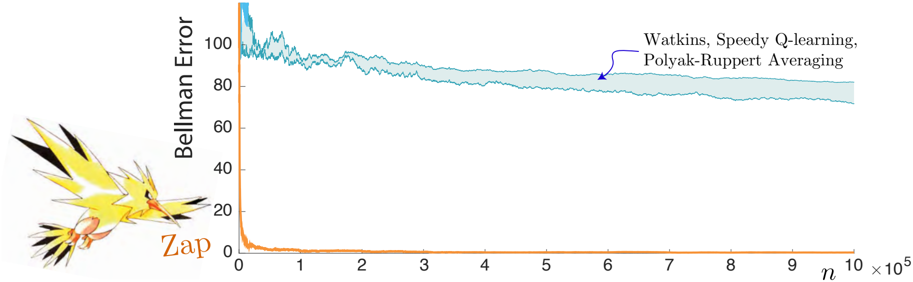
[Recall Eric's Tuesday plenary for defn of Σ^*]

Zap Q-Learning

Optimize Walk to Cafe



Convergence with Zap gain $\beta_n = n^{-0.85}$



Convergence of Zap-Q Learning

Discount factor: $\gamma = 0.99$

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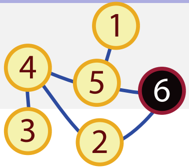

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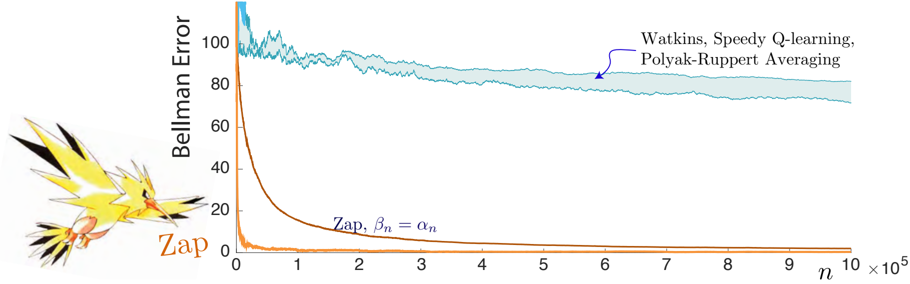

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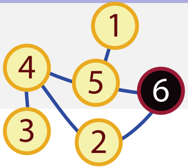

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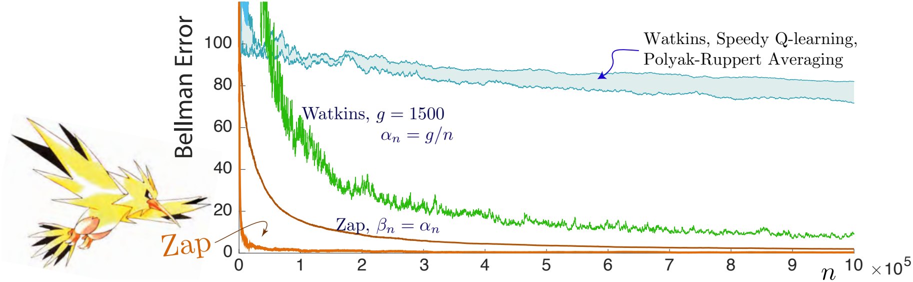

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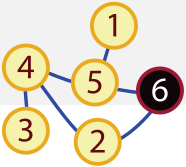

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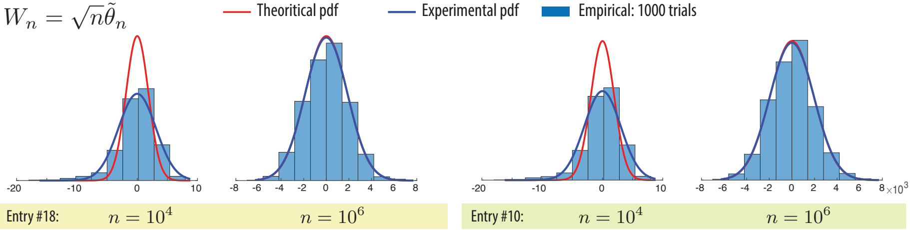

