Random landscape built by superposition of plane waves in high dimension

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EPSRC Grant EP/V002473/1

J. Math. Phys. 63 (9) (2022) & work in preparation

Contents

- Introduction
- Results on complexity
- Results on ground-state energy
- Conclusion

High-dimensional random landscapes

Random landscape $\mathscr{H}_N(\mathbf{x})$:

Random function of a large number N of degrees of freedom $\mathbf{x} = \{x_1, \cdots, x_N\}$

Important topic in physics, mathematics and beyond:

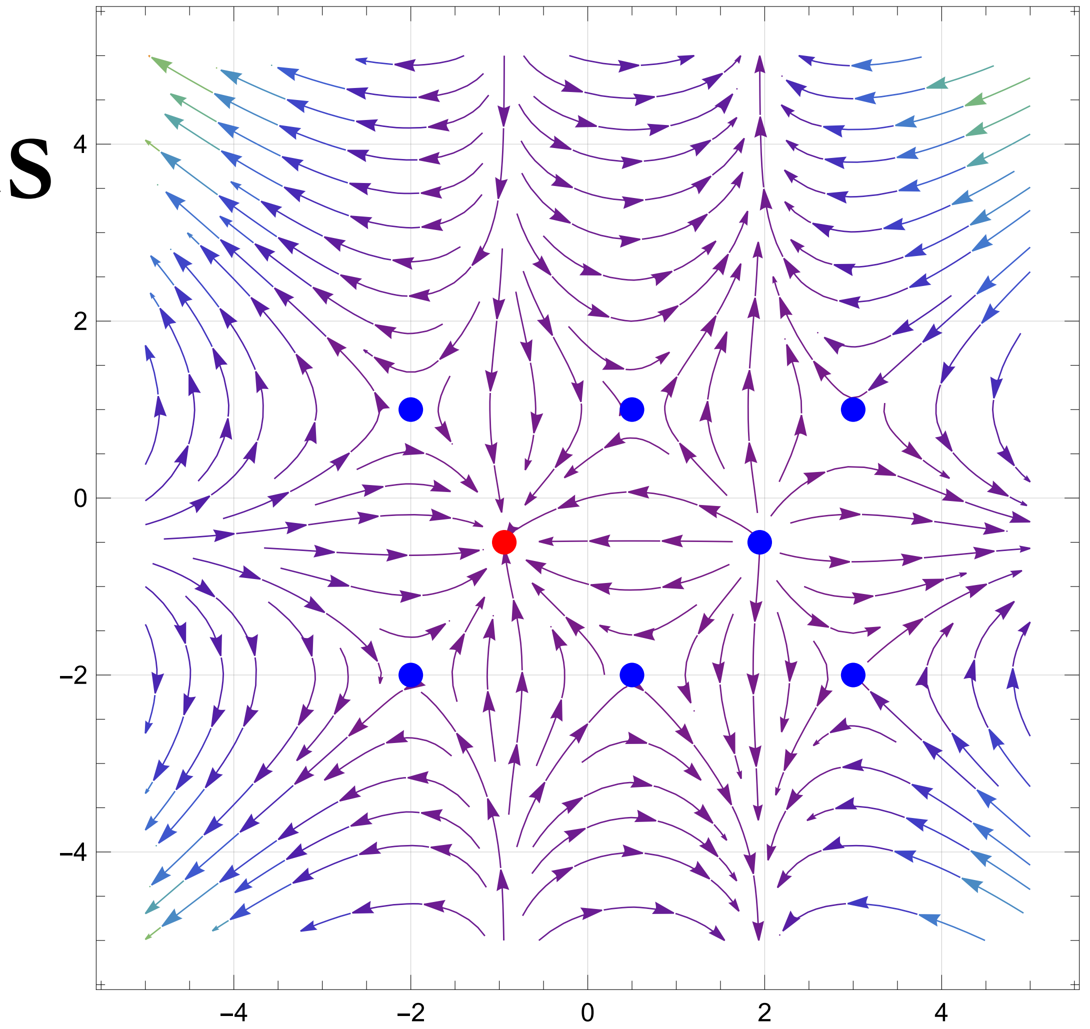
- Spin-glass energy landscape
- Utility function in economics
- Cost function in machine learning
- Fitness landscape in evolution

(Review by Ros & Fyodorov '22)

⇒In this talk, focus on static aspects analysed via RMT and tools from statistical physics

Number of stationary points / complexity

A first natural observable of importance is the number of stationary points (minima, maxima, saddles) of the landscape



The natural self-averaging observable is the quenched complexity

$$\xi_{\mathrm{tot},N} = \frac{1}{N} \ln \mathcal{N}_{\mathrm{tot},N} \qquad \lim_{N\to\infty} \xi_{\mathrm{tot},N} \underset{a.s.}{=} \lim_{N\to\infty} \mathbb{E}\left[\xi_{\mathrm{tot},N}\right] = \Xi_{\mathrm{tot}}$$

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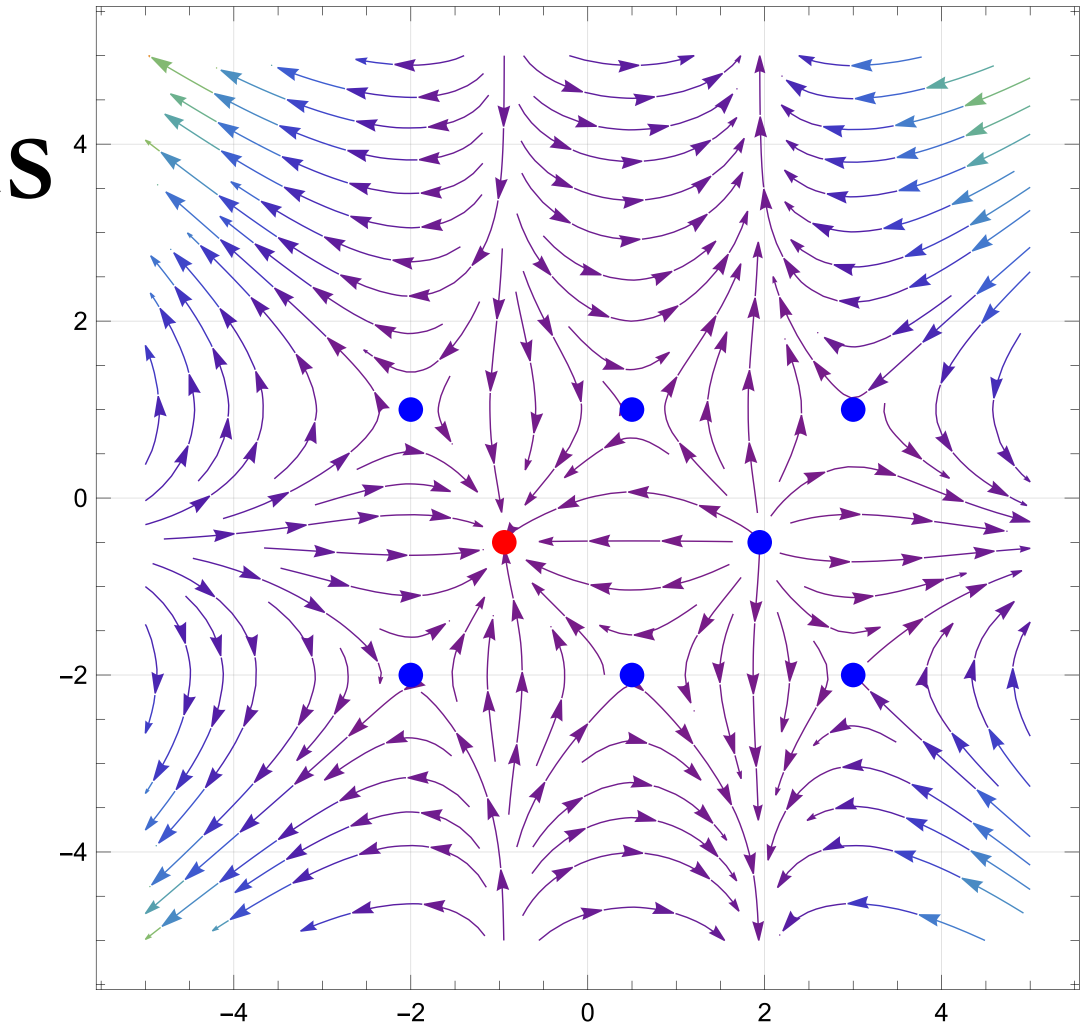

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Ergodicity breaking translates in a positive complexity $\Xi_{\text{tot}} > 0$

The latter is difficult to compute in most cases (see however Subag '17 & Ros et al. '19)

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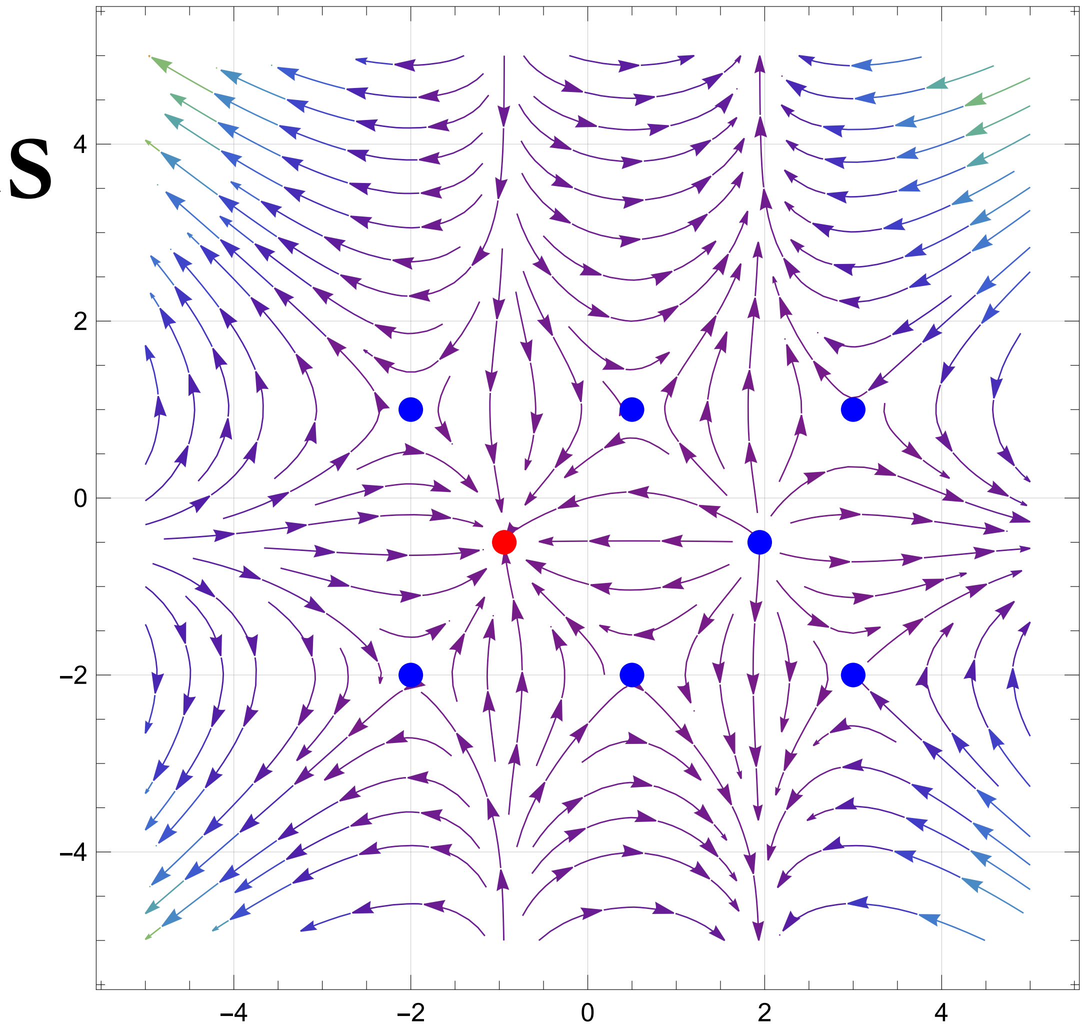

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The annealed complexity provides an upper bound and can be computed explicitly

$$\Sigma_{\text{tot}} = \lim_{N\to\infty} \frac{1}{N} \ln \mathbb{E}\left[\mathcal{N}_{\text{tot},N}\right] \geq \Xi_{\text{tot}} \qquad \mathbb{E}\left[\mathcal{N}_{\text{tot},N}\right] = \int d\mathbf{x}\, \mathbb{E}\left[\prod_{i=1}^{N} \delta((\nabla \mathscr{H}_N(\mathbf{x}))_i) \left|\det \nabla^2 \mathscr{H}_N(\mathbf{x})\right|\right]$$

Topology trivialisation transition

For one of the simplest random landscape (toy model in $d = 0$ of elastic manifold)

$$\mathscr{H}_N(\mathbf{x}) = \frac{\mu}{2}\mathbf{x}^2 + V(\mathbf{x})$$

with a Gaussian disordered potential

$$\mathbb{E}\left[V(\mathbf{x})\right] = 0 \qquad \mathbb{E}\left[V(\mathbf{x}_1)V(\mathbf{x}_2)\right] = N F\left(\frac{(\mathbf{x}_1 - \mathbf{x}_2)^2}{2N}\right)$$

(Thermodynamics: Mezard & Parisi ‘90 ‘91 ‘92, Engel ’93, Fyodorov & Sommers ’07

Dynamics: Franz & Mezard ‘94, Cugliandolo & Le Doussal ‘96

Complexity: Fyodorov ’04, Bray & Dean ‘07)

Recent exact results on $d = 0$ and finite d : Ben Arous, Bourgade, McKenna ’24, Ben Arous, Kivimae ‘24

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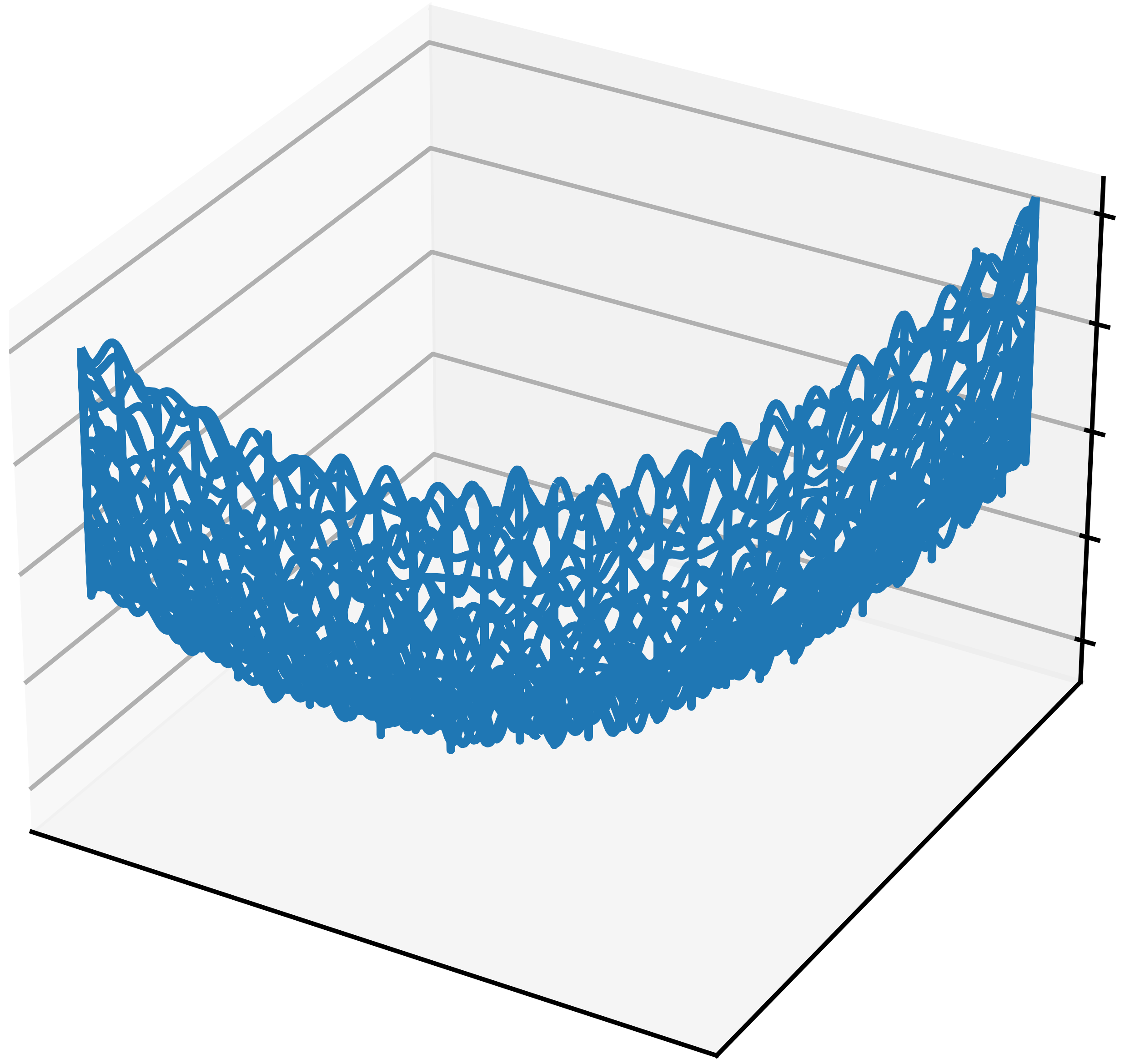
There exists a topology trivialisation transition as a function of μ (Fyodorov '04)

$$\Sigma_{\text{tot}} = \begin{cases} \frac{1}{2}\left(\frac{\mu^2}{\mu_c^2} - 1 - \ln\frac{\mu^2}{\mu_c^2}\right) & , \ \mu < \mu_c = \sqrt{F''(0)} \\ 0 & , \ \mu \geq \mu_c = \sqrt{F''(0)} \end{cases}$$