CLT gives good prediction of finite-n performance

Discount factor: $\gamma = 0.99$

Zap with Neural Networks

$$0 = \bar{f}(\theta^*) = \mathsf{E}\Big[\big\{c(X_n, U_n) + \gamma \underline{Q}^{\theta^*}(X_{n+1}) - Q^{\theta^*}(X_n, U_n)\big\}\zeta_n\Big]$$

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A few things to note:

- As far as we know, the empirical success of plain vanilla DQN is *extraordinary* (however, nobody reports failure)

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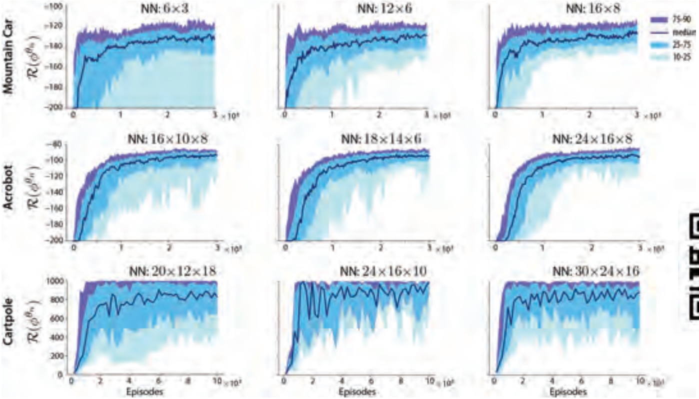
A few things to note:

- As far as we know, the empirical success of plain vanilla DQN is *extraordinary* (however, nobody reports failure)
- Zap Q-learning is the only approach for which convergence has been established (under mild conditions)
- We can expect stunning transient performance, based on coupling with the Newton-Raphson flow.

Zap with Neural Networks

VI. Stunning reliability with Q^θ parameterized by various neural networks



Reliability and stunning transient performance

—from coupling with the Newton-Raphson flow.

Challenges

Q-learning: $\{Q^\theta(x,u) : \theta \in \mathbb{R}^d\,,\ u \in \mathsf{U}\,,\ x \in \mathsf{X}\}$

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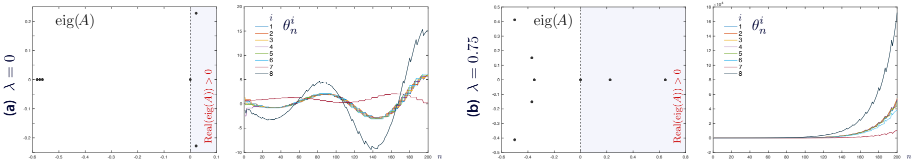
The Projected Bellman Equation



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Stability & The Projected Bellman Equation

Theory and Practice

Most of the elegant theory for tabular Q-learning: training is *oblivious*

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In practice we follow the intelligent mouse

I only need to see the cat *once*

Theory and Practice

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Approaches to exploration, $U_k \sim \widetilde{\phi}_k(\,\cdot \mid X_k)$:

- ε-greedy, $U_k = \phi^\theta(X_k)$ probability $1-\varepsilon$ small $\varepsilon > 0$
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$$\kappa_\theta \begin{cases} = \frac{1}{\|\theta\|}\kappa_0 & \|\theta\| \ge 1 \\ \ge \frac{1}{2}\kappa_0 & \textit{else} \end{cases}$$

SA recursion satisfies all the assumptions New in 2023

Theory for Tamed Gibbs $\widetilde{\phi}_k(u \mid x) \stackrel{\text{def}}{=} \mathsf{P}\{U_k = u \mid \mathcal{F}_k\,; X_k = x\}$

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For oblivious policy ($\varepsilon = 1$):

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- There is a unique invariant pmf π_θ for $(\boldsymbol{X}, \boldsymbol{U})$.
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Theory

$$\bar{f}(\theta) \stackrel{\text{def}}{=} \mathsf{E}\big[\{c(X_n,U_n) + \gamma \underline{Q}^\theta(X_{n+1}) - Q^\theta(X_n,U_n)\}\zeta_n\big]$$

Stability with sufficient optimism.