Large universality:

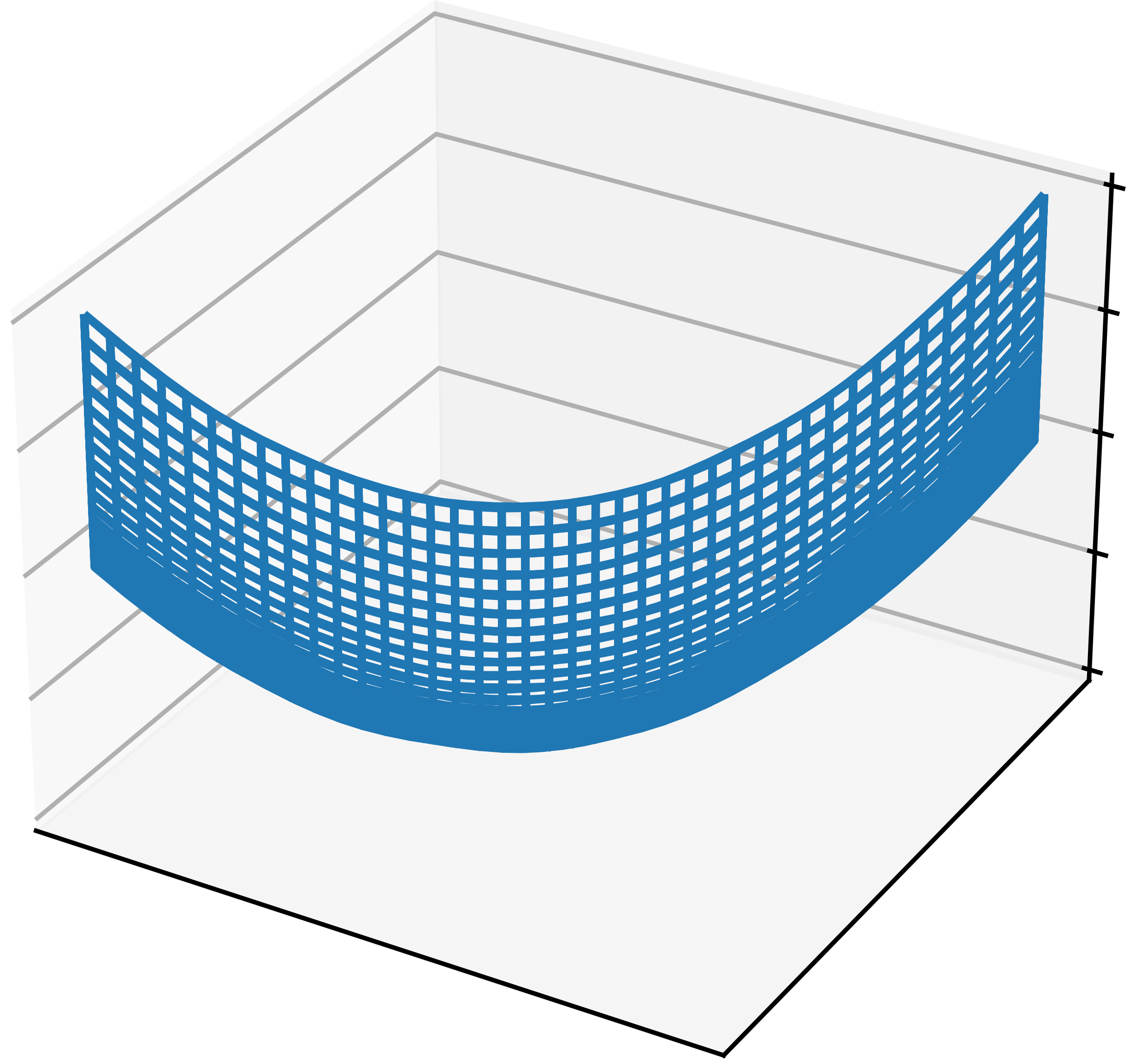
Only depends on $F''(0)$

Topology trivialisation transition



Topology
trivialisation
transition

μ_c



$\Sigma_{\text{tot}} > 0$

Complex phase

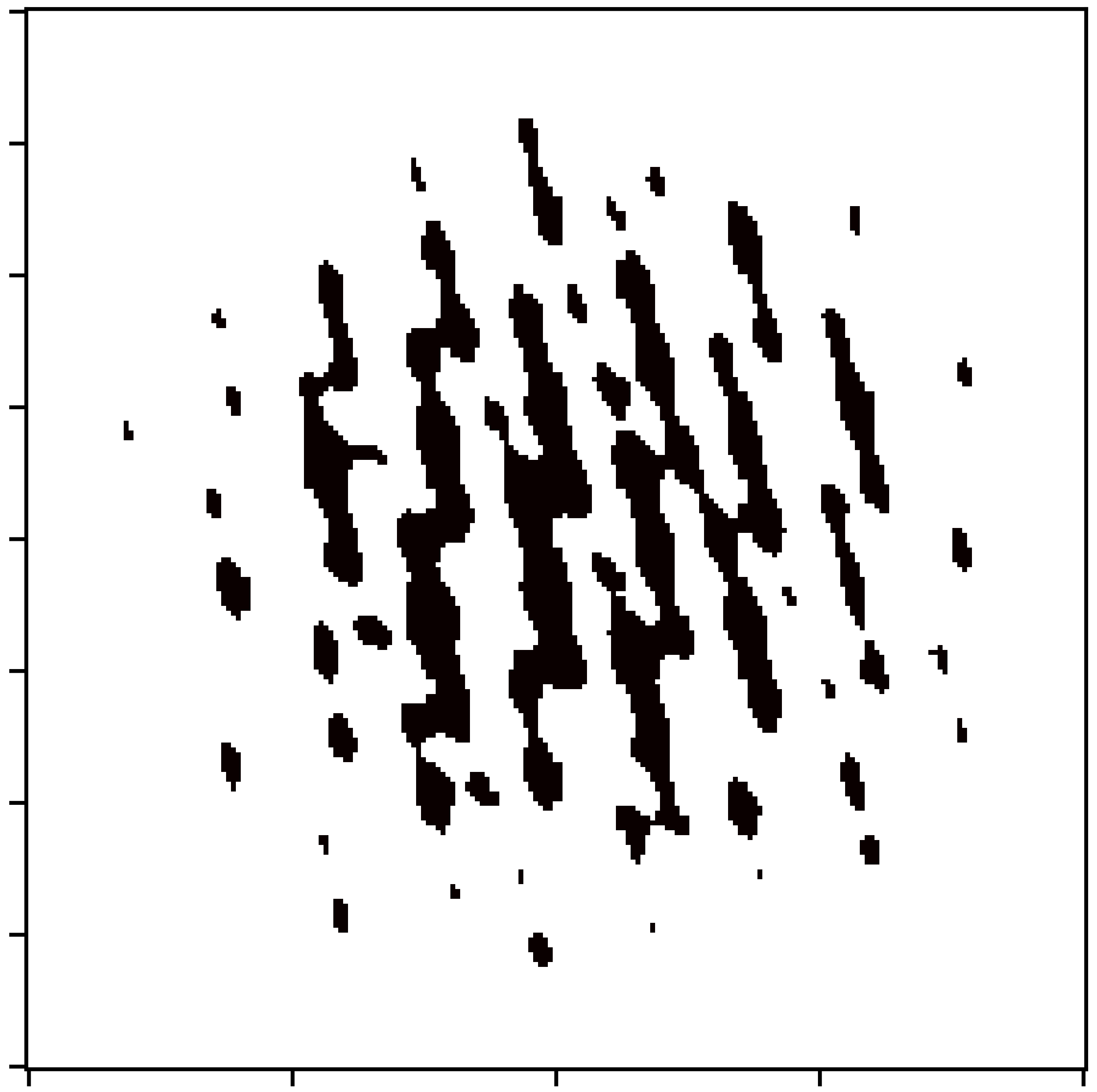
$\Sigma_{\text{tot}} = \Xi_{\text{tot}} = 0$

Trivial phase

μ

Topology trivialisation transition

$$S(E) = \{\mathbf{x} : \mathscr{H}_N(\mathbf{x}) \leq E\}$$



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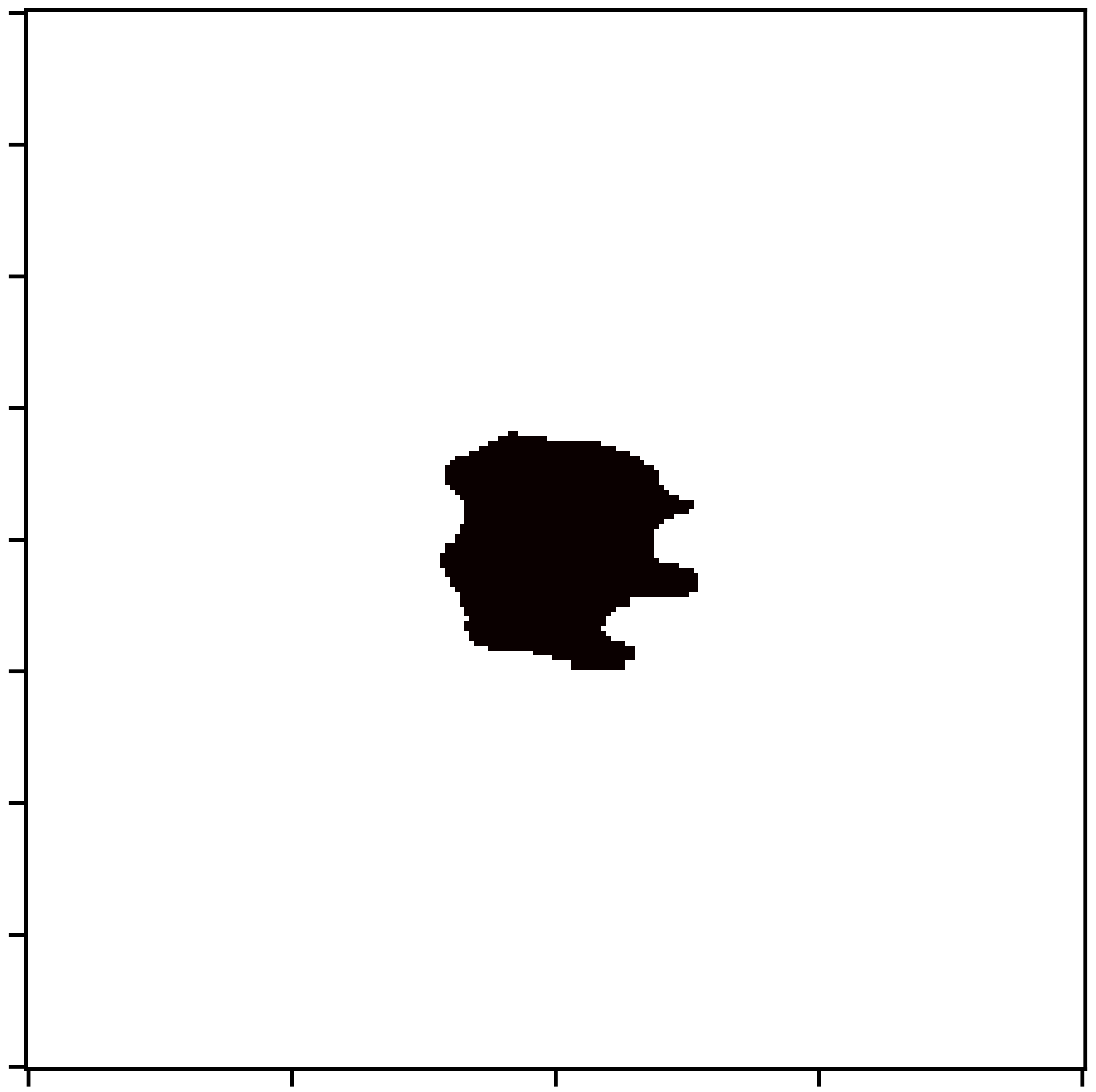

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Complex phase

$\Sigma_{\text{tot}} = \Xi_{\text{tot}} = 0$

Trivial phase

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Ground-state energy

A second natural observable of importance is the ground-state energy (GSE) of the landscape

This observable is also self-averaging

$$e_{\min,N} = \frac{1}{N} \min_{\mathbf{x}} \mathscr{H}_N(\mathbf{x})$$

$$\lim_{N\to\infty} e_{\min,N} \underset{a.s.}{=} \lim_{N\to\infty} \mathbb{E}\left[e_{\min,N}\right] = e_{\mathrm{typ}}$$

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Its average value (and the probability of atypical fluctuations) are computed in the physics literature via the replica method

$$e_{\min,N} = -\lim_{\beta\to\infty} \frac{1}{N\beta} \ln \mathcal{Z}_{\mathcal{N}}(\beta) \qquad \mathcal{Z}_{\mathcal{N}}(\beta) = \int d\mathbf{x}\, e^{-\beta \mathcal{H}_N(\mathbf{x})} \qquad \mathbb{E}\left[\ln \mathcal{Z}_N(\beta)\right] = \lim_{n\to 0} \frac{1}{n} \ln \mathbb{E}\left[\mathcal{Z}_N(\beta)^n\right]$$

There is a considerable literature in mathematics to compute the GSE rigorously

(Guerra '03, Talagrand '06, …)

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Ergodicity breaking translates in replica symmetry breaking (RSB)

In many instances the criterion for RSB matches that of positive annealed complexity (Fyodorov & Williams ’07)

Ground-state energy

For one of the simplest random landscape (toy model in $d = 0$ of elastic manifold)

$$\mathcal{H}_N(\mathbf{x}) = \frac{\mu}{2}\mathbf{x}^2 + V(\mathbf{x})$$

with a Gaussian disordered potential

$$\mathbb{E}\left[V(\mathbf{x})\right] = 0 \qquad\qquad \mathbb{E}\left[V(\mathbf{x}_1)V(\mathbf{x}_2)\right] = N\,F\left(\frac{(\mathbf{x}_1 - \mathbf{x}_2)^2}{2N}\right)$$