There is $\varepsilon_\gamma > 0$ (lower bound given in paper) for which the following hold:

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For each $0 < \varepsilon < \varepsilon_\gamma$, there is $\kappa_{\varepsilon,\gamma}$ such that

- The mean flow $\frac{d}{dt}\vartheta = \bar{f}(\vartheta)$ is *ultimately bounded.*

Proof follows Van Roy's analysis of TD-learning,

$$\frac{d}{dt}\|\vartheta_t\| \leq -\delta\|\vartheta_t\|\,, \qquad \textit{if} \;\; \|\vartheta_t\| \geq 1/\delta$$

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$$\bar{f}(\theta^*) = 0$$

Proof follows from the stability proof:

Denote $T(\theta) = \theta + \varepsilon_0 \bar{f}(\theta)$ for $\theta \in \mathbb{R}^d$, with $\varepsilon_0 > 0$ sufficiently small.

Goal: solve $T(\theta^*) = \theta^*$

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$$\|T(\theta)\| \le 1/\delta\,, \qquad \textit{if} \;\; \|\theta\| \le 1/\delta$$

Brouwer's fixed-point theorem tells us $T(\theta^*) = \theta^*$ has at least one solution.

See also de Farias & Van Roy [18]

Theory

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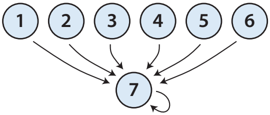
$$\bar{f}(\theta^*) = 0$$

- Under some additional assumptions θ^* is *locally* asymptotically stable

Baird's Example

$$\widetilde{\phi}_k(u \mid x) = (1-\varepsilon)\widetilde{\phi}_0^{\theta_k}(u \mid x) + \varepsilon \nu_{\mathcal{W}}(u)$$



$$h^\theta(x) = \theta^{\intercal}\psi(x) = \begin{cases} \theta^8 + 2\theta^k & x = k \le 6 \\ 2\theta^8 + \theta^7 & x = 7 \end{cases}$$

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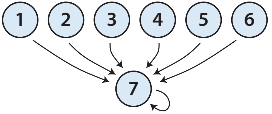

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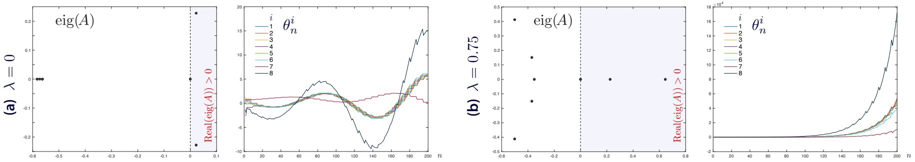

From CS&RL   Results for TD(λ)-learning, $\varepsilon = 1$

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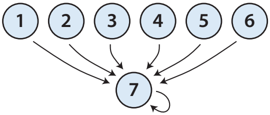

$$h^\theta(x) = \theta^T \psi(x) = \begin{cases} \theta^8 + 2\theta^k & x = k \le 6 \\ 2\theta^8 + \theta^7 & x = 7 \end{cases}$$

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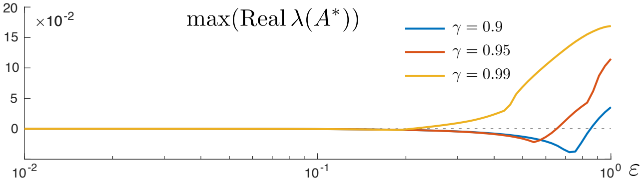

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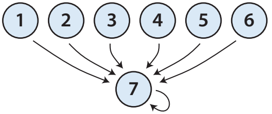

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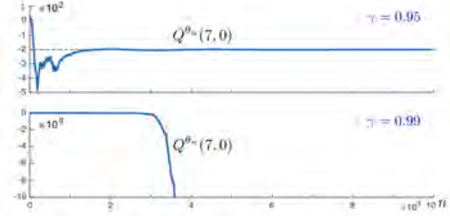

Fig. 1. Evolution of the Q-function approximations for two values of discount factor, and using an ε-greedy policy with common value of $\varepsilon = 0.5$.