The RS expression for the ground-state energy reads

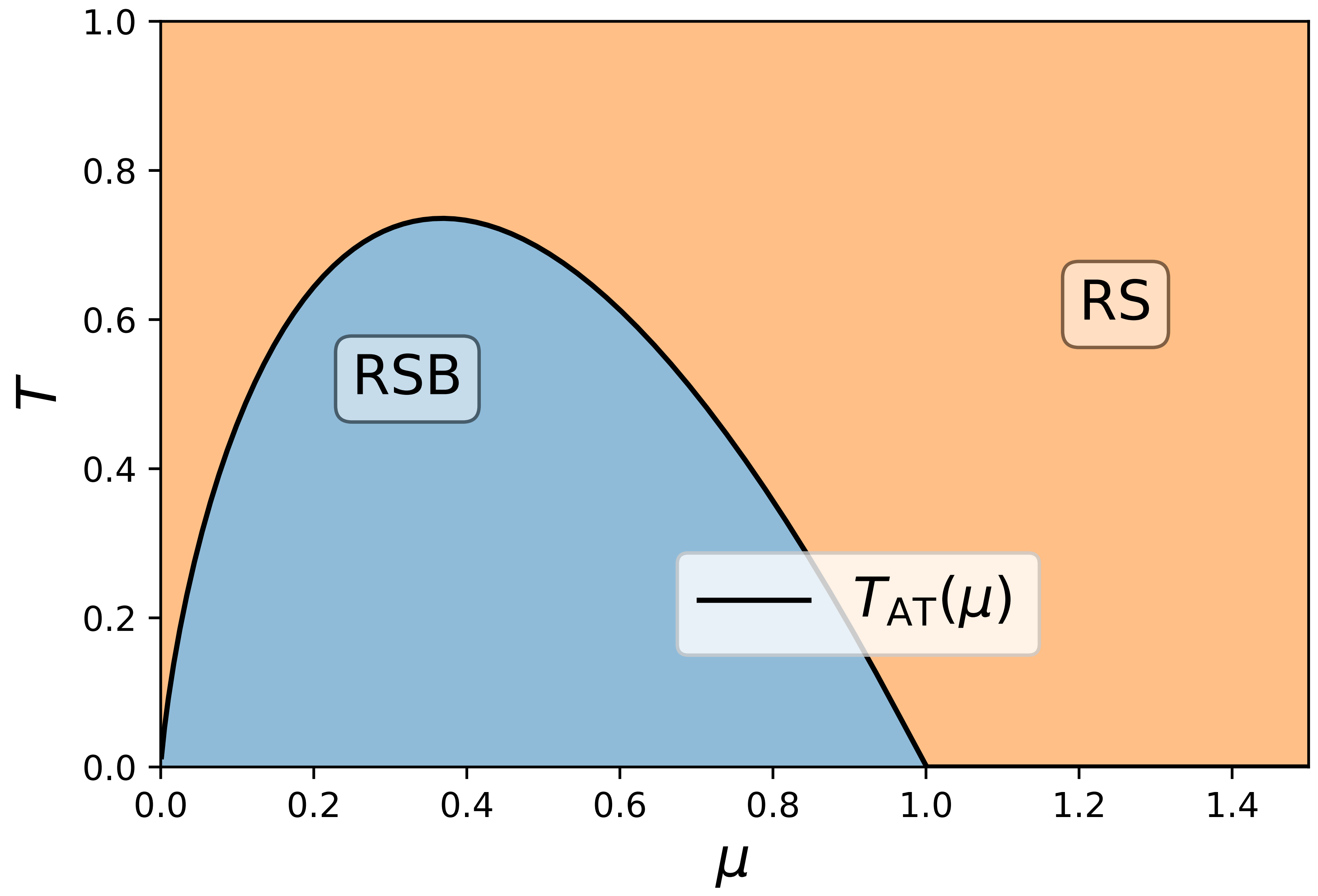
$$e_{\text{typ}} = \frac{F'(0)}{2\mu}$$

which becomes unstable (AT-line) for

$$\mu > \mu_c = \sqrt{F''(0)}$$

matching that of the complexity

See e.g. (Fyodorov & Sommers '07)



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The transition is towards a FRSB/1RSB phase for a positive/negative Schwarzian derivative

$$\mathcal{S}[F'(q)] = \frac{F^{(4)}(q)}{F''(q)} - \frac{3}{2}\left(\frac{F^{(3)}(q)}{F''(q)}\right)^2$$

See e.g. (Fyodorov & Sommers '07)

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In the 1RSB phase (negative Schwarzian derivative)

$$\mathcal{S}[F'(q)] < 0$$

Local minima are isolated, separated by high barriers

Only local minima are found in a small range of energy around e_{typ}

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In the FRSB phase (positive Schwarzian derivative)

$$\mathcal{S}[F'(q)] > 0$$

The landscape displays many flat directions

All types of saddles are found in a small range of energy around e_{typ}

Digression and motivation for the model: Semi-classical chaos

Semi-classical chaos

Consider a Riemmanian manifold $\mathscr{D}$ with strongly chaotic classical flow.

The eigenfunctions of the quantum Laplacian

$$-\Delta\psi_n(\mathbf{x}) = E_n\psi_n(\mathbf{x}) \qquad E_1 \leq E_2 \leq \cdots \qquad \mathbf{x} \in \mathscr{D}$$

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Are conjectured by Berry '77 to be expressed, in the semi-classical limit $n \gg 1$, as superpositions of plane waves

$$\psi_n(\mathbf{x}) = \sum_{l=1}^{M} \gamma_l \cos(\mathbf{k}_{n,l}\mathbf{x} + \theta_l) \qquad \mathbf{k}_{n,l}^2 = E_n \qquad \begin{array}{l} \gamma_l : \mathcal{N}(0,\sigma_l^2) \\ \theta_l : \mathrm{U}[0,2\pi) \end{array}$$

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Many properties of these eigenstates have been investigated (especially in 2D):

Nodal domains: Blum, Gnutzmann, & Smilansky '02; Bogomolny & Schmit '02

Critical points: Beliaev, Cammarota & Wigman '19

Maximum norm: Aurich, Bäcker, Schubert, Taglieber '99

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This type of eigenfunctions will be used here to construct a high-dimensional random landscape

The model

Let us consider the following generally non Gaussian random landscape

$$\mathcal{H}_N(\mathbf{x}) = \frac{\mu}{2}\mathbf{x}^2 + V_M(\mathbf{x}) \qquad V_M(\mathbf{x}) = \sum_{l=1}^{M} \phi_l(\mathbf{k}_l\mathbf{x}) \qquad \phi_l(z) = \sum_{n=1}^{\infty} \gamma_{n,l}\cos(n(z+\theta_{n,l}))$$

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where the wave vectors $\mathbf{k}_l$'s are either:

- Gaussian i.i.d. random variables
- Uniform vectors on the N-sphere

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- Gaussian i.i.d. random variables
- Uniform vectors on the N-sphere

For a non-random function $\phi(z)$ and $\mu = 0$:

Maillard, Ben Arous, Biroli '20

The i.i.d. random functions $\phi_l(x)$'s have zero average $\mathbb{E}_\phi \left[\phi(x)\right] = 0$

and their statistics is translationally invariant (θ_n : U[0,2π))

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We are interested in the limit $N, M \to \infty$ with $0 < \alpha = \frac{M}{N} < \infty$

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Average number of stationary points

For that problem let us now compute the average number of stationary points:

$$\mathbb{E}\left[\mathcal{N}_{\mathrm{tot},N}\right] = \int d\mathbf{x}\, \mathbb{E}\left[\prod_{i=1}^{N} \delta((\nabla \mathcal{H}_N(\mathbf{x}))_i) \left|\det \nabla^2 \mathcal{H}_N(\mathbf{x})\right|\right]$$

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$$\partial_{x_i} \mathcal{H}_N(\mathbf{x}) = \mu x_i - \sum_{l=1}^{M} k_{li} G_l \qquad G_l = -\phi_l'(\mathbf{k}_l \mathbf{x}) = \gamma_l \sin(\mathbf{k}_l \mathbf{x} + \theta_l)$$

$$\partial^2_{x_i, x_j} \mathcal{H}_N(\mathbf{x}) = \mu \delta_{ij} - \sum_{l=1}^{M} k_{li} k_{lj} T_l \qquad T_l = -\phi_l''(\mathbf{k}_l \mathbf{x}) = \gamma_l \cos(\mathbf{k}_l \mathbf{x} + \theta_l)$$

The statistics of the G_l's and T_l's

is independent of $\mathbf{x}$ and $\mathbf{k}_l$

Average number of stationary points

For that problem let us now compute the average number of stationary points:

$$\mathbb{E}\left[\mathcal{N}_{\text{tot},N}\right] = \int d\mathbf{x}\, \mathbb{E}_{\mathbf{G},\mathbf{T},\mathbf{k}}\left[\prod_{i=1}^{N} \delta\left(\mu x_i - \sum_{l=1}^{M} k_{li}\, G_l\right) \left|\det\left(\mu\delta_{ij} - \sum_{l=1}^{M} k_{li}k_{lj}\, T_l\right)\right|\right]$$

$$= \mathbb{E}_{\mathbf{G},\mathbf{T},\mathbf{k}}\left[\int d\mathbf{x} \prod_{i=1}^{N} \delta\left(\mu x_i - \sum_{l=1}^{M} k_{li}\, G_l\right) \left|\det\left(\mu\delta_{ij} - \sum_{l=1}^{M} k_{li}k_{lj}\, T_l\right)\right|\right]$$

$$= \frac{1}{\mu^N}\mathbb{E}_{\mathbf{T},\mathbf{k}}\left[\left|\det\left(\mu\delta_{ij} - \sum_{l=1}^{M} k_{li}k_{lj}\, T_l\right)\right|\right]$$

Strong self-averaging

In order to compute the annealed complexity, we suppose the strong self-averaging property

$$\lim_{N\to\infty} \frac{1}{N} \ln \mathbb{E}_{\mathbf{k}} \left[\left| \det \left(\mu\delta_{ij} - \sum_{l=1}^{M} k_{li}k_{lj} T_l \right) \right| \right] = \lim_{N\to\infty} \mathbb{E}_{\mathbf{k}} \left[\frac{1}{N} \ln \left| \det \left(\mu\delta_{ij} - \sum_{l=1}^{M} k_{li}k_{lj} T_l \right) \right| \right]$$

$$= \int d\lambda \, \rho_{KTK^T}(\lambda) \, \ln |\mu - \lambda|$$

$$\rho_{KTK^T}(\lambda) = \lim_{N\to\infty} \frac{1}{N} \mathrm{Tr} \left[\delta(\lambda \mathbb{I} - KTK^T) \right]$$

Results from Marchenko-Pastur ('67)

To characterise the limiting density, it is convenient to introduce its Stieltjes transform

$$m(z) = \lim_{N\to\infty} \frac{1}{N} \mathrm{Tr}\left[(z\mathbb{I} - KTK^T)^{-1}\right] \qquad \rho_{KTK^T}(\lambda) = \frac{1}{\pi} \lim_{\epsilon\to 0} m_i(\lambda + i\epsilon)$$

The Stieltjes transform satisfies the following self-consistent equation

$$\frac{1}{m(z)} = z - \alpha \int dt \, \frac{t\, p(t)}{1 - t\, m(z)} \qquad p(t) = \lim_{M\to\infty} \frac{1}{M} \sum_{l=1}^{M} \delta(t - T_l)$$

⇒An unbounded distribution $p(t)$ yields an unbounded spectrum $\rho_{KTK^T}(\lambda)$

Annealed complexity

Under the strong self-averaging property, the average number of stationary points can be expressed as a functional integral over the probability measure

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$$p(t) = \lim_{M\to\infty} \frac{1}{M} \sum_{l=1}^{M} \delta(t - T_l) \qquad p_0(t) = \mathbb{E}\left[\delta(t - T_l)\right]$$

$$\mathbb{E}\left[\mathcal{N}_{\text{tot},N}\right] \approx \int \prod_{l=1}^{M} dt_l\, p_0(t_l)\, e^{N\left[\int d\lambda\, \rho(\lambda) \ln|\mu - \lambda| - \ln \mu\right]} = \int_{\int dt\, p(t)=1} \mathcal{D}p(t)\, e^{N\Phi_\alpha[p(t),p_0(t)]}$$

$$\Phi_\alpha[p(t), p_0(t)] = -\alpha \int dt\, p(t)\, \ln \frac{p(t)}{p_0(t)} + \int d\lambda\, \rho_{KTK^T}(\lambda) \ln|\mu - \lambda| - \ln \mu$$

$$\Sigma_{\text{tot}} = \lim_{N\to\infty} \frac{1}{N} \ln \mathbb{E}\left[\mathcal{N}_{\text{tot},N}\right] = \max_{p(t):\int dt\, p(t)=1} \Phi_\alpha[p(t), p_0(t)]$$

Results

The annealed complexity can be expressed as

$$\Sigma_{\text{tot}}(\mu) = \max_{p(t):\int dt\, p(t)=1} \Phi_\alpha[p(t), p_0(t)] = \int_\mu^\infty d\nu \left(\frac{1}{\nu} + m_r(-\nu) \right)$$

$$\Phi_\alpha[p(t), p_0(t)] = -\alpha \int dt\, p(t)\, \ln \frac{p(t)}{p_0(t)} + \int d\lambda\, \rho_{KTK^T}(\lambda) \ln|\mu - \lambda| - \ln \mu$$

where the function

$$m(-\nu) = m_r(-\nu) + i\, m_i(-\nu)$$

$$\frac{1}{m(-\nu)} = -\nu - \alpha \frac{\int dt\, t\, p_0(t) \frac{|1 - t\, m(-\nu)|}{1 - t\, m(-\nu)}}{\int dt\, p_0(t)\, |1 - t\, m(-\nu)|}$$

Explicit solution to the optimisation problem:

$$p_*(t) = \frac{p_0(t)\, |1 - t\, m(-\nu)|}{\int dr\, p_0(r)\, |1 - r\, m(-\nu)|}$$

Results

For an unbounded distribution $p_0(t) = \mathbb{E}\left[\delta(t - T_l)\right]$, no trivialisation transition

$$\Sigma_{\text{tot}}(\mu) > 0 \qquad \text{for any } \mu < \infty$$

Indication that ergodicity broken for any μ ?

Gaussian $p_0(t)$: LACT, Belga Fedeli, Fyodorov, J. Math. Phys. 63 (9) (2022)

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For a bounded and zero average distribution $p_0(t)$, there is a trivialisation transition

$$\Sigma_{\rm tot}(\mu) \begin{cases} > 0\,, & \mu < \mu_c \\ = 0\,, & \mu \geq \mu_c \end{cases}$$

Indication that ergodicity broken for $\mu < \mu_c$?

$$\alpha \left[\int dt \, \frac{\mu_c^2 \, p_*(t)}{(\mu_c - t)^2} - 1 \right] - 1 = \alpha \left[\int dt \, \frac{\mu_c \, p_0(t)}{|\mu_c - t|} - 1 \right] - 1 = 0$$

Results

For a bounded and zero average distribution $p_0(t) = \mathbb{E}\left[\delta(t - T_l)\right]$, the complexity vanishes

$$\alpha \left[\int dt \, \frac{\mu_c^2 \, p_*(t)}{(\mu_c - t)^2} - 1 \right] - 1 = \alpha \left[\int dt \, \frac{\mu_c \, p_0(t)}{|\,\mu_c - t\,|} - 1 \right] - 1 = 0$$

The complexity vanishes quadratically

$$\Sigma_{\text{tot}}(\mu) \approx C_2(\mu - \mu_c)^2$$

Results

Similar results can be obtained for the annealed complexity of minima:

For unbounded $p_0(t) = \mathbb{E}\left[\delta(t - T_l)\right]$ $\Sigma_{\text{tot}}(\mu) > \Sigma_{\text{min}}(\mu) > 0$ for any $\mu < \infty$

For bounded $p_0(t) = \mathbb{E}\left[\delta(t - T_l)\right]$ $\Sigma_{\text{min}}(\mu) \begin{cases} > 0\,, \ \mu < \mu_c \\ = 0\,, \ \mu \geq \mu_c \end{cases}$

The complexity of minima vanishes quadratically

$$\Sigma_{\text{min}}(\mu) \approx C_2'(\mu - \mu_c)^2$$

Results

The results for the annealed complexity only provide a bound for the quenched complexity and thus on the ergodicity breaking transition.

$$\Sigma_{\text{tot}} = \lim_{N\to\infty} \frac{1}{N} \ln \mathbb{E}\left[\mathcal{N}_{\text{tot},N}\right] \geq \Xi_{\text{tot}} \qquad\qquad \Xi_{\text{tot}} > 0 \Leftrightarrow \text{Ergodicity breaking}$$

From the results so far, ergodicity is NOT broken for any $\mu > \mu_c$

(however $\mu_c = +\infty$ for unbounded support)

Can these results be confirmed from the computation of the ground-state energy?

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- Results on complexity
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Ground-state energy

We are now interested in computing the average (and typical) ground-state energy

$$e_{\text{typ}} = \lim_{N\to\infty} \frac{1}{N} \min_{\mathbf{x}} \mathcal{H}_N(\mathbf{x})$$

We will compute its value using the replica method

$$e_{\text{typ}} = -\lim_{\beta\to\infty} \frac{1}{N\beta} \mathbb{E}\left[\ln \mathcal{Z}_N(\beta)\right] \qquad \mathcal{Z}_N(\beta) = \int d\mathbf{x}\, e^{-\beta\mathcal{H}_N(\mathbf{x})} \qquad \mathbb{E}\left[\ln \mathcal{Z}_N(\beta)\right] = \lim_{n\to 0} \frac{1}{n} \ln \mathbb{E}\left[\mathcal{Z}_N(\beta)^n\right]$$

We first need to evaluate the quantity

$$\lim_{N\to\infty} \frac{1}{N} \ln \mathbb{E}\left[\mathcal{Z}_N(\beta)^n\right]$$

Replicated partition function

The first step to obtain the average ground-state energy is to compute the replicated partition function

$$\mathbb{E}\left[\mathcal{Z}_N(\beta)^n\right] = \int \prod_{a=1}^{n} d\mathbf{x}_a \, e^{-\frac{\beta\mu}{2}\sum_{a=1}^{n}\mathbf{x}_a^2} \prod_{l=1}^{M} \mathbb{E}\left[e^{-\beta\sum_{a=1}^{n}\phi_l(\mathbf{k}_l\mathbf{x}_a)}\right]$$

Each term of the product can be re-expressed in terms of inverse overlap as

$$\mathbb{E}\left[e^{-\beta\sum_{a=1}^{n}\phi(\mathbf{k}\mathbf{x}_a)}\right] = \frac{\sqrt{\det(Q)}}{(2\pi)^{\frac{n}{2}}} P_n(Q)$$

$$(Q^{-1})_{ab} = \frac{\mathbf{x}_a \cdot \mathbf{x}_b}{N}$$

$$P_n(Q) = \int d\mathbf{z} \, e^{-\frac{\mathbf{z}Q\mathbf{z}}{2}} \mathbb{E}_\phi\left[e^{-\beta\sum_{a=1}^{n}\phi(z_a)}\right]$$