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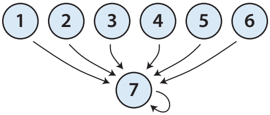

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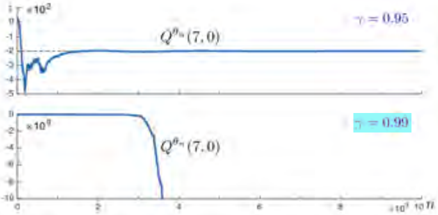

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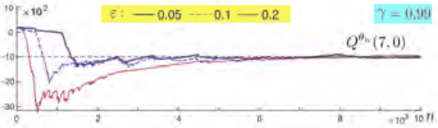

Fig. 2. Evolution of the Q-function approximations when using an ε-greedy policy. Convergence holds when $\varepsilon > 0$ is sufficiently small.

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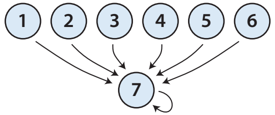

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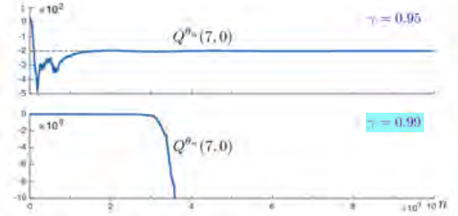

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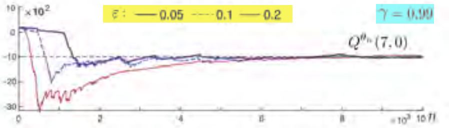

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Recent application to change detection, using Zap: $A^* = \partial_\theta \bar{f}(\theta^*)$ is not Hurwitz [7].

Conclusions & Future Directions

- Reinforcement Learning is cursed by dimension, variance, and nonlinear (algorithm) dynamics

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- Second order methods can ensure stability—use them when you can

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Future work:

- Beyond the projected Bellman error for Q-learning [45, 46, 47, 48]

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- Second order methods can ensure stability—use them when you can

Future work:

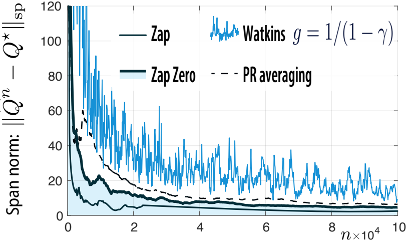
- Beyond the projected Bellman error for Q-learning [45, 46, 47, 48]
- Zap with optimism:

$$
\begin{aligned}
A(\theta) &= \partial_\theta \mathsf{E}_{\pi_\theta}[f(\theta, \Xi_n)] \\
&= \mathsf{E}_{\pi_\theta}[\partial_\theta f(\theta, \Xi_n)] + \mathsf{E}_{\pi_\theta}[f(\theta, \Xi_n)\Lambda_\theta(\Xi_n)^{\intercal}]
\end{aligned}
$$

- Reinforcement Learning is cursed by dimension, variance, and nonlinear (algorithm) dynamics
- Second order methods can ensure stability—use them when you can

Future work:

- Beyond the projected Bellman error for Q-learning [45, 46, 47, 48]
- Zap with optimism
- Acceleration techniques (momentum and matrix momentum)
  See Zap-Zero in CS&RL and [3]:



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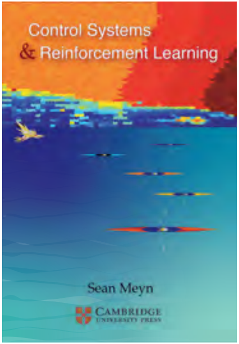
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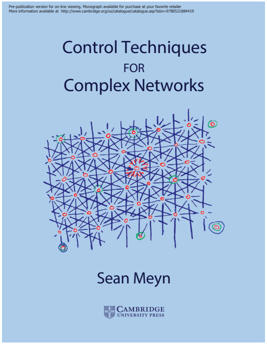
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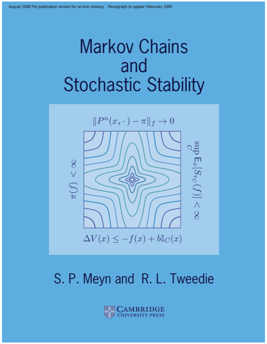
August 2008 Pre-publication version for on-line viewing. Monograph to appear Februrary 2009
Markov Chains
and
Stochastic Stability
$\|P^n(x,\cdot) - \pi\|_f \to 0$
$\pi(f) < \infty$
$\sup_C \mathsf{E}_x[S_{\tau_C}(f)] < \infty$
$\Delta V(x) \le -f(x) + b\mathbb{I}_C(x)$
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