Replicated partition function

One can now obtain

$$\mathbb{E}\left[\mathcal{Z}_N(\beta)^n\right] = \int \prod_{a=1}^{n} d\mathbf{x}_a \, e^{-\frac{\beta\mu}{2}\sum_{a=1}^{n}\mathbf{x}_a^2} \prod_{l=1}^{M} \mathbb{E}\left[e^{-\beta \sum_{a=1}^{n} \phi_l(\mathbf{k}_l \mathbf{x}_a)}\right]$$

$$= c_{N,n} \int Q^{-1} dQ Q^{-1} (\det(Q))^{-\frac{(n+1)}{2}} \, e^{N\Psi_{n,\alpha}(Q)}$$

$$\Psi_{n,\alpha}(Q) = -\frac{\beta\mu}{2}\mathrm{Tr}\left(Q^{-1}\right) - \frac{1-\alpha}{2}\ln\det Q + \alpha \ln P_n(Q) + \frac{n}{2}\left[(1-\alpha)\ln(2\pi) + 1\right]$$

Replicated partition function

One can now obtain

$$\mathbb{E}\left[\mathcal{Z}_N(\beta)^n\right] = \int \prod_{a=1}^{n} d\mathbf{x}_a \, e^{-\frac{\beta\mu}{2}\sum_{a=1}^{n}\mathbf{x}_a^2} \prod_{l=1}^{M} \mathbb{E}\left[e^{-\beta\sum_{a=1}^{n}\phi_l(\mathbf{k}_l\mathbf{x}_a)}\right]$$
$$= c_{N,n}\int Q^{-1}dQQ^{-1}(\det(Q))^{-\frac{(n+1)}{2}}\, e^{N\Psi_{n,\alpha}(Q)}$$

$$\Psi_{n,\alpha}(Q) = -\frac{\beta\mu}{2}\mathrm{Tr}\left(Q^{-1}\right) - \frac{1-\alpha}{2}\ln\det Q + \alpha\ln P_n(Q) + \frac{n}{2}\left[(1-\alpha)\ln(2\pi)+1\right]$$

The average and typical GSE is obtained as

$$e_{\mathrm{typ}} = -\operatorname*{ext}_{Q>0}\lim_{n\to 0}\frac{\Psi_{n,\alpha}(Q)}{n\beta}$$

Parisi formula

The average GSE is obtained from the Parisi formula

$$e_{\text{typ}} = \sup_{l, w(l')} \left[\frac{\mu}{2} \int_0^l \frac{dt}{\left(\mu + \int_0^t w(\tau)\, d\tau\right)^2} - \frac{1-\alpha}{2} \int_0^l \frac{dt}{\mu + \int_0^t w(\tau)\, d\tau} - \alpha \ln \mathbb{E}_\phi \left[f(0,0) \right] \right]$$

Parisi formula

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The function $f(t, h)$ satisfies Parisi's PDE

$$\partial_t f = -\frac{1}{2} \left[\partial_h^2 f + w(t) \left(\partial_h f \right)^2 \right]$$

with the random boundary condition

$$f(t \geq l, h) = -\epsilon_{\min} \left(\mu + \int_0^l w(\tau)\, d\tau, h \right) \quad \text{where} \quad \epsilon_{\min}(\nu, h) = \min_z \left[\frac{\nu}{2} z^2 - hz + \phi(z) \right]$$

Replica-symmetric solution

The simplest solution corresponds to a replica-symmetric solution

$$(Q^{-1})_{ab} = \frac{\mathbf{x}_a \cdot \mathbf{x}_b}{N} = \begin{cases} r\,, & a \neq b \\ r_d\,, & a = b \end{cases}$$

If that solution is correct, the system is ergodic

Replica-symmetric solution

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If that solution is correct, the system is ergodic

The properties of that solution can be expressed in term of an effective 1D disordered system

$$H_{\mu,h}(z) = \frac{\mu}{2} z^2 - hz + \phi(z)$$

In particular

$$e_{\text{typ}} = \alpha\, \mathbb{E}_\phi \left[\epsilon_{\min}(\mu,0)\right]$$

$$r = \alpha\, \mathbb{E}_\phi \left[z^2_{\min}(\mu,0)\right]$$

$$\lim_{\beta\to\infty} \beta(r_d - r) = \frac{1}{\mu}$$

$$\epsilon_{\min}(\mu, h) = \min_z H_{\mu,h}(z)$$

$$z_{\min}(\mu, h) = \operatorname*{argmin}_z H_{\mu,h}(z)$$

De-Almeida-Thouless line

For the RS solution to be stable, one needs to ensure that the solution corresponds indeed to a maximum, i.e. the eigenvalues of the quadratic form

$$A^{(n)}_{(ab),(cd)} = \frac{\partial^2 \Psi_{n,\alpha}(Q)}{\partial Q_{ab} \partial Q_{cd}}$$

are all negative as $n \to 0$

The replicon (i.e. largest eigenvalue) reads

$$\lambda_{\mathrm{RS}}(\mu) = \alpha \, \mathbb{E}_\phi \left[\frac{\mu^2}{(\mu + \phi''[z_{\min}(\mu)])^2} - 1 \right] - 1$$

De-Almeida-Thouless line

In particular, using that

$$\phi(z) = \gamma \cos(z+\theta)$$

$$\phi''(z) = -\gamma \cos(z+\theta) = -\phi(z)$$

and denoting $p_*(t)$ the PDF of $\phi[z_{\min}(\mu)]$

The marginality criterion for the replicon reads

$$\lambda_{RS}(\mu_c) = 0 = \alpha\, \mathbb{E}_\phi \left[\frac{\mu_c^2}{(\mu_c + \phi''[z_{\min}(\mu_c)])^2} - 1 \right] - 1 = \alpha \left[\int dt\, \frac{\mu_c^2 p_*(t)}{(\mu_c - t)^2} - 1 \right] - 1$$

and matches the criterion for the complexity to vanish

$$\alpha \left[\int dt\, \frac{\mu_c^2\, p_*(t)}{(\mu_c - t)^2} - 1 \right] - 1 = \alpha \left[\int dt\, \frac{\mu_c\, p_0(t)}{|\mu_c - t|} - 1 \right] - 1 = 0$$

Ergodicity breaking

As the two criterion concur, one can safely conclude that:

Ergodicity is broken for any value of μ for an unbounded distribution of $\phi(z)$

Ergodicity breaking

As the two criterion concur, one can safely conclude that:

Ergodicity is broken for any value of μ for an unbounded distribution of $\phi(z)$

For a bounded support, there exist a finite value μ_c which satisfies

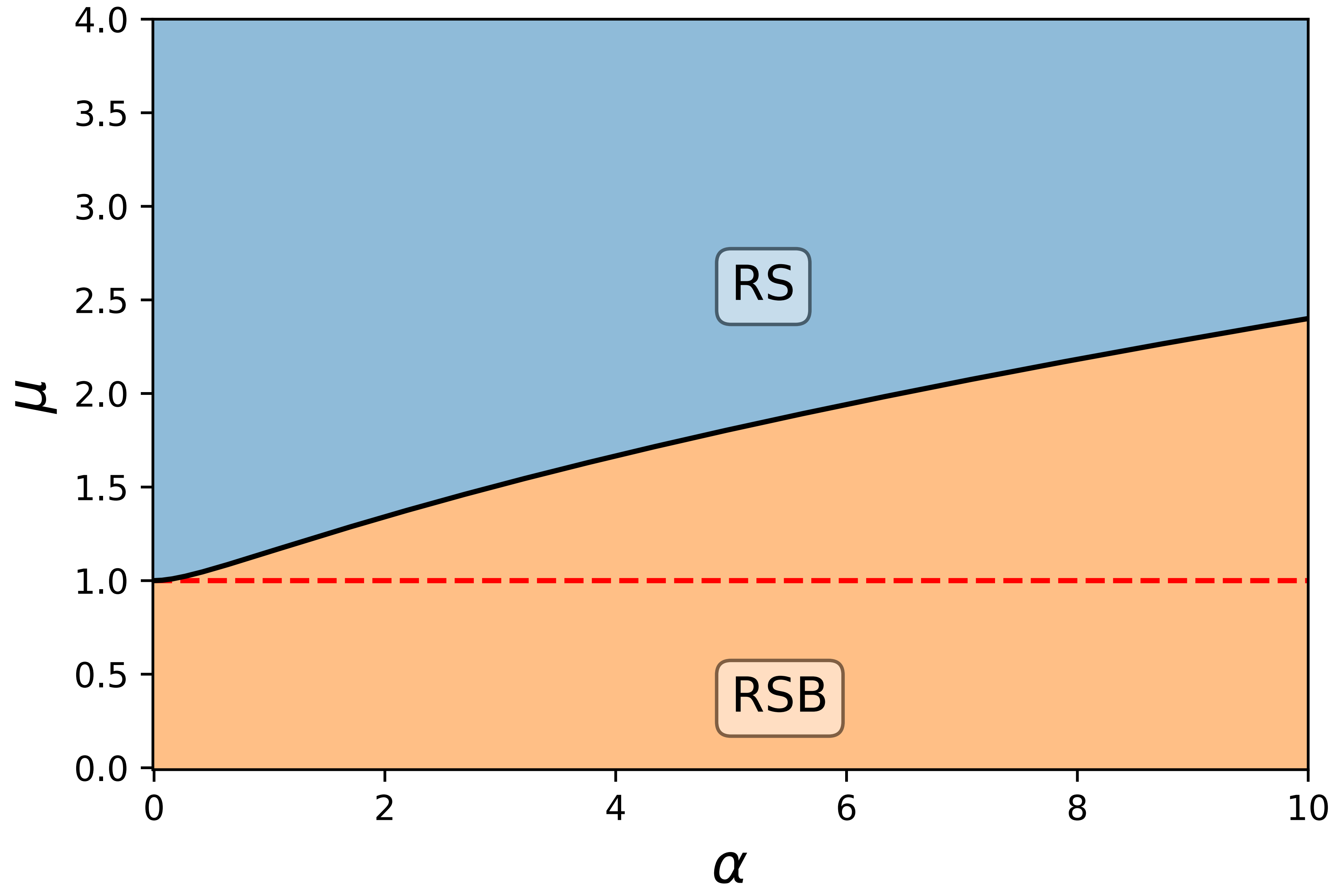
$$\alpha\left[\int dt\,\frac{\mu_c\,p_0(t)}{|\mu_c - t|} - 1\right] - 1 = 0$$

below which ergodicity is broken

Ergodicity breaking

For the simplest case with bounded support

$$\phi(z) = \cos(z + \theta) \qquad p_0(t) = \frac{1}{\pi\sqrt{1 - t^2}}$$



$$\mu_c(\alpha) = \frac{1 + \alpha}{\sqrt{1 + 2\alpha}}$$

Ergodicity breaking

The transition is expected to be continuous if the rescaled “breaking point” is positive

$$w_{\mathrm{AT}} = \frac{\alpha\mu^3\,\mathbb{E}\left[\mathscr{C}_3^2\right]}{2\left[\alpha\mu^3\,\mathbb{E}\left[\mathscr{C}_2^3\right] - (\alpha+2)\right]}$$

$$\epsilon_{\min}(\mu,0) - \epsilon_{\min}(\mu,h) = \lim_{\beta\to\infty}\frac{1}{\beta}\ln\langle e^{\beta h z}\rangle_H \qquad \mathscr{C}_k = -\,\partial_h^k\,\epsilon_{\min}(\mu,h)$$

Ergodicity breaking

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and the transition is towards a FRSB/1RSB phase if the following is positive/negative

$$w'_{\mathrm{AT}} = \frac{\alpha\mu^4\left(\mathbb{E}\left[\mathscr{C}_4^2\right] - 12w_{\mathrm{AT}}\,\mathbb{E}\left[\mathscr{C}_3^2\mathscr{C}_2\right] + 6w_{\mathrm{AT}}^2\,\mathbb{E}\left[\mathscr{C}_2^4\right]\right) - 6(\alpha+3)w_{\mathrm{AT}}^2}{2\left[\alpha\mu^3\,\mathbb{E}\left[\mathscr{C}_2^3\right] - (\alpha+2)\right]}$$

For the simplest model $w_{\mathrm{AT}} > 0$ while $w'_{\mathrm{AT}} > 0$ for $\alpha < 22.9\ldots$ and $w'_{\mathrm{AT}} < 0$ otherwise

1RSB solution

In addition to the AT line, a so-called random first order transition (RFOT) may occur when the ground-state energy obtained from a 1RSB solution matches that of the RS solution

1RSB solution

In addition to the AT line, a so-called random first order transition (RFOT) may occur when the ground-state energy obtained from a 1RSB solution matches that of the RS solution

The ground-state energy difference reads

$$\Delta e_{\text{typ}} = \underset{l,m\geq 0}{\text{ext}} \left[\frac{1}{2}\left(\frac{l}{\mu + ml} - \frac{1-\alpha}{m} \ln\left(1 + \frac{ml}{\mu}\right)\right) - \frac{\alpha}{m}\mathbb{E}\left[\ln\left(\int \frac{dh}{\sqrt{2\pi l}} e^{-\frac{h^2}{2l} - m\epsilon_{\min}(\mu+ml,h)} \right) - m\epsilon_{\min}(\mu)\right]\right]$$

The difference vanishes both for:

$l \to 0$ (continuous transition)

$m \to 0$ (discontinuous transition)

where the properties depend on the effective 1D model

$$H_{\mu,h}(z) = \frac{\mu}{2}z^2 - hz + \phi(z)$$

$$\epsilon_{\min}(\mu, h) = \min_z H_{\mu,h}(z)$$

$$z_{\min}(\mu, h) = \underset{z}{\text{argmin}}\, H_{\mu,h}(z)$$

1RSB solution

The RFOT transition occurs as $m \to 0$ which is obtained by solving

$$A(l_*) = 0 \text{ and } A'(l_*) = 0$$

$$A(l) = -\frac{l^2(1+\alpha)}{4\mu^2} + \alpha\,\mathbb{E}\left[\left(\int \frac{dh}{\sqrt{2\pi l}} e^{-\frac{h^2}{2l}} \epsilon_{\min}(\mu, h)\right)^2 - \int \frac{dh}{\sqrt{2\pi l}} e^{-\frac{h^2}{2l}} \left[\epsilon_{\min}^2(\mu, h) - l\, z_{\min}^2(\mu, h)\right]\right]$$

1RSB solution

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$$A(l) = -\frac{l^2(1+\alpha)}{4\mu^2} + \alpha\,\mathbb{E}\left[\left(\int \frac{dh}{\sqrt{2\pi l}} e^{-\frac{h^2}{2l}} \epsilon_{\min}(\mu, h)\right)^2 - \int \frac{dh}{\sqrt{2\pi l}} e^{-\frac{h^2}{2l}} \left[\epsilon^2_{\min}(\mu, h) - l\, z^2_{\min}(\mu, h)\right]\right]$$

The AT line is recovered as $l \to 0$

$$\alpha\,\mathbb{E}_\phi\left[\frac{\mu_c^2}{(\mu_c + \phi''[z_{\min}(\mu_c)])^2} - 1\right] - 1 = 0$$

The domain of stability of the RS ansatz is reduced

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Conclusion

This model of superposition of plane waves offers a new type of random landscapes

$$\mathcal{H}_N(\mathbf{x}) = \frac{\mu}{2}\mathbf{x}^2 + V_M(\mathbf{x}) \qquad V_M(\mathbf{x}) = \sum_{l=1}^{M} \phi_l(\mathbf{k}_l\mathbf{x})$$

- The annealed complexity can be computed for a large class of i.i.d. zero-mean translationally invariant random functions $\phi(x)$
- For unbounded support there is no topology trivialisation transition
- For bounded support there exists a critical value μ_c above which the landscape is topologically trivial
- These results are confirmed and extended from the computation of the ground-state energy

To go further

This model of superposition of plane waves offers a new type of random landscapes

$$\mathscr{H}_N(\mathbf{x}) = \frac{\mu}{2}\mathbf{x}^2 + V_M(\mathbf{x}) \qquad V_M(\mathbf{x}) = \sum_{l=1}^{M} \phi_l(\mathbf{k}_l\mathbf{x})$$

- The large deviation function of the ground-state energy can be computed for this model

$$\mathscr{L}(e) = -\lim_{N\to\infty} \frac{1}{N} \ln \mathbb{E}\left[\delta(e - e_{\min,N})\right]$$

- The annealed complexity of minima at fixed energy can be computed and provides a lower bound

$$\mathscr{L}(e) \geq -\Sigma_{\min}(e) = -\lim_{N\to\infty} \frac{1}{N} \ln \mathbb{E}\left[\rho_{\min}(e)\right]$$

SK model: Parisi & Rizzo '08,

2-spin: Fyodorov & Le Doussal '14, Dembo & Zeitouni '15

General spherical: LACT, Fyodorov & Le Doussal '24